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ON THE $(p, q)-$ FIBONACCI $N-$DIMENSIONAL RECURRENCES

Orhan Dişkaya and Hamza Menken

Abstract. In this study, one-dimensional, two-dimensional, three-dimensional and $n-$dimensional recurrences of the $(p, q)-$Fibonacci sequence are examined and their some identities are given.

1. Introduction

A rabbit problem is included in the book "Liber Abaci" written by Fibonacci in 1202. An interesting number pattern was encountered in the solution of this problem. This number sequence is the most important number sequence discovered, namely the Fibonacci number sequence. The Fibonacci sequence F_n is defined by the recurrence

$$F_{n+2} = F_{n+1} + F_n, \quad n \geqslant 0$$

with the initial values $F_0 = 0$ and $F_1 = 1$. For the recent process of Fibonacci and Fibonacci-like sequences, we refer the reader to [**2, 6, 7, 8, 9, 10, 12**]. In these studies, interesting properties of this number sequence were noticed. The Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas, Tribonacci, Padovan, and Perrin number sequences are some other number sequences defined similarly to the Fibonacci number sequence. In addition, the Fibonacci sequence has been generalized until today and has been associated with other mathematical structures and interesting results have been obtained. One of the generalizations of Fibonacci sequence is the $(p, q)-$Fibonacci sequence. For $n \in \mathbb{N}$ and $p^2 + 4q > 0$, the $(p, q)-$Fibonacci

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sequence $F_n(p,q)$ is defined by the recurrence

$$F_{n+2}(p,q) = pF_{n+1}(p,q) + qF_n(p,q)$$

with the initial values $F_0(p,q) = 0$ and $F_1(p,q) = 1$ [**11**].
In this study, some components of the $(p,q)-$Fibonacci sequence are mentioned. These components are given below.

2. Two-dimensional recurrences of the $(p,q)-$Fibonacci sequence

In this section, we first introduce the two-dimensional recurrences of the $(p,q)-$Fibonacci sequence based on the one-dimensional recurrence.

Definition 2.1. For $n, m \in \mathbb{N}$ and $p^2 + 4q > 0$, the two-dimensional of $(p,q)-$Fibonacci sequence $F_{p,q}^{n,m}$ is defined by the recurrences

$$F_{p,q}^{n+2,m} = pF_{p,q}^{n+1,m} + qF_{p,q}^{n,m} \tag{2.1}$$

$$F_{p,q}^{n,m+2} = pF_{p,q}^{n,m+1} + qF_{p,q}^{n,m} \tag{2.2}$$

with the initial values $F_{p,q}^{0,0} = 0$, $F_{p,q}^{1,0} = 1$, $F_{p,q}^{0,1} = i$, and $F_{p,q}^{1,1} = 1 + i$.

Proposition 2.1. *The following properties are apply:*

1. $F_{p,q}^{n,0} = F_n(p,q)$,
2. $F_{p,q}^{0,m} = iF_m(p,q)$,
3. $F_{p,q}^{1,m} = F_{m+1}(p,q) + iF_m(p,q)$,
4. $F_{p,q}^{n,1} = F_n(p,q) + iF_{n+1}(p,q)$.
5. $F_{p,q}^{n,m} = F_n(p,q)F_{m+1}(p,q) + iF_{n+1}(p,q)F_m(p,q)$.

Proof. 1. By the recurrence $F_{p,q}^{n+2,m} = pF_{p,q}^{n+1,m} + qF_{p,q}^{n,m}$, $m = 0$ and the Mathematical Principle of Induction, we can prove the first assertion. With this way, for $n = 0$:

$$F_{p,q}^{0,0} = F_0(p,q) = 0$$

Hence $F_{p,q}^{0,0} = 0$ and the equality is true. Suppose that the desired equality is true for any $k \leqslant n$. Thus, we write

$$F_{p,q}^{k,0} = F_k(p,q).$$

Let's show that it is true for $k + 1$. Then

$$\begin{aligned} F_{p,q}^{k+3,0} &= pF_{p,q}^{k+2,0} + qF_{p,q}^{k+1,0} \\ &= pF_{k+2}(p,q) + qF_{k+1}(p,q) \\ &= F_{k+3}(p,q). \end{aligned}$$

Thus, the validity of Proposition 2.1 (1) has been confirmed.

2. The recurrence $F_{p,q}^{n,m+2} = pF_{p,q}^{n,m+1} + qF_{p,q}^{n,m}$, $n = 0$ and the Mathematical Principle of Induction are used for the proof of this part. For $m = 0$:

$$F_{p,q}^{0,0} = iF_0(p,q) = 0$$

the equality is true, since $F_{p,q}^{0,0} = iF_0(p,q)$. Suppose it is true for any $k \leqslant m$, Thus

$$F_{p,q}^{0,k} = iF_k(p,q)$$

Let's indicate that it is accurate for $k+1$. Then

$$\begin{aligned} F_{p,q}^{0,k+3} &= pF_{p,q}^{0,k+2} + qF_{p,q}^{0,k+1} \\ &= piF_{k+2}(p,q) + qiF_{k+1}(p,q) \\ &= iF_{k+3}(p,q). \end{aligned}$$

So, the validity of Proposition 2.1 (2) has been confirmed.

3. By the recurrence $F_{p,q}^{n+2,m} = pF_{p,q}^{n+1,m} + qF_{p,q}^{n,m}$, $n=1$ and the Mathematical Principle of Induction, we can prove the third assertion. With this way, for $m=0$:

$$F_{p,q}^{1,0} = F_1(p,q) + iF_0(p,q) = 1$$

the equality is true. Suppose that the desired equality is true for any $k \leqslant m$. Thus, we write

$$F_{p,q}^{1,k} = F_{k+1}(p,q) + iF_k(p,q).$$

Let's show that it is true for $k+1$. Then

$$\begin{aligned} F_{p,q}^{1,k+3} &= pF_{p,q}^{1,k+2} + qF_{p,q}^{1,k+1} \\ &= p\left(F_{k+3}(p,q) + iF_{k+2}(p,q)\right) + q\left(F_{k+2}(p,q) + iF_{k+1}(p,q)\right) \\ &= \left(pF_{k+3}(p,q) + qF_{k+2}(p,q)\right) + i\left(pF_{k+2}(p,q) + qF_{k+1}(p,q)\right) \\ &= F_{k+4}(p,q) + iF_{k+3}(p,q) \end{aligned}$$

Thus, the validity of Proposition 2.1 (3) has been confirmed.

4. The recurrence $F_{p,q}^{n+2,m} = pF_{p,q}^{n+1,m} + qF_{p,q}^{n,m}$, $m=1$ and the Mathematical Induction Principle are used for the proof of this part. For $n=0$:

$$F_{p,q}^{0,1} = F_0(p,q) + iF_1(p,q) = i$$

the equality is true. Suppose it is true for any $k \leqslant n$, Thus

$$F_{p,q}^{k,1} = F_k(p,q) + iF_{k+1}(p,q)$$

Let's indicate that it is accurate for $k+1$. Then

$$\begin{aligned} F_{p,q}^{k+3,1} &= pF_{p,q}^{k+2,1} + qF_{p,q}^{k+1,1} \\ &= p\left(F_{k+2}(p,q) + iF_{k+3}(p,q)\right) + q\left(F_{k+1}(p,q) + iF_{k+2}(p,q)\right) \\ &= \left(pF_{k+2}(p,q) + qF_{k+1}(p,q)\right) + i\left(pF_{k+3}(p,q) + qF_{k+2}(p,q)\right) \\ &= F_{k+3}(p,q) + iF_{k+4}(p,q). \end{aligned}$$

So, the validity of Proposition 2.1 (4) has been confirmed.

5. By the recurrence $F_{p,q}^{n+2,m} = pF_{p,q}^{n+1,m} + qF_{p,q}^{n,m}$ and $F_{p,q}^{n,m+2} = pF_{p,q}^{n,m+1} + qF_{p,q}^{n,m}$ the Mathematical Principle of Induction, we can prove the fifth assertion. With this way, for $m = 0$:

$$\begin{aligned} F_{p,q}^{n,0} &= F_n(p,q)F_1(p,q) + iF_{n+1}(p,q)F_0(p,q) \\ &= F_n(p,q) \end{aligned}$$

and for $n = 0$:

$$\begin{aligned} F_{p,q}^{0,m} &= F_0(p,q)F_{m+1}(p,q) + iF_1(p,q)F_m(p,q) \\ &= iF_m(p,q) \end{aligned}$$

the equalities are true. Suppose that the desired equalities are true for any $k \leqslant n$ and $k \leqslant m$, respectively, thus, we write

$$F_{p,q}^{k,m} = F_k(p,q)F_{m+1}(p,q) + iF_{k+1}(p,q)F_m(p,q)$$

and

$$F_{p,q}^{n,k} = F_n(p,q)F_{k+1}(p,q) + iF_{n+1}(p,q)F_k(p,q).$$

Let's show that these equalities is true. Then for $n = k+1$,

$$\begin{aligned} F_{p,q}^{k+3,m} &= pF_{p,q}^{k+2,m} + qF_{p,q}^{k+1,m} \\ &= p\left(F_{k+2}(p,q)F_{m+1}(p,q) + iF_{k+3}(p,q)F_m(p,q)\right) \\ &+ q\left(F_{k+1}(p,q)F_{m+1}(p,q) + iF_{k+2}(p,q)F_m(p,q)\right) \\ &= \left(pF_{k+2}(p,q) + qF_{k+1}(p,q)\right)F_{m+1}(p,q) \\ &+ i(p,q)\left(pF_{k+3}(p,q) + qF_{k+2}(p,q)\right)F_m \\ &= F_{k+3}(p,q)F_{m+1}(p,q) + iF_{k+4}(p,q)F_m(p,q) \end{aligned}$$

for $m = k+1$,

$$\begin{aligned} F_{p,q}^{n,k+3} &= pF_{p,q}^{n,k+2} + qF_{p,q}^{n,k+1} \\ &= p\left(F_n(p,q)F_{k+3}(p,q) + iF_{n+1}(p,q)F_{k+2}(p,q)\right) \\ &+ q\left(F_n(p,q)F_{k+2}(p,q) + iF_{n+1}(p,q)F_{k+1}(p,q)\right) \\ &= F_n(p,q)\left(pF_{k+3}(p,q) + qF_{k+2}(p,q)\right) \\ &+ iF_{n+1}(p,q)\left(pF_{k+2}(p,q) + qF_{k+1}(p,q)\right) \\ &= F_n(p,q)F_{k+4}(p,q) + iF_{n+1}(p,q)F_{k+3}(p,q) \end{aligned}$$

Thus, the validity of Proposition 2.1 (5) has been confirmed.

□

3. Three-dimensional Recurrences of the $(p,q)-$Fibonacci sequence

In this section the three-dimensional recurrence of the $(p,q)-$Fibonacci sequence is defined.

DEFINITION 3.1. For $n,m,r \in \mathbb{N}$ and $p^2+4q>0$, the three-dimensional of $(p,q)-$Fibonacci sequence $F_{p,q}^{n,m,r}$ is defined by the recurrences

$$F_{p,q}^{n+2,m,r} = pF_{p,q}^{n+1,m,r} + qF_{p,q}^{n,m,r} \tag{3.1}$$

$$F_{p,q}^{n,m+2,r} = pF_{p,q}^{n,m+1,r} + qF_{p,q}^{n,m,r} \tag{3.2}$$

$$F_{p,q}^{n,m,r+2} = pF_{p,q}^{n,m,r+1} + qF_{p,q}^{n,m,r} \tag{3.3}$$

with the initial values $F_{p,q}^{0,0,0}=0$, $F_{p,q}^{1,0,0}=1$, $F_{p,q}^{0,1,0}=i$, $F_{p,q}^{0,0,1}=j$, $F_{p,q}^{1,1,0}=1+i$, $F_{p,q}^{1,0,1}=1+j$, $F_{p,q}^{0,1,1}=i+j$, and $F_{p,q}^{1,1,1}=1+i+j$, where $i^2=j^2=-1$.

PROPOSITION 3.1. *The following properties are apply:*

1. $F_{p,q}^{n,0,0} = F_n(p,q)$,
2. $F_{p,q}^{0,m,0} = iF_m(p,q)$,
3. $F_{p,q}^{0,0,r} = jF_r(p,q)$,
4. $F_{p,q}^{n,1,0} = F_n(p,q) + iF_{n+1}(p,q)$,
5. $F_{p,q}^{n,0,1} = F_n(p,q) + jF_{n+1}(p,q)$,
6. $F_{p,q}^{n,1,1} = F_n(p,q) + iF_{n+1}(p,q) + jF_{n+1}(p,q)$,
7. $F_{p,q}^{1,m,0} = F_{m+1}(p,q) + iF_m(p,q)$,
8. $F_{p,q}^{0,m,1} = iF_m(p,q) + jF_{m+1}(p,q)$,
9. $F_{p,q}^{1,m,1} = F_{m+1}(p,q) + iF_m(p,q) + jF_{m+1}(p,q)$,
10. $F_{p,q}^{1,0,r} = F_{r+1}(p,q) + jF_r(p,q)$,
11. $F_{p,q}^{0,1,r} = iF_{r+1}(p,q) + jF_r(p,q)$,
12. $F_{p,q}^{1,1,r} = F_{r+1}(p,q) + iF_{r+1}(p,q) + jF_r(p,q)$,
13. $F_{p,q}^{n,m,0} = F_n(p,q)F_{m+1}(p,q) + iF_{n+1}(p,q)F_m(p,q)$,
14. $F_{p,q}^{n,0,r} = F_n(p,q)F_{r+1}(p,q) + jF_{n+1}(p,q)F_r(p,q)$,
15. $F_{p,q}^{0,m,r} = iF_m(p,q)F_{r+1}(p,q) + jF_{m+1}(p,q)F_r(p,q)$,
16. $F_{p,q}^{n,m,1} = F_n(p,q)F_{m+1}(p,q)+iF_{n+1}(p,q)F_m(p,q)+jF_{n+1}(p,q)F_{m+1}(p,q)$,
17. $F_{p,q}^{n,1,r} = F_n(p,q)F_{r+1}(p,q) + iF_{n+1}(p,q)F_{r+1}(p,q) + jF_{n+1}(p,q)F_r(p,q)$,
18. $F_{p,q}^{1,m,r} = F_{m+1}(p,q)F_{r+1}(p,q)+iF_m(p,q)F_{r+1}(p,q)+jF_{m+1}(p,q)F_r(p,q)$,
19. $F_{p,q}^{n,m,r} = F_n(p,q)F_{m+1}(p,q)F_{r+1}(p,q) + iF_{n+1}(p,q)F_m(p,q)F_{r+1}(p,q) + jF_{n+1}(p,q)F_{m+1}(p,q)F_r(p,q)$,

PROOF. The proofs are easily shown by the Mathematical Principle of Induction as in Proposition 2.1. □

The fourth-dimensional recurrences of the $(p,q)-$Fibonacci sequence are called quaternions. Various investigation on the quaternions can be found in [**14, 15, 1, 3, 13, 4, 5**].

4. $n-$dimensional recurrences of the $(p,q)-$Fibonacci sequence

In this section the $n-$dimensional recurrences of the $(p,q)-$Fibonacci sequence is defined.

Definition 4.1. For $n_0, n_1, ..., n_{n-1} \in \mathbb{N}$ and $p^2+4q>0$, the $n-$dimensional of $(p,q)-$Fibonacci sequence $F_{p,q}^{n_0,n_1,\ldots,n_{n-1}}$ is defined by the recurrences

$$F_{p,q}^{n_0+2,n_1,\ldots,n_{n-1}} = pF_{p,q}^{n_0+1,n_1,\ldots,n_{n-1}} + qF_{p,q}^{n_0,n_1,\ldots,n_{n-1}} \tag{4.1}$$

$$F_{p,q}^{n_0,n_1+2,\ldots,n_{n-1}} = pF_{p,q}^{n_0,n_1+1,\ldots,n_{n-1}} + qF_{p,q}^{n_0,n_1,\ldots,n_{n-1}} \tag{4.2}$$

$$\vdots = \vdots \tag{4.3}$$

$$F_{p,q}^{n_0,n_1,\ldots,n_{n-1}+2} = pF_{p,q}^{n_0,n_1,\ldots,n_{n-1}+1} + qF_{p,q}^{n_0,n_1,\ldots,n_{n-1}}$$

with the initial values

$$\begin{aligned}
F_{p,q}^{0,0,0,\ldots,0} &= 0,\\
F_{p,q}^{1,0,0,\ldots,0} &= 1,\\
F_{p,q}^{0,1,0,\ldots,0} &= e_2,\\
F_{p,q}^{0,0,1,\ldots,0} &= e_3,\\
\vdots &= \vdots\\
F_{p,q}^{0,0,0,\ldots,1} &= e_n,\\
\vdots &= \vdots\\
F_{p,q}^{0,1,1,\ldots,1} &= e_1+e_2+...+e_{n-1},\\
F_{p,q}^{1,0,1,\ldots,1} &= 1+e_2+...+e_{n-1},\\
F_{p,q}^{1,1,0,\ldots,1} &= 1+e_1+...+e_{n-1},\\
\vdots &= \vdots\\
F_{p,q}^{1,1,1,\ldots,0} &= 1+e_1+e_2+...+e_{n-2}\\
F_{p,q}^{1,1,1,\ldots,1} &= 1+e_1+e_2+...+e_{n-1}.
\end{aligned}$$

Note that here the unit vectors are $1=e_0$, $i=e_,$ $e_2=j,\ldots,e_{n-1}$.

Proposition 4.1. *The following properties are apply:*

1. $F_{p,q}^{n_0,0,0,\ldots,0} = F_{n_0}(p,q)$,
2. $F_{p,q}^{0,n_1,0,\ldots,0} = e_1F_{n_1}(p,q)$,
3. $F_{p,q}^{0,0,0,\ldots,n_{n-1}} = e_{n-1}F_{n_{n-1}}(p,q)$,
4. $F_{p,q}^{n_0,1,0,\ldots,0} = F_{n_0}(p,q) + e_1F_{n_0+1}(p,q)$,
5. $F_{p,q}^{n_0,0,0,\ldots,1} = F_{n_0}(p,q) + e_{n-1}F_{n_0+1}(p,q)$,
6. $F_{p,q}^{n_0,1,1,\ldots,1} = F_{n_0}(p,q) + e_1F_{n_0+1}(p,q) + ... + e_{n-1}F_{n_0+1}(p,q)$,
7. $F_{p,q}^{1,n_1,0,\ldots,0} = F_{n_1+1}(p,q) + e_1F_{n_1}(p,q)$,
8. $F_{p,q}^{0,n_1,0,\ldots,1} = e_1F_{n_1}(p,q) + e_{n-1}F_{n_1+1}(p,q)$,
9. $F_{p,q}^{1,n_1,1,\ldots,1} = F_{n_1+1}(p,q) + e_1F_{n_1}(p,q) + ... + e_{n-1}F_{n_1+1}(p,q)$,
10. $F_{p,q}^{1,0,0,\ldots,n_{n-1}} = F_{n_n+1}(p,q) + e_{n-1}F_{n_{n-1}}(p,q)$,
11. $F_{p,q}^{0,1,0,\ldots,n_{n-1}} = e_1F_{n_n+1}(p,q) + e_{n-1}F_{n_{n-1}}(p,q)$,

12. $F_{p,q}^{1,1,1,\ldots,n_{n-1}} = F_{n_n+1}(p,q) + e_1 F_{n_n+1}(p,q) + \ldots + e_{n-1} F_{n_{n-1}}(p,q)$,
13. $F_{p,q}^{n_0,n_1,n_2,\ldots,0} = F_{n_0}(p,q) F_{n_1+1}(p,q) \ldots F_{n_{n-2}+1}(p,q)$
$+ e_1 F_{n_0+1}(p,q) F_{n_1}(p,q) \ldots F_{n_{n-2}+1}(p,q) + \ldots +$
$+ e_{n-2} F_{n_0+1}(p,q) F_{n_1+1}(p,q) \ldots F_{n_{n-2}}(p,q)$,
14. $F_{p,q}^{n_0,0,n_2,\ldots,n_{n-1}} = F_{n_0}(p,q) F_{n_2+1}(p,q) \ldots F_{n_{n-1}+1}(p,q)$
$+ e_2 F_{n_0+1}(p,q) F_{n_2}(p,q) \ldots F_{n_{n-1}+1}(p,q) + \ldots +$
$+ e_{n-1} F_{n_0+1}(p,q) F_{n_2+1}(p,q) \ldots F_{n_{n-1}}(p,q)$,
15. $F_{p,q}^{0,n_1,n_2,\ldots,n_{n-1}} = e_1 F_{n_1}(p,q) F_{n_2+1}(p,q) \ldots F_{n_{n-1}+1}(p,q)$
$+ e_2 F_{n_1+1}(p,q) F_{n_2}(p,q) \ldots F_{n_{n-1}+1}(p,q) + \ldots +$
$+ e_{n-2} F_{n_1+1}(p,q) F_{n_2+1}(p,q) \ldots F_{n_{n-1}}(p,q)$,
16. $F_{p,q}^{n_0,n_1,n_2,\ldots,1} = F_{n_0}(p,q) F_{n_1+1}(p,q) \ldots F_{n_{n-2}+1}(p,q)$
$+ e_1 F_{n_0+1}(p,q) F_{n_1}(p,q) \ldots F_{n_{n-2}+1}(p,q) + \ldots +$
$+ e_{n-2} F_{n_0+1}(p,q) F_{n_1+1}(p,q) \ldots F_{n_{n-2}}(p,q)$
$+ e_{n-1} F_{n_0+1}(p,q) F_{n_1+1}(p,q) \ldots F_{n_{n-1}+1}(p,q)$,
17. $F_{p,q}^{n_0,1,n_2,\ldots,n_{n-1}} = F_{n_0}(p,q) F_{n_2+1}(p,q) \ldots F_{n_{n-1}+1}(p,q)$
$+ e_1 F_{n_0+1}(p,q) F_{n_2+1}(p,q) \ldots F_{n_{n-1}+1}(p,q)$
$+ e_2 F_{n_0+1}(p,q) F_{n_2}(p,q) \ldots F_{n_{n-1}+1}(p,q) + \ldots +$
$+ e_{n-1} F_{n_0+1}(p,q) F_{n_2+1}(p,q) \ldots F_{n_{n-1}}(p,q)$,
18. $F_{p,q}^{0,n_1,n_2,\ldots,n_{n-1}} = F_{n_1+1}(p,q) F_{n_2+1}(p,q) \ldots F_{n_{n-1}+1}(p,q)$
$+ e_1 F_{n_1}(p,q) F_{n_2+1}(p,q) \ldots F_{n_{n-1}+1}(p,q)$
$+ e_2 F_{n_1+1}(p,q) F_{n_2}(p,q) \ldots F_{n_{n-1}+1}(p,q) + \ldots +$
$+ e_{n-2} F_{n_1+1}(p,q) F_{n_2+1}(p,q) \ldots F_{n_{n-1}}(p,q)$,
19. $F_{p,q}^{n_0,n_1,n_2,\ldots,n_{n-1}} = F_{n_0}(p,q) F_{n_1+1}(p,q) \ldots F_{n_{n-1}+1}(p,q)$
$+ e_1 F_{n_0+1}(p,q) F_{n_1}(p,q) \ldots F_{n_{n-1}+1}(p,q) + \ldots +$
$+ e_{n-1} F_{n_0+1}(p,q) F_{n_1+1}(p,q) \ldots F_{n_{n-1}}(p,q)$,

Proof. The proofs are easily shown by the Mathematical Principle of Induction as in Proposition 2.1 . □

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Orhan Dişkaya, Department of Mathematics, Mersin University, Mersin, Turkey
Email address: `orhandiskaya@mersin.edu.tr`

Hamza Menken, Department of Mathematics, Mersin University, Mersin, Turkey
Email address: `hmenken@mersin.edu.tr`