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APPLICATIONS OF SCHWARZ LEMMA FOR THE CLASS $\mathcal{M}(\alpha,\theta)$

Bülent Nafi Örnek and Nurbanu Tuğçe Akca

Abstract. We shall introduce the class of analytical functions known as $\mathcal{M}(\alpha,\theta)$ and examine the various characteristics of the functions that belong to this class.

1. Introduction

Let $\mathcal{S}$ denote the class of functions $h(z) = z + \sum\limits_{p=1}^{\infty} b_{p+1}z^{p+1}$ that are analytic in the unit disk $\mathcal{K} = \{z : |z| < 1\}$. Also, let $\mathcal{M}(\alpha,\theta)$ be the subclass of $\mathcal{S}$ satisfying the condition

$$\sum_{p=1}^{\infty}\left(p+1+\left|p+1-2\alpha e^{-i\theta}\right|\right)|b_{p+1}| \leqslant 1-\left|1-2\alpha e^{-i\theta}\right|, \tag{1.1}$$

for some $|\theta| < \frac{\pi}{2}$, and $0 < \alpha \leqslant \cos\theta$. Let $h(z) \in \mathcal{M}(\alpha,\theta)$ and think about the following functions

$$\vartheta(z) = \frac{2\alpha e^{-i\theta}}{f(z)} - 1,\ f(z) = \frac{zh'(z)}{h(z)},$$

where $f(z) = 1 + b_2z + \left(2b_3 - b_2^2\right)z^2 + \left(b_2^3 - 3b_2b_3 + 3b_4\right)z^3 + ...$

Here, $\vartheta(z)$ is an analytic function in $\mathcal{K}$ and $\vartheta(0) = 2\alpha e^{-i\theta} - 1$. Now, let us show that $|\vartheta(z)| < 1$ for $z \in \mathcal{K}$. From the definition of $\vartheta(z)$ and $f(z)$, we have

$$\vartheta(z) = \frac{2\alpha e^{-i\theta}}{f(z)} - 1 = \frac{2\alpha e^{-i\theta} - f(z)}{f(z)} = \frac{2\alpha h(z) - e^{i\theta}zh'(z)}{e^{i\theta}zh'(z)}.$$

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If we substitute the value of the function $h(z)$ and take its modulus, we obtain

$$\begin{aligned} |\vartheta(z)| &= \left| \frac{2\alpha\left(z+\sum_{p=1}^{\infty} b_{p+1}z^{p+1}\right) - e^{i\theta}z\left(1+\sum_{p=1}^{\infty}(p+1)b_{p+1}z^{p}\right)}{e^{i\theta}z\left(1+\sum_{p=1}^{\infty}(p+1)b_{p+1}z^{p+1}\right)} \right| \\ &= \left| \frac{2\alpha e^{-i\theta} - 1 + \sum_{p=1}^{\infty}\left(2\alpha e^{-i\theta}-(p+1)\right)b_{p+1}z^{p}}{1+\sum_{p=1}^{\infty}(p+1)b_{p+1}z^{p+1}} \right| \end{aligned}$$

and

$$|\vartheta(z)| \leqslant \frac{\left|1-2\alpha e^{-i\theta}\right| + \sum_{p=1}^{\infty}\left|p+1-2\alpha e^{-i\theta}\right| |b_{p+1}|}{1-\sum_{p=1}^{\infty}(p+1)|b_{p+1}|}.$$

From (1.1), we have

$$\left|1-2\alpha e^{-i\theta}\right| \leqslant 1 - \sum_{p=1}^{\infty}\left(p+1+\left|p+1-2\alpha e^{-i\theta}\right|\right)|b_{p+1}|.$$

If we use this expression in the above inequality, we find that $|\vartheta(z)| < 1$ for $|z| < 1$.

In this study, we will determine an upper bound for the second coefficient in the Taylor expansion of the function $h(z)$. In other words, we will find a value that bounds this coefficient from above. Additionally, considering the non-zero zeros of the function $1-\frac{zh'(z)}{h(z)}$, we will establish a different upper bound for this coefficient.

Now, let us consider the function.

$$w(z) = \frac{\vartheta(z)-\vartheta(0)}{1-\overline{\vartheta(0)}\vartheta(z)}.$$

Here, $w(z)$ is an analytic function, $w(0)=0$ and $|w(z)|<1$ for $z\in\mathcal{K}$. Thus, the function $w(z)$ satisfies the conditions of the Schwarz lemma ([**5**], p.329). Applying the Schwarz lemma, we obtain

$$|w'(0)| = \frac{|\vartheta'(0)|}{1-|\vartheta(0)|^2} \leqslant 1.$$

Also, we have $|\vartheta(0)| = \left|2\alpha - e^{i\theta}\right|$ and $|\vartheta'(0)| = 2\alpha\,|f'(0)|$. Therefore, we obtain

$$\frac{2\alpha\,|f'(0)|}{1-\left|2\alpha-e^{i\theta}\right|^2} \leqslant 1$$

and

$$|f'(0)| \leqslant \frac{1-\left|2\alpha-e^{i\theta}\right|^2}{2\alpha}.$$

Since $1-\left|2\alpha-e^{i\theta}\right|^2=1-\left(4\alpha^2-4\alpha\cos\theta+1\right)=4\alpha\cos\theta-4\alpha^2=4\alpha\left(\cos\theta-\alpha\right)$ and $|f'(0)|=|b_2|$, we take

$$|b_2|\leqslant 2\left(\cos\theta-\alpha\right).$$

LEMMA 1.1. *If $h\in\mathcal{M}(\alpha,\theta)$, then we have the inequality*

$$|b_2|\leqslant 2\left(\cos\theta-\alpha\right).$$

Now let us consider the following function, taking into account the non-zero zeros of the function $1-\frac{zh'(z)}{h(z)}$,

$$\Theta(z)=\frac{w(z)}{\prod\limits_{i=1}^{n}\frac{z-c_i}{1-\overline{c_i}z}}.$$

Since $w(z)$ function satisfies the conditions of the Schwarz lemma, we obtain

$$\Theta(z)=\frac{\vartheta(z)-\vartheta(0)}{1-\overline{\vartheta(0)}\vartheta(z)}\frac{1}{\prod\limits_{i=1}^{n}\frac{z-c_i}{1-\overline{c_i}z}}$$

$$=\frac{-2\alpha e^{-i\theta}\left(b_2z+(2b_3-b_2^2)z^2+\ldots\right)}{1+b_2z+(2b_3-b_2^2)z^2+\ldots-|1-2\alpha e^{i\theta}|^2+(2\alpha e^{i\theta}-1)\left(b_2z+(2b_3-b_2^2)z^2+\ldots\right)}\frac{1}{\prod\limits_{i=1}^{n}\frac{z-c_i}{1-\overline{c_i}z}},$$

$$\frac{\Theta(z)}{z}=\frac{-2\alpha e^{-i\theta}\left(b_2+(2b_3-b_2^2)z+\ldots\right)}{1+b_2z+(2b_3-b_2^2)z^2+\ldots-|1-2\alpha e^{i\theta}|^2+(2\alpha e^{i\theta}-1)\left(b_2z+(2b_3-b_2^2)z^2+\ldots\right)}\frac{1}{\prod\limits_{i=1}^{n}\frac{z-c_i}{1-\overline{c_i}z}},$$

$$|w'(0)|=\frac{2\alpha\left|b_2\right|}{\left(1-|1-2\alpha e^{i\theta}|^2\right)\prod\limits_{i=1}^{n}|c_i|}\leqslant 1$$

and

$$|b_2|\leqslant\frac{\left(1-\left|1-2\alpha e^{i\theta}\right|^2\right)\prod\limits_{i=1}^{n}|c_i|}{2\alpha}=2\left(\cos\theta-\alpha\right)\prod_{i=1}^{n}|c_i|.$$

We thus obtain the following lemma.

LEMMA 1.2. *Let $h\in\mathcal{M}(\alpha,\theta)$ and $c_1,c_2,\ldots,c_n$ be zeros of the function $1-\frac{zh'(z)}{h(z)}$ in D that are different from zero. Then we have the inequality*

$$|b_2|\leqslant 2\left(\cos\theta-\alpha\right)\prod_{i=1}^{n}|c_i|.$$

A stronger upper bound for the coefficient b_2 is achieved according to this lemma when the zeros of the function $1-\frac{zh'(z)}{h(z)}$ are taken into account. These two lemmas are also derived from analytical functions within the unit disc.

To analyse the behaviour of the derivative of this function at the boundary of the unit disc, the following lemma is required [**10, 15**].

LEMMA 1.3. *Let $g(z)$ be an analytic function in $\mathcal{K}$, $g(0)=0$ and $|g(z)|<1$ for $z\in\mathcal{K}$. If $g(z)$ extends continuously to boundary point $1\in\partial\mathcal{K}=\{z:|z|=1\}$, and if $|g(1)|=1$ and $g'(1)$ exists, then*

$$|g'(1)|\geqslant\frac{2}{1+|g'(0)|} \tag{1.2}$$

and

$$|g'(1)|\geqslant 1. \tag{1.3}$$

Moreover, the equality in (1.2) *holds if and only if*

$$g(z)=z\frac{z-\tau}{1-\tau z}$$

for some $\tau\in(-1,0]$. Also, the equality in (1.3) *holds if and only if $g(z)=ze^{i\gamma}$.*

Inequality (1.5) and its generalizations have significant applications in the geometric theory of functions, and they remain active topics in the mathematics literature [**1**–**4**, **6**–**13**].

The following lemma, known as the Julia-Wolff lemma, is needed in the sequel (see, [**14**]).

LEMMA 1.4 (Julia-Wolff lemma). *Let g be an analytic function in $\mathcal{K}$, $g(0)=0$ and $g(\mathcal{K})\subset\mathcal{K}$. If, in addition, the function g has an angular limit $g(1)$ at $1\in\partial\mathcal{K}$, $|g(1)|=1$, then the angular derivative $g'(1)$ exists and $1\leqslant|g'(1)|\leqslant\infty$.*

COROLLARY 1.1. *The analytic function g has a finite angular derivative $g'(1)$ if and only if g' has the finite angular limit $g'(1)$ at $1\in\partial\mathcal{K}$.*

2. Main results

In this section, we discuss various versions of the boundary Schwarz lemma for the class $\mathcal{M}(\alpha,\theta)$ class. Furthermore, we have obtained estimates for the modulus of the angular derivative within a class of analytic functions on the unit disc, assuming the existence of an angular limit at a boundary point.

THEOREM 2.1. *Let $h\in\mathcal{M}(\alpha,\theta)$. Assume that, for $1\in\partial\mathcal{K}$, h has an angular limit $h(1)$ at the points 1, $h'(1)=\frac{\alpha}{\cos\theta}h(1)$. Then we have the inequality*

$$\left|\left(\frac{zh'(z)}{h(z)}\right)'_{z=1}\right|\geqslant\frac{\cos\theta-\alpha}{2\cos^2\theta}. \tag{2.1}$$

PROOF. Let

$$w(z)=\frac{\vartheta(z)-\vartheta(0)}{1-\overline{\vartheta(0)}\vartheta(z)},\vartheta(z)=\frac{2\alpha e^{-i\theta}}{f(z)}-1,\ f(z)=\frac{zh'(z)}{h(z)}.$$

If we take the derivative of the function $w(z)$, we get

$$w'(z)=\frac{1-|\vartheta(0)|^2}{\left(1-\overline{\vartheta(0)}\vartheta(z)\right)^2}\vartheta'(z).$$

Also, since $h'(1)=\frac{\alpha}{\cos\theta}h(1)$, $\vartheta(0)=2\alpha e^{-i\theta}-1$, $\vartheta(1)=e^{-2i\theta}$, we have $|w(1)|=1$. Therefore, $w(z)$ function satisfies the conditions of Lemma 1.3. That is,

$$\begin{aligned} 1 &\leqslant |w'(1)|=\frac{1-|\vartheta(0)|^2}{\left|1-\overline{\vartheta(0)}\vartheta(1)\right|^2}|\vartheta'(1)|=\frac{1-\left|2\alpha e^{-i\theta}-1\right|^2}{\left|1-\left(2\alpha e^{i\theta}-1\right)e^{-2i\theta}\right|^2}2\alpha\frac{|f'(1)|}{|f(1)|^2} \\ &= \frac{4\alpha\left(\cos\theta-\alpha\right)}{\left(2\alpha-2\cos\theta\right)^2}2\alpha\frac{|f'(1)|}{\left(\frac{\alpha}{\cos\theta}\right)^2}=\frac{2\cos^2\theta}{\cos\theta-\alpha}\left|f'(1)\right|. \end{aligned}$$

Thus, we obtain

$$|f'(1)|\geqslant\frac{\cos\theta-\alpha}{2\cos^2\theta}$$

and

$$\left|\left(\frac{zh'(z)}{h(z)}\right)'_{z=1}\right|\geqslant\frac{\cos\theta-\alpha}{2\cos^2\theta}.$$

□

The inequality (2.1) can be strengthened from below by taking into account, $b_2=\frac{h''(0)}{2}$, the second coefficient of the expansion of the function $h(z)=z+b_2z^2+b_3z^3+....$

THEOREM 2.2. *Under the same assumptions as in Theorem 2.1, we have*

$$\left|\left(\frac{zh'(z)}{h(z)}\right)'_{z=1}\right|\geqslant\frac{1}{\cos^2\theta}\frac{2\left(\cos\theta-\alpha\right)^2}{2\left(\cos\theta-\alpha\right)+|b_2|}. \tag{2.2}$$

PROOF. Let the function $w(z)$ be as given above. From the Lemma 1.3, we obtain

$$\frac{2}{1+|w'(0)|}\leqslant|w'(1)|=\frac{2\cos^2\theta}{\cos\theta-\alpha}\left|f'(1)\right|.$$

Since

$$|w'(0)|=\frac{|b_2|}{2\left(\cos\theta-\alpha\right)},$$

we take

$$\frac{2}{1+\frac{|b_2|}{2(\cos\theta-\alpha)}}\leqslant\frac{2\cos^2\theta}{\cos\theta-\alpha}\left|f'(1)\right|$$

and

$$|f'(1)|\geqslant\frac{1}{\cos^2\theta}\frac{2\left(\cos\theta-\alpha\right)^2}{2\left(\cos\theta-\alpha\right)+|b_2|}.$$

□

The inequality (2.2) can be strengthened as below by taking into account $b_3=\frac{h'''(0)}{3!}$ which is the coefficient in the expansion of the function $h(z)=z+b_2z^2+b_3z^3+....$

THEOREM 2.3. *Let $h \in \mathcal{M}(\alpha, \theta)$. Assume that, for $1 \in \partial\mathcal{K}$, h has an angular limit $h(1)$ at the points 1, $h'(1) = \frac{\alpha}{\cos\theta} h(1)$. Then we have the inequality*

(2.3)

$$\left| \left(\frac{zh'(z)}{h(z)} \right)'_{z=1} \right| \geqslant \frac{\cos\theta - \alpha}{2\cos^2\theta} \left(1 + \frac{2\left(2(\cos\theta - \alpha) - |b_2|\right)^2}{4(\cos\theta - \alpha)^2 - |b_2|^2 + \left|2(2b_3 - b_2^2)(\cos\theta - \alpha) - b_2^2 e^{i\theta}\right|} \right).$$

PROOF. Let the function $w(z)$ be as given above and $d(z) = z$. By the maximum principle, for each $z \in \mathcal{K}$, we have the inequality $|w(z)| \leqslant |d(z)|$. Therefore, we take

$$\begin{aligned}
\phi(z) &= \frac{w(z)}{d(z)} = \frac{1}{z}\left(\frac{\vartheta(z) - \vartheta(0)}{1 - \overline{\vartheta(0)}\vartheta(z)} \right) \\
&= \frac{1}{z} \frac{-2\alpha e^{-i\theta}\left(b_2 z + (2b_3 - b_2^2)z^2 + \ldots\right)}{1 + b_2 z + (2b_3 - b_2^2)z^2 + \ldots - |1 - 2\alpha e^{i\theta}|^2 + (2\alpha e^{i\theta} - 1)\left(b_2 z + (2b_3 - b_2^2)z^2 + \ldots\right)} \\
&= \frac{-2\alpha e^{-i\theta}\left(b_2 + (2b_3 - b_2^2)z + \ldots\right)}{1 + b_2 z + (2b_3 - b_2^2)z^2 + \ldots - |1 - 2\alpha e^{i\theta}|^2 + (2\alpha e^{i\theta} - 1)\left(b_2 z + (2b_3 - b_2^2)z^2 + \ldots\right)}
\end{aligned}$$

is an analytic function in $\mathcal{K}$ and $|\phi(z)| < 1$ for $z \in \mathcal{K}$. In particular, we have

$$|\phi(0)| = \frac{2\alpha |b_2|}{1 - |1 - 2\alpha e^{i\theta}|^2} = \frac{|b_2|}{2(\cos\theta - \alpha)} \leqslant 1 \tag{2.4}$$

and

$$|\phi'(0)| = \frac{\left|2\left(2b_3 - b_2^2\right)(\cos\theta - \alpha) - b_2^2 e^{i\theta}\right|}{4(\cos\theta - \alpha)^2}.$$

The auxiliary function

$$\varphi(z) = \frac{\phi(z) - \phi(0)}{1 - \overline{\phi(0)}\phi(z)}$$

is analytic in $\mathcal{K}$, $\varphi(0) = 0$, $|\varphi(z)| < 1$ for $|z| < 1$ and $|\varphi(1)| = 1$ for $1 \in \partial\mathcal{K}$. Moreover, since the expression $\frac{1.w'(1)}{w(1)}$ is a real number greater than or equal to 1 (see [**3**]) and $h'(1) = \frac{\alpha}{\cos\theta} h(1)$ yields $|w(1)| = 1$, we take

$$\frac{1.w'(1)}{w(1)} = \left| \frac{1.w'(1)}{w(1)} \right| = |w'(1)|.$$

From Lemma 1.3, we obtain

$$\begin{aligned}
\frac{2}{1 + |\varphi'(0)|} &\leqslant |\varphi'(1)| = \frac{1 - |\phi(0)|^2}{\left|1 - \overline{\phi(0)}\phi(1)\right|^2} |\phi'(1)| \\
&\leqslant \frac{1 + |\phi(0)|}{1 - |\phi(0)|} \left| \frac{w'(1)}{d(1)} - \frac{w(1)d'(1)}{d^2(1)} \right| \\
&= \frac{1 + |\phi(0)|}{1 - |\phi(0)|} \left| \frac{w(1)}{d(1)} \right| \left| \frac{w'(1)}{w(1)} - \frac{d'(1)}{d(1)} \right| \\
&\leqslant \frac{1 + |\phi(0)|}{1 - |\phi(0)|} \left\{ |w'(1)| - |d'(1)| \right\} \\
&= \frac{2(\cos\theta - \alpha) + |b_2|}{2(\cos\theta - \alpha) - |b_2|} \left(\frac{2\cos^2\theta}{\cos\theta - \alpha} |f'(1)| - 1 \right).
\end{aligned}$$

Since

$$|\varphi'(0)| = \frac{|\phi'(0)|}{1-|\phi(0)|^2} = \frac{\frac{\left|2\left(2b_3-b_2^2\right)(\cos\theta-\alpha)-b_2^2e^{i\theta}\right|}{4(\cos\theta-\alpha)^2}}{1-\left(\frac{|b_2|}{2(\cos\theta-\alpha)}\right)^2} = \frac{\left|2\left(2b_3-b_2^2\right)(\cos\theta-\alpha)-b_2^2e^{i\theta}\right|}{4(\cos\theta-\alpha)^2-|b_2|^2},$$

we obtain

$$\frac{2}{1+\frac{\left|2\left(2b_3-b_2^2\right)(\cos\theta-\alpha)-b_2^2e^{i\theta}\right|}{4(\cos\theta-\alpha)^2-|b_2|^2}} \leqslant \frac{2(\cos\theta-\alpha)+|b_2|}{2(\cos\theta-\alpha)-|b_2|}\left(\frac{2\cos^2\theta}{\cos\theta-\alpha}|f'(1)|-1\right),$$

$$\frac{2(2(\cos\theta-\alpha)-|b_2|)^2}{4(\cos\theta-\alpha)^2-|b_2|^2+\left|2\left(2b_3-b_2^2\right)(\cos\theta-\alpha)-b_2^2e^{i\theta}\right|} \leqslant \frac{2\cos^2\theta}{\cos\theta-\alpha}|f'(1)|-1$$

and

$$|f'(1)| \geqslant \frac{\cos\theta-\alpha}{2\cos^2\theta}\left(1+\frac{2(2(\cos\theta-\alpha)-|b_2|)^2}{4(\cos\theta-\alpha)^2-|b_2|^2+\left|2\left(2b_3-b_2^2\right)(\cos\theta-\alpha)-b_2^2e^{i\theta}\right|}\right). \qquad \square$$

If $1-\frac{zh'(z)}{h(z)}$ have zeros different from $z=0$, taking into account these zeros, the inequality (2.3) can be strengthened in another way. This is given by the following Theorem.

THEOREM 2.4. *Let $h\in\mathcal{M}(\alpha,\theta)$ and $c_1,c_2,...,c_n$ be zeros of the function $1-\frac{zh'(z)}{h(z)}$ in $\mathcal{K}$ that are different from zero. Assume that, for $1\in\partial\mathcal{K}$, h has an angular limit $h(1)$ at the points 1, $h'(1)=\frac{\alpha}{\cos\theta}h(1)$. Then we have the inequality*

$$\left|\left(\frac{zh'(z)}{h(z)}\right)'_{z=1}\right| \geqslant \frac{\cos\theta-\alpha}{2\cos^2\theta}\left(1+\sum_{i=1}^{n}\frac{1-|c_i|^2}{|1-c_i|^2}\right. \tag{2.5}$$
$$\left.+\frac{\left(2(\cos\theta-\alpha)\prod_{i=1}^{n}|c_i|-|b_2|\right)^2}{\left(\left((2(\cos\theta-\alpha))\prod_{i=1}^{n}|c_i|\right)^2-|b_2|^2\right)+\prod_{i=1}^{n}|c_i|\left|2(\cos\theta-\alpha)\left(2b_3-b_2^2+b_2\sum_{i=1}^{n}\frac{1-|c_i|^2}{c_i}\right)-b_2^2e^{i\theta}\right|}\right).$$

PROOF. Let the function $w(z)$ be as given above and $c_1,c_2,...,c_n$ be zeros of the function $1-\frac{zh'(z)}{h(z)}$ in $\mathcal{K}$ that are different from zero. Also, consider the function

$$\Upsilon(z)=z\prod_{i=1}^{n}\frac{z-c_i}{1-\overline{c_i}z}.$$

By the maximum principle for each $z\in\mathcal{K}$, we have

$$|w(z)|\leqslant|\Upsilon(z)|.$$

Consider the function

$$s(z)=\frac{w(z)}{\Upsilon(z)}=\left(\frac{\vartheta(z)-\vartheta(0)}{1-\overline{\vartheta(0)}\vartheta(z)}\right)\frac{1}{z\prod_{i=1}^{n}\frac{z-c_i}{1-\overline{c_i}z}}$$

$$=\frac{-2\alpha e^{-i\theta}\left(b_2z+(2b_3-b_2^2)z^2+...\right)}{1+b_2z+(2b_3-b_2^2)z^2+...-|1-2\alpha e^{i\theta}|^2+(2\alpha e^{i\theta}-1)\left(b_2z+(2b_3-b_2^2)z^2+...\right)}\frac{1}{z\prod_{i=1}^{n}\frac{z-c_i}{1-\overline{c_i}z}}$$

$$=\frac{-2\alpha e^{-i\theta}\left(b_2+(2b_3-b_2^2)z+...\right)}{1+b_2z+(2b_3-b_2^2)z^2+...-|1-2\alpha e^{i\theta}|^2+(2\alpha e^{i\theta}-1)\left(b_2z+(2b_3-b_2^2)z^2+...\right)}\frac{1}{\prod_{i=1}^{n}\frac{z-c_i}{1-\overline{c_i}z}}.$$

In particular, we have

$$|s(0)| = \frac{|b_2|}{2\left(\cos\theta - \alpha\right)} \frac{1}{\prod_{i=1}^{n} |c_i|}$$

and

$$|s'(0)| = \frac{\left| 2\left(\cos\theta - \alpha\right)\left(2b_3 - b_2^2 + b_2 \sum_{i=1}^{n} \frac{1-|c_i|^2}{c_i} \right) - b_2^2 e^{i\theta} \right|}{4\left(\cos\theta - \alpha\right)^2 \prod_{i=1}^{n} |c_i|}.$$

The auxiliary function

$$r(z) = \frac{s(z) - s(0)}{1 - \overline{s(0)}s(z)}$$

is analytic in $\mathcal{K}$, $|r(z)| < 1$ for $|z| < 1$ and $r(0) = 0$. For $1 \in \partial\mathcal{K}$, we take $|r(1)| = 1$.

From Lemma 1.3, we obtain

$$\begin{aligned} \frac{2}{1 + |r'(0)|} &\leqslant |r'(1)| = \frac{1 - |s(0)|^2}{\left|1 - \overline{s(0)}s(1)\right|} |s'(1)| \\ &\leqslant \frac{1 + |s(0)|}{1 - |s(0)|} \left(|w'(1)| - |\Upsilon'(1)|\right). \end{aligned}$$

It can be seen that

$$|r'(0)| = \frac{|s'(0)|}{1 - |s(0)|^2}$$

and

$$\begin{aligned} |r'(0)| &= \frac{\frac{\left|2(\cos\theta-\alpha)\left(2b_3-b_2^2+b_2\sum_{i=1}^{n}\frac{1-|c_i|^2}{c_i}\right)-b_2^2e^{i\theta}\right|}{4(\cos\theta-\alpha)^2\prod_{i=1}^{n}|c_i|}}{1-\left(\frac{|b_2|}{2(\cos\theta-\alpha)}\frac{1}{\prod_{i=1}^{n}|c_i|}\right)^2} \\ &= \prod_{i=1}^{n}|c_i| \frac{\left|2(\cos\theta-\alpha)\left(2b_3-b_2^2+b_2\sum_{i=1}^{n}\frac{1-|c_i|^2}{c_i}\right)-b_2^2e^{i\theta}\right|}{\left(\left((2(\cos\theta-\alpha))\prod_{i=1}^{n}|c_i|\right)^2-|b_2|^2\right)} \end{aligned}$$

Also,we have

$$|\Upsilon'(1)| = 1 + \sum_{i=1}^{n} \frac{1 - |c_i|^2}{|1 - c_i|^2}, \ 1 \in \partial\mathcal{K}.$$

Therefore, we obtain

$$\begin{aligned} &\frac{2}{1+\prod_{i=1}^{n}|c_i|\frac{\left|2(\cos\theta-\alpha)\left(2b_3-b_2^2+b_2\sum_{i=1}^{n}\frac{1-|c_i|^2}{c_i}\right)-b_2^2e^{i\theta}\right|}{\left(\left((2(\cos\theta-\alpha))\prod_{i=1}^{n}|c_i|\right)^2-|b_2|^2\right)}} \\ &\leqslant \frac{2(\cos\theta-\alpha)\prod_{i=1}^{n}|c_i|+|b_2|}{2(\cos\theta-\alpha)\prod_{i=1}^{n}|c_i|-|b_2|}\left(\frac{2\cos^2\theta}{\cos\theta-\alpha}|f'(1)|-1-\sum_{i=1}^{n}\frac{1-|c_i|^2}{|1-c_i|^2}\right), \end{aligned}$$

$$\frac{\left(2(\cos\theta-\alpha)\prod_{i=1}^{n}|c_i|-|b_2|\right)^2}{\left(\left((2(\cos\theta-\alpha))\prod_{i=1}^{n}|c_i|\right)^2-|b_2|^2\right)+\prod_{i=1}^{n}|c_i|\left|2(\cos\theta-\alpha)\left(2b_3-b_2^2+b_2\sum_{i=1}^{n}\frac{1-|c_i|^2}{c_i}\right)-b_2^2e^{i\theta}\right|}$$
$$\leqslant \frac{2\cos^2\theta}{\cos\theta-\alpha}|f'(1)|-1-\sum_{i=1}^{n}\frac{1-|c_i|^2}{|1-c_i|^2}$$

and so, we get inequality (2.5). □

The relationship between the coefficients b_2 and b_3 in the $h(z)=z+\sum_{p=1}^{\infty}b_{p+1}z^{p+1}$ function's Maclaurin expansion is demonstrated in the following theorem.

Theorem 2.5. *Let $h\in\mathcal{M}(\alpha,\theta)$, $1-\frac{zh'(z)}{h(z)}$ has no zeros in $\mathcal{K}$ except $z=0$ and $b_2>0$. Then we have the inequality*

$$\left|2\left(2b_3-b_2^2\right)(\cos\theta-\alpha)-b_2^2e^{i\theta}\right|\leqslant 4\left|(\cos\theta-\alpha)\,b_2\ln\left(\frac{b_2}{2(\cos\theta-\alpha)}\right)\right|. \tag{2.6}$$

Proof. Let $b_2>0$ in the expression of the function $h(z)$. Having in mind the inequality (2.4) and the function $1-\frac{zh'(z)}{h(z)}$ has no zeros in $\mathcal{K}$ except $z=0$, we denote by $\ln\phi(z)$ the analytic branch of the logarithm normed by the condition

$$\ln\phi(0)=\ln\left(\frac{b_2}{2(\cos\theta-\alpha)}\right)<0.$$

The auxiliary function

$$q(z)=\frac{\ln\phi(z)-\ln\phi(0)}{\ln\phi(z)+\ln\phi(0)}$$

is analytic in the unit disc $\mathcal{K}$, $|q(z)|<1$ for $z\in\mathcal{K}$, $q(0)=0$.

By Schwarz lemma, we obtain

$$\begin{aligned}1 &\geqslant |q'(0)|=\frac{|2\ln\phi(0)|}{|\ln\phi(0)+\ln\phi(0)|^2}\left|\frac{\phi'(0)}{\phi(0)}\right| \\ &= \frac{-1}{2\ln\phi(0)}\left|\frac{\phi'(0)}{\phi(0)}\right|. \\ &= -\frac{\frac{\left|2\left(2b_3-b_2^2\right)(\cos\theta-\alpha)-b_2^2e^{i\theta}\right|}{4(\cos\theta-\alpha)^2}}{2\ln\left(\frac{b_2}{2(\cos\theta-\alpha)}\right)\frac{b_2}{2(\cos\theta-\alpha)}}\end{aligned}$$

and

$$\left|2\left(2b_3-b_2^2\right)(\cos\theta-\alpha)-b_2^2e^{i\theta}\right|\leqslant 4\left|(\cos\theta-\alpha)\,b_2\ln\left(\frac{b_2}{2(\cos\theta-\alpha)}\right)\right|.$$

□

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Bülent Nafi ÖRNEK, Department of Computer Engineering, Amasya University, Merkez-Amasya 05100, Türkiye
Email address: nafiornek@gmail.com, nafi.ornek@amasya.edu.tr

Nurbanu Tuğçe AKCA, Department of Mathematics, Amasya University, Merkez-Amasya, 05100, Türkiye
Email address: nurbanutugce@icloud.com