COMMON FIXED POINTS FOR TWO WEAK SUBSEQUENTIAL CONTINUOUS MAPPINGS

Hakima Bouhadjera

Abstract. In this paper we are concerned with the existence and uniqueness of common fixed points for a pair of mappings satisfying an implicit relation under new concepts. Our results present an interesting contribution in the fixed point theory's area.

1. Introduction

In 1986, Jungck [5] introduced the notion of compatible mappings. Inspired by the above work, many authors developed much weaker conditions. One of the most interesting generalization is subcompatibility introduced in [2] ([3]). Again, Pathak et al. [10] introduced the notions of $R$-weak commutativity of type ($A_f$) and ($A_g$) for obtaining common fixed point theorems. Motivated by the above concepts, Kumar ([6], [7]) gave the notion of $R$-weak commutativity of type ($P$). On the other hand, Pant [8] initiated the study of fixed points for discontinuous mappings by using the concept of reciprocal continuity. Recently, in [2] ([3]), we suggested the notion of subsequential continuity which represents a legitimate generalization of the concept of reciprocal continuity. More recently, Pant et al. [9] introduced the notion of weak reciprocal continuity and obtained fixed point theorems by employing the new notion. Quite recently, Gopal et al. [4] presented their new notions of sequential continuity of type ($A_f$) and ($A_g$). Appeared in 2016 in [1], the new concept of weak subsequential continuity represents a genuine reasonable generalization of weak reciprocal continuity (resp. sequential continuity of type...)

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(A_f) or (A_g)). This definition makes up an addition to develop the literature of fixed point theory.

2. Preliminaries

Let us start by stating some needed definitions.

**Definition 2.1.** ([5]) Two self-mappings \( f \) and \( g \) of a metric space \((X, d)\) are called compatible if and only if

\[
\lim_{n \to \infty} d(fgx_n, gfx_n) = 0,
\]

whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t \) for some \( t \in X \).

**Definition 2.2.** ([2, 3]) Two self-mappings \( f \) and \( g \) of a metric space \((X, d)\) are called subcompatible if and only if there exists a sequence \( \{x_n\} \) in \( X \) such that

\[
\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t, \quad t \in X
\]

and which satisfy

\[
\lim_{n \to \infty} d(fgx_n, gfx_n) = 0.
\]

**Definition 2.3.** ([8]) Two self-mappings \( f \) and \( g \) of a metric space \((X, d)\) are called reciprocally continuous if

\[
\lim_{n \to \infty} fgx_n = ft \quad \text{and} \quad \lim_{n \to \infty} gfx_n = gt,
\]

whenever \( \{x_n\} \) is a sequence such that \( \lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t \) for some \( t \) in \( X \).

**Definition 2.4.** ([2, 3]) Two self-mappings \( f \) and \( g \) of a metric space \((X, d)\) are said to be subsequentially continuous if and only if there exists a sequence \( \{x_n\} \) such that

\[
\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t
\]

for some \( t \) in \( X \) and satisfy

\[
\lim_{n \to \infty} fgx_n = ft \quad \text{and} \quad \lim_{n \to \infty} gfx_n = gt.
\]

**Definition 2.5.** ([9]) Two self-mappings \( f \) and \( g \) of a metric space \((X, d)\) will be called weakly reciprocally continuous if \( \lim_{n \to \infty} fgx_n = ft \) or \( \lim_{n \to \infty} gfx_n = gt \), whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t \) for some \( t \) in \( X \).

**Definition 2.6.** ([4]) A pair \((f, g)\) of self-mappings defined on a metric space \((X, d)\) is said to be subsequentially continuous of type \((A_f)\) if and only if there exists a sequence \( \{x_n\} \) in \( X \) such that \( \lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t \) for some \( t \in X \) and

\[
\lim_{n \to \infty} fgx_n = ft \quad \text{and} \quad \lim_{n \to \infty} gfx_n = gt.
\]

**Definition 2.7.** ([4]) A pair \((f, g)\) of self-mappings defined on a metric space \((X, d)\) is said to be subsequentially continuous of type \((A_g)\) if and only if there exists a sequence \( \{x_n\} \) in \( X \) such that \( \lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t \) for some \( t \in X \) and

\[
\lim_{n \to \infty} gfx_n = gt \quad \text{and} \quad \lim_{n \to \infty} fgx_n = ft.
\]
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Definition 2.8. ([10]) Let \((\mathcal{X}, d)\) be a metric space and let \(f, g\) be self-mappings of \(\mathcal{X}\). The mappings \(f\) and \(g\) are said to be \(R\)-weakly commuting of type \((A_f)\) if there exists a positive real number \(R\) such that

\[
d(fgx, ggx) \leq Rd(fx, gx)
\]

for all \(x \in \mathcal{X}\). \(f\) and \(g\) are said to be \(R\)-weakly commuting of type \((A_f)\) if (2.1) holds for some real number \(R > 0\).

Definition 2.9. ([10]) Let \((\mathcal{X}, d)\) be a metric space and let \(f, g\) be self-mappings of \(\mathcal{X}\). The mappings \(f\) and \(g\) are said to be \(R\)-weakly commuting of type \((A_g)\) if there exists a positive real number \(R\) such that

\[
d(gfx, ffx) \leq Rd(fx, gx)
\]

for all \(x \in \mathcal{X}\). \(f\) and \(g\) are said to be \(R\)-weakly commuting of type \((A_g)\) if (2.2) holds for some real number \(R > 0\).

Definition 2.10. ([6, 7]) A pair of self-mappings \((f, g)\) of a metric space \((\mathcal{X}, d)\) is said to be \(R\)-weakly commuting of type \((P)\) if there exists some \(R > 0\) such that

\[
d(ffx, ggx) \leq Rd(fx, gx)
\]

for all \(x \in \mathcal{X}\).

Definition 2.11. ([1]) Two self-mappings \(f\) and \(g\) of a metric space \((\mathcal{X}, d)\) are called weakly subsequentially continuous if and only if there exists a sequence \(\{x_n\}\) in \(\mathcal{X}\) such that \(\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t\) for some \(t\) in \(\mathcal{X}\) and which satisfy

\[
\lim_{n \to \infty}gfx_n = ft \quad \text{or} \quad \lim_{n \to \infty}ffx_n = gt.
\]

According to definitions 2.5, 2.6, 2.7, 2.11, it can easily seen that weakly reciprocally continuous mappings are weakly subsequentially continuous mappings. Also, if in our definition we have \(\lim_{n \to \infty} gfx_n = ft\), then evidently sequentially continuous of type \((A_f)\) mappings imply our definition (alternately, if we have \(\lim_{n \to \infty} gfx_n = gt\), then sequentially continuous of type \((A_g)\) mappings imply our definition). To see that the converse implications are not true in general, let us give the next example which fulfills our desire.

Example 2.1. Let \(\mathcal{X} = [0, 6]\) and let \(d\) be the usual metric on \(\mathcal{X}\). We define \(f, g : \mathcal{X} \to \mathcal{X}\) as follows:

\[
fx = \begin{cases} 
3 - x & \text{if } x \in [0, 3] \\
\frac{9}{x} & \text{if } x \in (3, 6]
\end{cases}, \quad gx = \begin{cases} 
3 + x & \text{if } x \in [0, 3) \\
\frac{27}{x^2} & \text{if } x \in [3, 6].
\end{cases}
\]

First, we readily see that \(f\) and \(g\) are not continuous at \(x = 3\). It can also be noted that \(f\) and \(g\) are weakly subsequentially continuous. To see this, let \(\{x_n\}\) be the sequence in \(\mathcal{X}\) given by \(x_n = 3 + \frac{1}{n}\) for \(n = 1, 2, \ldots\). Then

\[
fx_n = \frac{9}{x_n} \to 3 \text{ as } n \to \infty,
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\[ gx_n = \frac{27}{x_n^2} \rightarrow 3 = t \] as \( n \rightarrow \infty \)

and

\[ fgx_n = f\left(\frac{27}{x_n^2}\right) = 3 - \frac{27}{x_n^2} \rightarrow 0 = f(3) \] as \( n \rightarrow \infty \),

but

\[ gfx_n = g\left(\frac{9}{x_n}\right) = 3 + \frac{9}{x_n} \rightarrow 6 \neq 3 = g(3) \] as \( n \rightarrow \infty \).

Again, it is obvious that \( f \) and \( g \) are not sequentially continuous of type \((A_f)\) because

\[ ggx_n = g\left(\frac{27}{x_n^2}\right) = 3 + \frac{27}{x_n^2} \rightarrow 6 \neq 3 = g(3) \] as \( n \rightarrow \infty \).

Finally, we can check that \( f \) and \( g \) are not weakly reciprocally continuous by giving the sequence \( x_n = \frac{1}{n} \) for \( n = 1, 2, \ldots \). Then

\[ fx_n = 3 - x_n \rightarrow 3 = t \] as \( n \rightarrow \infty \),

\[ gx_n = 3 + x_n \rightarrow 3 = t \]

but

\[ fgx_n = f(3 + x_n) = \frac{9}{3 + x_n} \rightarrow 3 \neq 0 = f(3) \] as \( n \rightarrow \infty \)

and

\[ gfx_n = g(3 - x_n) = 6 - x_n \rightarrow 6 \neq 3 = g(3) \] as \( n \rightarrow \infty \).

3. Implicit Relations

Motivated by [11], let us consider \( F \) the set of all continuous functions \( F : \mathbb{R}^6 \rightarrow \mathbb{R} \) such that

(1) \((F_1)\): \( F \) is increasing in variable \( t_6 \),

(2) \((F_2)\): there exists \( h \in [0, 1) \) such that for all \( u, v \geq 0 \),

\[ F(u, v, v, u, v, u) \leq 0 \implies u \leq hv, \]

(3) \((F_3)\): \( F(t, t, 0, 0, t, t) > 0 \) for all \( t > 0 \),

(4) \((F_4)\): \( F(t, 0, t, 0, t, 0) > 0 \) for all \( t > 0 \).

Example 3.1. \( F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - k \max\left\{ \frac{t_2 + t_3 + t_4}{3}, \frac{t_5 + t_6}{2} \right\} \), where \( k \in [0, 1) \).

(1) \((F_1)\): Obvious.

(2) \((F_2)\): Let \( u, v \geq 0 \),

\[ F(u, v, v, u, 0, u + v) = u - k \max\left\{ \frac{u + 2v}{3}, \frac{u + v}{2} \right\} \leq 0. \]

If \( u > v \), then \( v < u \leq \frac{k}{2 - k} v < v \), a contradiction. Hence \( u \leq v \) which implies \( u \leq hv \), where \( 0 \leq h = \frac{2k}{3 - k} < 1 \).

(3) \((F_3)\): \( F(t, t, 0, 0, t, t) = t - k \max\left\{ \frac{t}{3}, \frac{t}{2} \right\} = t(1 - k) > 0 \) for all \( t > 0 \).

(4) \((F_4)\): \( F(t, 0, t, 0, t, 0) = t - k \max\left\{ \frac{t}{3}, \frac{t}{2} \right\} = t(1 - k) > 0 \) for all \( t > 0 \).
Example 3.2. \( F(t_1, t_2, t_3, t_4, t_5, t_6) = kt_1 - t_2 - t_3 - t_4 - t_5 - t_6 \), where \( k > 5 \).

1. \((F_1)\): Obvious.
2. \((F_2)\): Let \( u, v \geq 0 \) and \( F(u, v, v, u, 0, u + v) = u(k - 2) - 3v \leq 0 \) which implies \( u \leq hv \), where \( h = \frac{3}{k - 2} \in [0, 1) \).
3. \((F_3)\): \( F(t, t, 0, 0, t, t) = t(k - 3) > 0 \) for all \( t > 0 \).
4. \((F_4)\): \( F(t, 0, 0, t, 0) = t(k - 2) > 0 \) for all \( t > 0 \).

Example 3.3. \( F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - k \frac{t_3 t_4 + t_5 t_6}{1 + t_2} \), where \( k \in [0, 1) \).

1. \((F_1)\): Obvious.
2. \((F_2)\): Let \( u, v \geq 0 \) be and \( F(u, v, v, u, 0, u + v) = u^2 - k \frac{uv}{1 + v} \leq 0 \). If \( u > 0 \), then \( u \leq k \frac{v}{1 + v} \), which implies \( u \leq hv \), where \( 0 \leq h = k < 1 \). If \( u = 0 \), then \( u \leq hv \).
3. \((F_3)\): \( F(t, t, 0, 0, t, t) = t^2(1 - \frac{k}{1 + t}) > 0 \) for all \( t > 0 \).
4. \((F_4)\): \( F(t, 0, 0, t, 0) = t^2 > 0 \) for all \( t > 0 \).

Now, let \( f \) and \( g \) be self-mappings of a metric space \((X, d)\). Let us define the set
\[ S = \{ \{x_n\} \subseteq X : \text{if there holds \( \lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_n = t \), then there holds} \] \[ \lim_{n \to \infty} f g x_n = ft \text{ or } \lim_{n \to \infty} g f x_n = gt \}. \]

Suppose that \( f X \subseteq g X \), there exists a sequence \( \{x_n\}_{n=0}^\infty \), such that \( x_{n+1} \) is the pre-image under \( g \) of \( f x_n \), that is
\[
(a) \, f x_0 = g x_1, f x_1 = g x_2, \ldots, f x_n = g x_{n+1}, \ldots
\]

Let us define the set \( U \) to be the set of all sequences \( \{x_n\} \) defined by \((a)\). Let us define the sequence \( \{y_n\} \subseteq X \) by \( y_n = f x_n = g x_{n+1}, n = 0, 1, 2, \ldots \).

4. Main Results

Theorem 4.1. Let \( f \) and \( g \) be weakly subsequentially continuous self-mappings of a complete metric space \((X, d)\) such that \( f X \subseteq g X \), \( U \cap S \neq \emptyset \) and
\[
(4.1) \, F(d(x, y), d(x, y), d(x, g x), d(f x, g x), d(f y, g y), d(f x, g y), d(g x, f y)) \leq 0
\]
for all \( x, y \) in \( X \) and \( F \in \mathcal{F} \). If \( f \) and \( g \) are either \( R \)-weakly commuting of type \((A)\) or \( R \)-weakly commuting of type \((A_f)\) or \( R \)-weakly commuting of type \((P)\) then \( f \) and \( g \) have a unique common fixed point.

Proof. We choose an arbitrary \( x_0 \) such that the corresponding sequence \( \{x_n\} \) defined in \((a)\) belongs to \( U \cap S \). Then, as in [9], by a routine calculation it follows that \( \{y_n\} \) defined above is a Cauchy sequence. Since \( X \) is complete, \( \{y_n\} \) converges to a point \( t \) in \( X \). Moreover, \( \lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_{n+1} = t \).
Now, suppose that $f$ and $g$ are $R$-weakly commuting of type $(A_g)$. Weak subsequential continuity of $f$ and $g$ implies that $\lim_{n \to \infty} fx_n = ft$ or $\lim_{n \to \infty} gx_n = gt$. Let us first assume that $\lim_{n \to \infty} fx_n = gt$. Then $R$-weak commutativity of type $(A_g)$ yields $d(fx_n, gx_n) \leq Rd(fx_n, gx_n)$. Taking the limit as $n \to \infty$, we obtain
\[ \lim_{n \to \infty} ff_n = 0. \]
Making $n \to \infty$ we get
\[ F(f, f) = 0, \]
that is $ft = gt$. Again, by virtue of $R$-weak commutativity of type $(A_g)$,
\[ d(f, g) \leq Rd(f, g), \]
which yields $ff = gft$ or $fg = ft = gft$. By (4.1) we have
\[ F(f, f), 0, d(f, f), 0) \leq 0, \]
that is $ft = gft$.

Next, assume that $\lim_{n \to \infty} fx_n = ft$. If $A \subseteq gA$ implies that there is a point $u \in X$ such that $ft = gu$. Then $\lim_{n \to \infty} fx_n = gu$. By virtue of (a) this also yields $\lim_{n \to \infty} fx_n = gu$. Hence $R$-weak commutativity of type $(A_g)$ implies that $d(fx_n, gx_n) \leq Rd(fx_n, gx_n)$. Taking the limit as $n \to \infty$ we get
\[ \lim_{n \to \infty} gx_n = ft = gu. \]
On using (4.1), we find
\[ F(f, f), 0, d(f, f), 0) \leq 0, \]
that is $ft = gft$.

Finally, on using (4.1), we get
\[ F(d(f, g), 0, d(f, g), 0) \leq 0, \]
which is a contradiction. Thus $fu = ffu = gfu$. So we obtain
\[ F(fu, f), d(gu, gfu), d(fu, gfu), d(gu, ffu), 0) \leq 0, \]
which is a contradiction. Thus $fu = ffu = gfu$.

Suppose that $f$ and $g$ are $R$-weakly commuting of type $(A_f)$. Weak subsequential continuity of $f$ and $g$ implies that $\lim_{n \to \infty} fx_n = ft$ or $\lim_{n \to \infty} gx_n = gt$. Let $\lim_{n \to \infty} gx_n = gt$. $R$-weak commutativity of type $(A_f)$ yields
d(ggx_n, fgx_n) \leq Rd(fx_n, gx_n). Taking the limit as n \to \infty and by virtue of (a), we obtain \( \lim_{n \to \infty} fgx_n = gt \). On using (4.1), we get

\[
F(d(ft, fgx_n), d(gt, ggx_n), d(ft, gt), d(fgx_n, ggx_n),
d(ft, ggx_n), d(gt, fgx_n)) \leq 0.
\]

Making n \to \infty, we get

\[
F(d(ft, gt), 0, d(ft, gt), 0, d(ft, gt), 0) \leq 0,
\]
a contradiction. Hence ft = gt. Again, by virtue of R-weak commutativity of type \((A_f)\), \(d(ggt, fglt) \leq Rd(fx_n, gx_n)\). This yields ggt = fglt and \(fglt = ggt = ffft\). By (4.1), we have

\[
F(d(ft, ffft), d(gt, gfft), d(ft, gt), d(ffft, gfft), d(ft, gfft), d(gt, ffft))
= F(d(ft, ffft), d(ft, ffft), 0, 0, d(ft, ffft), d(ft, ffft)) \leq 0,
\]
that is \(ft = ffft = ggt\).

Next, assume that \(\lim_{n \to \infty} fgx_n = ft\). Then \(fX \subseteq gX\) implies that there is an element \(u \in X\) such that \(ft = gu\). Hence R-weak commutativity of type \((A_f)\) implies that \(d(ggx_n, fgx_n) \leq Rd(fx_n, gx_n)\). Letting n \to \infty we get

\[
\lim_{n \to \infty} ggx_n = ft = gu.
\]

On using (4.1), we get

\[
F(d(fu, fgx_n), d(gu, ggx_n), d(fu, gu), d(fgx_n, ggx_n),
d(fu, ggx_n), d(gu, fgx_n)) \leq 0.
\]

Taking the limit as n \to \infty, we obtain

\[
F(d(fu, gu), 0, d(fu, gu), 0, d(fu, gu), 0) \leq 0,
\]
a contradiction so that \(fu = gu\). Again, using R-weak commutativity of type \((A_f)\) we have \(d(ggu, fgu) \leq Rd(fu, gu)\). This yields ggu = fggu and \(gfu = ggu = fggu = ffft\). By (4.1), we have

\[
F(d(fu, ffft), d(gu, gfft), d(fu, gu), d(ffft, gfft), d(fu, gfft), d(gu, ffft))
= F(d(fu, ffft), d(fu, ffft), 0, 0, d(fu, ffft), d(fu, ffft)) \leq 0,
\]
i.e., \(fu = ffft = ggu\).

Finally, suppose that f and g are R-weakly commuting of type \((P)\). Now, weak subsequential continuity of f and g implies that \(\lim_{n \to \infty} fx_n = ft\) or \(\lim_{n \to \infty} gx_n = gt\). Let us first assume that \(\lim_{n \to \infty} fx_n = gt\). By virtue of (a) and R-weak commutativity of type \((P)\), we have \(\lim_{n \to \infty} ffx_n = \lim_{n \to \infty} ggx_n = gt\). Using (4.1), we get

\[
F(d(ft, ffx_n), d(gt, gfx_n), d(ft, gt), d(ffx_n, gfx_n),
d(ft, gfx_n), d(gt, ffx_n)) \leq 0.
\]

On letting n \to \infty, we obtain

\[
F(d(ft, gt), 0, d(ft, gt), 0, d(ft, gt), 0) \leq 0,
\]
that is \(ft = gt\). Again, by virtue of R-weak commutativity of type \((P)\),
\(d(fflt = ggt) \leq Rd(ft, gt)\). This implies that \(fflt = ggt\)
and \( fgt = fft = gft \). Also, using (4.1), we find
\[
F(d(ft, fft), d(gt, gft), d(ft, gt), d(fft, gft), d(fft, gft), d(gt, fft)) = F(d(ft, fft), d(ft, fft), 0, 0, d(ft, fft), d(ft, fft)) \leq 0,
\]
a contradiction. Hence \( ft = fft = gft \).

Now, assume that \( \lim_{n \to \infty} fgx_n = ft \). \( f\mathcal{X} \subseteq g\mathcal{X} \) implies that there exists a point \( u \in \mathcal{X} \) which verifies \( ft = gu \). By virtue of (a) and \( R \)-weak commutativity of type \((\mathcal{P})\), we get \( \lim_{n \to \infty} ffx_n = \lim_{n \to \infty} ggx_n = ft = gu \). We assert that \( fu = gu \). Let on contrary that \( fu \neq gu \). Using (4.1), we obtain
\[
F(d(fu, fgx_n), d(gu, ggx_n), d(fu, gu), d(fgx_n, ggx_n), d(fu, ggx_n), d(gu, fgx_n)) \leq 0.
\]
At infinity, we get
\[
F(d(fu, gu), 0, d(fu, gu), 0, d(fu, gu), 0) \leq 0,
\]
i.e., \( fu = gu \). Again, by virtue of \( R \)-weak commutativity of type \((\mathcal{P})\), \( d(ffu, ggu) \leq Rd(fu, gu) \). This yields \( ffu = ggu \) and \( gu = fffu = ggu = gfu \). On using (4.1), we get
\[
F(d(fu, ffu), d(gu, gfu), d(fu, gu), d(ffu, gfu), d(fu, gfu), d(gu, ffu)) = F(d(fu, ffu), d(fu, ffu), 0, 0, d(fu, ffu), d(fu, ffu)) \leq 0,
\]
that is, \( fu = ffu = gfu \).

Uniqueness of the common fixed point follows immediately by \((F_3)\) and (4.1).

To illustrate our Theorem, we give the following example.

**Example 4.1.** Let \( \mathcal{X} = [1, 3] \) and let \( d \) be the usual metric on \( \mathcal{X} \). Define \( f, g : \mathcal{X} \to \mathcal{X} \) as follows:
\[
fx = \begin{cases} 
1 & \text{if } x = 1 \\
3 & \text{if } x \in (1, 2] \\
1 & \text{if } x \in (2, 3],
\end{cases}
gx = \begin{cases} 
1 & \text{if } x = 1 \\
3 & \text{if } x \in (1, 2] \\
\frac{x}{2} & \text{if } x \in (2, 3].
\end{cases}
\]
Then \( f \) and \( g \) satisfy all conditions of the above theorem and have the common fixed point \( x = 1 \). It can be verified that \( f \) and \( g \) satisfy condition (4.1) with \( F = t_1 - k \max\{t_2, t_3, t_4, t_5, t_6\} \) where \( k \in \left[1, \frac{1}{4}, \frac{1}{2}\right) \). Furthermore, \( f \) and \( g \) are \( R \)-weakly commuting of type \((\mathcal{A}_b)\). It can be noted that \( f \) and \( g \) are weakly subcontinuously continuous. At this end, let \( \{x_n\} \) be a sequence in \( \mathcal{X} \) such that \( x_n = 2 + \frac{1}{n} \) for \( n = 1, 2, \ldots \). Then, \( fx_n = 1 \to 1 = t, gx_n = \frac{x_n}{2} \to 1 = t \) and \( ggx_n = g(1) = 1 \), but \( fgx_n = f(\frac{x_n}{2}) = \frac{3}{2} \neq 1 = f(1) \). On the other hand, we have \( f\mathcal{X} = \{1, \frac{3}{2}\} \subseteq g\mathcal{X} = [\frac{3}{2}, 3] \cup \{3\} \).

Now, we give some results.
Corollary 4.1. Let \( f \) and \( g \) be weakly subsequentially continuous mappings from a complete metric space \( (X, d) \) into itself such that \( fX \subseteq gX \), \( U \cap S \neq \emptyset \) and 
\[
d(fx, fy) \leq k \max\{ \frac{d(gx, gy) + d(fx, gx) + d(fy, gy)}{3}, \frac{d(fx, gy) + d(gx, fy)}{2} \}
\]
for all \( x, y \) in \( X \), where \( k \in [0, 1) \). If \( f \) and \( g \) are either \( R \)-weakly commuting of type \((A_g)\) or \( R \)-weakly commuting of type \((A_f)\) or \( R \)-weakly commuting of type \((P)\) then \( f \) and \( g \) have a unique common fixed point.

Proof. Use Theorem 4.1 and Example 3.1.

Corollary 4.2. Let \( f \) and \( g \) be weakly subsequentially continuous mappings from a complete metric space \( (X, d) \) into itself such that \( fX \subseteq gX \), \( U \cap S \neq \emptyset \) and 
\[
d(fx, fy) \leq \frac{1}{k}d(gx, gy) + d(fx, gx) + d(fy, gy) + d(fx, gy) + d(gx, fy)
\]
for all \( x, y \) in \( X \), where \( k > 5 \). If \( f \) and \( g \) are either \( R \)-weakly commuting of type \((A_g)\) or \( R \)-weakly commuting of type \((A_f)\) or \( R \)-weakly commuting of type \((P)\) then \( f \) and \( g \) have a unique common fixed point.

Proof. Use Theorem 4.1 and Example 3.2.

Corollary 4.3. Let \( f \) and \( g \) be weakly subsequentially continuous mappings from a complete metric space \( (X, d) \) into itself such that \( fX \subseteq gX \), \( U \cap S \neq \emptyset \) and 
\[
d^2(fx, fy) \leq k[d(fx, gx)d(fy, gy) + d(fx, gy)d(gx, fy)]
\]
for all \( x, y \) in \( X \), where \( k \in [0, 1) \). If \( f \) and \( g \) are either \( R \)-weakly commuting of type \((A_g)\) or \( R \)-weakly commuting of type \((A_f)\) or \( R \)-weakly commuting of type \((P)\) then \( f \) and \( g \) have a unique common fixed point.

Proof. Use Theorem 4.1 and Example 3.3.

In the following, we will prove a common fixed point theorem for a subcompatible pair of self-mappings.

Theorem 4.2. Let \( f \) and \( g \) be weakly reciprocally continuous subcompatible self-mappings of a metric space \((X, d)\) satisfying \( fX \subseteq gX \) and
\[
F(d(fx, fy), d(gx, gy), d(fx, gx), d(fy, gy), d(fx, gy), d(gx, fy)) \leq 0
\]
for all \( x, y \) in \( X \), where \( F \) is continuous and satisfies only \((F_3)\) and \((F_3)\). If \( f \) and \( g \) are \( R \)-weakly commuting of type \((A_g)\) or \( R \)-weakly commuting of type \((A_f)\) then \( f \) and \( g \) have a unique common fixed point.

Proof. Since \( f \) and \( g \) are weakly reciprocally continuous there exists a sequence \( \{x_n\} \) in \( X \) such that \( \lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t \) for some \( t \) in \( X \) and which satisfy \( \lim_{n \to \infty} ggx_n = ft \) or \( \lim_{n \to \infty} gfx_n = gt \). Let \( \lim_{n \to \infty} gfx_n = gt \). Then \( R \)-weak commutativity of type \((A_g)\) yields \( d(fgx_n, gfx_n) \leq Rd(fx_n, gx_n) \). Taking the limit as
\( n \to \infty \), we get \( \lim_{n \to \infty} ffx_n = gt \). By (4.2) we have

\[
F(d(ft, ffx_n), d(gt, gfx_n), d(ft, gt), d(ffx_n, gfx_n),
\]
\[
d(ft, gfx_n), d(gt, ffx_n)) \leq 0.
\]

At infinity we get

\[
F(d(ft, gt), 0, d(ft, gt), 0, d(ft, gt), 0) \leq 0,
\]
a contradiction so that \( ft = gt \). Again, by virtue of \( R \)-weak commutativity of type \( (A_3) \), \( d(ffx, gft) \leq Rd(ft, gt) \), which yields \( ffx = gft \) and \( fgt = ffx = gft \). Using (4.2), we obtain

\[
F(d(ft, ffx), d(gt, gft), d(ft, gt), d(ffx, gft), d(gt, ffx)) = F(d(ft, ffx), d(ft, ffx), 0, 0, d(ft, ffx), d(ft, ffx)) \leq 0,
\]
a contradiction. Hence \( ftx = gtx \).

Next, assume that \( \lim_{n \to \infty} ftx_n = ft \). Since \( fX \subseteq gX \) then, there is an element \( u \in X \) such that \( ft = gu \). By virtue of subcompatibility and \( R \)-weak commutativity of type \( (A_3) \), \( \lim_{n \to \infty} ftx_n = \lim_{n \to \infty} gfx_n = ft = gu \). By (4.2), we have

\[
F(d(fu, ftx_n), d(gu, gfx_n), d(fu, gu), d(ftx_n, gfx_n),
\]
\[
d(fu, gfx_n), d(gu, ftx_n)) \leq 0.
\]

Making \( n \to \infty \), we obtain

\[
F(d(fu, gu), 0, d(fu, gu), 0, d(fu, gu), 0) \leq 0,
\]
a contradiction. Hence \( fu = gu \). Again, by virtue of \( R \)-weak commutativity of type \( (A_3) \) we get \( ffu = gfu \) and \( fgu = ffx = gfu = ggu \). On using (4.2), we obtain

\[
F(d(fu, ffx), d(gu, gfu), d(fu, gu), d(ffx, gfu), d(fu, ffu), d(gu, ffu))
\]
\[
= F(d(fu, ffx), d(fu, ffx), 0, 0, d(fu, ffx), d(fu, ffx)) \leq 0,
\]
that is, \( fu = ffx = fgu \).

Finally, suppose that \( f \) and \( g \) are \( R \)-weakly commuting of type \( (A_f) \), then, we have \( d(fgx_n, gx_n) \leq Rd(fx_n, gx_n) \). Now, weak reciprocal continuity implies that \( \lim_{n \to \infty} ftx_n = ft \) or \( \lim_{n \to \infty} gfx_n = gt \). Let \( \lim_{n \to \infty} ftx_n = gt \). By virtue of subcompatibility, we have \( \lim_{n \to \infty} gfx_n = gt \) and consequently \( \lim_{n \to \infty} gtx_n = gt \). Using (4.2), we get

\[
F(d(ft, ftx_n), d(gt, gtx_n), d(ft, gt), d(ftx_n, gtx_n),
\]
\[
d(ft, gtx_n), d(gt, ftx_n)) \leq 0.
\]

At infinity we get

\[
F(d(ft, gt), 0, d(ft, gt), 0, d(ft, gt), 0) \leq 0,
\]
i.e., \( ft = gt \). Again, by virtue of \( R \)-weak commutativity of type \( (A_f) \), \( gft = fgt \) and \( gft = gft = fft \). On using (4.2), we obtain

\[
F(d(ft, fft), d(gt, gft), d(ft, gt), d(fft, gft), d(ft, gft), d(gt, fft)) = F(d(ft, fft), d(ft, fft), 0, 0, d(ft, fft), d(ft, fft)) \leq 0,
\]

that is, \( ft = fft = gft \).

Next, suppose that \( \lim_{n \to \infty} fx_n = ft \). \( fX \subseteq gX \) implies that, there exists some \( u \in X \) such that \( ft = gu \). By virtue of \( R \)-weak commutativity of type \( (A_f) \), we have

\[
\lim_{n \to \infty} ggx_n = \lim_{n \to \infty} ffx_n = ft = gu.
\]

Using (4.2), we get

\[
F(d(fu, fx_n), d(gu, ggx_n), d(fu, gu), d(fgx_n, ggx_n), d(fu, ggx_n), d(gu, fx_n)) \leq 0.
\]

Taking the limit as \( n \to \infty \), we obtain

\[
F(d(fu, gu), 0, d(fu, gu), 0, d(fu, gu), 0) \leq 0,
\]

i.e., \( fu = gu \). Again, by virtue of \( R \)-weak commutativity of type \( (A_f) \) we get \( ggx = fgu \) and \( gfu = ggu = fgu = ffxu \). We assert that \( fx = ffu = gfu \). Let on contrary that \( fu \neq ffxu \). On using (4.2), we obtain

\[
F(d(fu, fx_u), d(gu, gfu), d(fu, gu), d(fu, ggu), d(fu, gfu), d(gu, ffxu)) = F(d(fu, fx_u), d(fu, ffxu), 0, 0, d(fu, ffxu), d(fu, ffxu)) \leq 0,
\]
a contradiction. Hence \( fu = ffxu = gfu \).

Uniqueness of the common fixed point follows easily by \((F_3)\) and (4.2).

The next example illustrates our result.

**Example 4.2.** Endow \( X = [0, 10] \) with the absolute value metric and define \( f, g : X \to X \) by

\[
fx = \begin{cases} 
1 & \text{if } x \in [0, 1] \\
4 & \text{if } x \in (1, 5] \\
1 & \text{if } x \in (5, 10], 
\end{cases} \quad gx = \begin{cases} 
1 & \text{if } x \in [0, 1] \\
7 & \text{if } x \in (1, 5] \\
x + \frac{1}{6} & \text{if } x \in (5, 10]. 
\end{cases}
\]

Then \( f \) and \( g \) are certainly \( R \)-weakly commuting of type \( (A_f) \) since

\[
d(ffx, gfx) \leq Rd(fx, gx) \text{ for all } x \in X.
\]

Moreover, \( f \) and \( g \) are subcompatible. To this end, consider the sequence \( x_n = 1 - \frac{1}{n} \) for \( n = 1, 2, \ldots \). Then \( fx_n = 1 - gx_n \) and \( gfx_n = gfx = 1 \). Thus \( |gfx_n - gfx| = 0 \). To see that \( f \) and \( g \) are weakly reciprocally continuous, consider \( x_n = \frac{5}{n} + \frac{1}{n} \) for \( n = 1, 2, \ldots \). Then \( fx_n = 1, gx_n = \frac{x_n + 1}{6} \to 1 \), and \( gfx_n = 1 = g(1) \); whereas

\[
gfx_n = f(\frac{x_n + 1}{6}) = \frac{4}{3} \neq 1 = f(1).
\]

On the other hand, observe that
Finally, we can check that condition (4.2) is verified for all $x, y \in X$ with $k \in [\frac{3}{35}, 1)$. Consequently, all conditions of theorem 4.2 are satisfied and $x = 1$ is the unique common fixed point.

Finally, we end our paper by giving some results.

**Corollary 4.4.** Let $f$ and $g$ be weakly reciprocally continuous subcompatible self-mappings of a metric space $(X, d)$ satisfying $fX \subseteq gX$ and the inequality

$$d(fx, fy) \leq k\max\left\{ \frac{d(gx, gy) + d(fx, gx) + d(fy, gy)}{3}, \frac{d(fx, gy) + d(gx, fy)}{2} \right\}$$

for all $x, y \in X$, where $k \in [0, 1)$. If $f$ and $g$ are $R$-weakly commuting of type $(A_g)$ or $R$-weakly commuting of type $(A_f)$ then $f$ and $g$ have a unique common fixed point.

**Proof.** Use Theorem 4.2 and Example 3.1.

**Corollary 4.5.** Let $f$ and $g$ be weakly reciprocally continuous subcompatible self-mappings of a metric space $(X, d)$ satisfying $fX \subseteq gX$ and the inequality

$$d(fx, fy) \leq \frac{1}{k}[d(gx, gy) + d(fx, gx) + d(fy, gy) + d(fx, gy) + d(gx, fy)]$$

for all $x, y \in X$, where $k > 3$. If $f$ and $g$ are $R$-weakly commuting of type $(A_g)$ or $R$-weakly commuting of type $(A_f)$ then $f$ and $g$ have a unique common fixed point.

**Proof.** Use Theorem 4.2 and Example 3.2.

**Corollary 4.6.** Let $f$ and $g$ be weakly reciprocally continuous subcompatible self-mappings of a metric space $(X, d)$ satisfying $fX \subseteq gX$ and the inequality

$$d^2(fx, fy) \leq k\frac{d(fx, gx)d(fy, gy) + d(fx, gy)d(gx, fy)}{1 + d(gx, gy)}$$

for all $x, y \in X$, where $k \in [0, 1)$. If $f$ and $g$ are $R$-weakly commuting of type $(A_g)$ or $R$-weakly commuting of type $(A_f)$ then $f$ and $g$ have a unique common fixed point.

**Proof.** Use Theorem 4.2 and Example 3.3.

**References**


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Laboratory of Applied Mathematics, Badji Mokhtar–Annaba University, P.O. Box 12, 23000 Annaba, Algeria
E-mail address: b_hakima2000@yahoo.fr