ON RELATIONS BETWEEN ATOM-BOND SUM-CONNECTIVITY INDEX AND OTHER CONNECTIVITY INDICES

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Abstract. In 2022, the atom-bond sum-connectivity index is introduced by Ali, Furtula, Redžepović and Gutman inspired by the Randić index, the sum-connectivity index and the atom-bond-connectivity index. Recently, the extreme values of the new index for a graph class is widely studied. In this paper, we pay more attention to the mathematical relations between the atom-bond sum-connectivity index and some other connectivity indices.

1. Introduction

In chemical graph theory, the topological index of a graph, also called molecular structure descriptor, is often used to predict the physico-chemical properties and biological activities of molecules. In particular, a large number of degree-based topological indices have been introduced and extensively studied [9] in mathematical chemistry.

Let $G$ be a simple connected undirected graph with the vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, $d_v$ denotes the degree of vertex $v$ in $G$. The minimum and the maximum degree of $G$ are denoted by $\delta$ and $\Delta$, respectively. A pendant vertex is a vertex of degree one and a quasi-pendant vertex is a vertex adjacent to a pendant vertex.

The Randić index [8], also called the connectivity index or the branching index, is one of the most famous and important degree-based topological indices, and
defined as

\[ R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}}. \]

The harmonic index \([6]\), the sum-connectivity index \([10]\) and the atom-bond-connectivity index \([5]\) are the class of successful variants of the connectivity index, and defined as

\[ H(G) = \sum_{uv \in E(G)} \frac{2}{d_u + d_v}, \]

\[ SCI(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u + d_v}}, \]

\[ ABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}}. \]

In 2022, Ali, Furtula, Redžepović and Gutman \([1]\) proposed a novel degree-based topological index called the atom-bond sum-connectivity index (ABS index for short) based on the above connectivity index, which is defined as

\[ ABS(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}} = \sum_{uv \in E(G)} \sqrt{1 - \frac{2}{d_u + d_v}}. \]

Recently, the extreme values of the ABS index is widely studied, see \([2, 3, 4, 7]\). A natural problem is the mathematical relationship between these connectivity indices. For a connected graph \( G \) with \( n \geq 4 \) vertices, it is not difficult to find that

\[ H(G) \leq R(G), \quad \text{(as mean value inequality)}, \]
\[ R(G) \leq SCI(G), \quad \text{for } \delta \geq 2, \]
\[ SCI(G) < ABC(G), \]
\[ ABC(G) \leq ABS(G), \quad \text{for } \delta \geq 2. \]

In this paper, the mathematical relations between the ABS index and other connectivity indices are investigated.

2. Main results

**Theorem 2.1.** Let \( G \) be a connected graph with the maximum degree \( \Delta \) and the minimum degree \( \delta \). Then

\[ \sqrt{\delta(\delta - 1)} R(G) \leq ABS(G) \leq \sqrt{\Delta(\Delta - 1)} R(G) \]

with equality if and only if \( G \) is regular.

**Proof.** Let \( f(x, y) = xy - \frac{2xy}{x+y} \). Then we have

\[ \frac{\partial f(x, y)}{\partial x} = \frac{y(x^2 + y^2 + 2xy - 2y)}{(x + y)^2} > 0, \]
\[ \frac{\partial f(x, y)}{\partial y} = \frac{x(x^2 + y^2 + 2xy - 2x)}{(x + y)^2} > 0. \]
This implies that \( f(x, y) \) is increasing for both \( x \geq 1 \) and \( y \geq 1 \). Thus we have

\[
\sqrt{\delta(\delta - 1)} \leq \sqrt{d_ud_v \left( 1 - \frac{2}{d_u + d_v} \right)} \leq \sqrt{\Delta(\Delta - 1)},
\]

that is,

\[
\frac{\sqrt{\delta(\delta - 1)}}{\sqrt{d_ud_v}} \leq \sqrt{1 - \frac{2}{d_u + d_v}} \leq \frac{\sqrt{\Delta(\Delta - 1)}}{\sqrt{d_ud_v}}
\]

for \( \delta \leq d_u \leq \Delta \) and \( \delta \leq d_v \leq \Delta \). Further, we have

\[
\sqrt{\delta(\delta - 1)} R(G) \leq \text{ABS}(G) \leq \sqrt{\Delta(\Delta - 1)} R(G)
\]

with equality if and only if \( G \) is regular. This completes the proof.

\( \square \)

**Theorem 2.2.** Let \( G \) be a connected graph. If the degree of quasi-pendant vertices of \( G \) is greater than or equal to three, then \( \text{ABS}(G) > R(G) \).

**Proof.** Let \( g(x, y) = x^2 y + xy^2 - 2xy - x - y \) for \( x \geq 1 \) and \( y \geq 1 \). Then we have

\[
\frac{\partial g(x, y)}{\partial x} = 2(xy - 1) + (y - 1)^2 > 0,
\]

\[
\frac{\partial g(x, y)}{\partial y} = 2(xy - 1) + (x - 1)^2 > 0
\]

for \( x \neq 1 \) and \( y \neq 1 \). Thus \( g(x, y) \) is increasing for \( x \neq 1 \) and \( y \neq 1 \). Since

\[
g(1, 3) = 3, \quad g(2, 2) = 4,
\]

we have

\[
g(1, y) \geq g(1, 3) > 0, \quad g(x, y) \geq g(2, 2) > 0
\]

for \( x \geq 2 \) and \( y \geq 3 \). Further, we have

\[
g(d_u, d_v) = d_u^2 d_v + d_u d_v^2 - 2d_u d_v - d_u - d_v > 0,
\]

that is,

\[
1 - \frac{2}{d_u + d_v} > \frac{1}{d_u d_v}
\]

for \( d_u \geq 2 \) and \( d_v \geq 3 \). This means that if the degree of quasi-pendant vertices of \( G \) is greater than or equal to three, then

\[
\text{ABS}(G) > R(G).
\]

This completes the proof.

\( \square \)

**Corollary 2.1.** Let \( G \) be a connected graph. If the degree of quasi-pendant vertices of \( G \) is greater than or equal to three, then \( \text{ABS}(G) > H(G) \).

**Theorem 2.3.** Let \( G \) be a connected graph with the maximum degree \( \Delta \) and the minimum degree \( \delta \). Then

\[
\sqrt{\delta(\delta - 1)} H(G) \leq \text{ABS}(G) \leq \sqrt{\Delta(\Delta - 1)} H(G)
\]

with equality if and only if \( G \) is regular.
Proof. Let \( h(x) = \frac{1}{x^2} - \frac{1}{x} \). Then \( h'(x) = \frac{x - 2}{x^3} \). This implies that \( h(x) \) is decreasing for \( x < 2 \). For \( uv \in E(G) \), we have
\[
h\left(\frac{1}{\delta}\right) \leq h\left(\frac{2}{d_u + d_v}\right) \leq h\left(\frac{1}{\Delta}\right),
\]
that is,
\[
\left(\delta - \delta\right) \left(\frac{2}{d_u + d_v}\right)^2 \leq 1 - \frac{2}{d_u + d_v} \leq (\Delta^2 - \Delta) \left(\frac{2}{d_u + d_v}\right)^2
\]
for \( \delta \leq d_u \leq \Delta \) and \( \delta \leq d_v \leq \Delta \). Further, we have
\[
\sqrt{\delta(\delta - 1)H(G)} \leq \text{ABS}(G) \leq \sqrt{(\Delta - 1)H(G)}
\]
with equality if and only if \( G \) is regular. This completes the proof. \( \square \)

By Theorems 2.1 and 2.3, we have

Theorem 2.4. Let \( G \) be a connected graph with the maximum degree \( \Delta \) and the minimum degree \( \delta \). Then
\[
\sqrt{\delta(\delta - 1)R(G)} \leq \text{ABS}(G) \leq \sqrt{(\Delta - 1)H(G)}
\]
with equality if and only if \( G \) is regular.

References


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