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ON S-2-ABSORBING FILTERS OF LATTICES

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ABSTRACT. Let $\pounds$ be a bounded distributive lattice and S a join closed subset of $\pounds$. Following the concept of S-2-absorbing submodules, we define S-2-absorbing filters of $\pounds$. Let $\mathbf{p}$ be a filter of $\pounds$ disjoint with S. We say that $\mathbf{p}$ is an S-2-absorbing filter of $\pounds$ if there is a fixed $s \in S$ such that for all $x, y, z \in \pounds$ if $x \vee y \vee z \in \mathbf{p}$, then $s \vee x \vee y \in \mathbf{p}$ or $s \vee y \vee z \in \mathbf{p}$ or $s \vee x \vee z \in \mathbf{p}$. We will make an intensive investigation of the basic properties and possible structures of these filters.

1. Introduction

All lattices considered in this paper are assumed to have a least element denoted by 0 and a greatest element denoted by 1, in other words they are bounded. Our objective in this paper is to extend the notion of S-2-absorbing property in modules theory to S-2-absorbing property in the lattices, and to investigate the relations between S-2-absorbing filters and 2-absorbing filters. Indeed, we are interested in investigating S-2-absorbing filters to use other notions of S-2-absorbing and associate which exist in the literature as laid forth in [11, 13].

The notion of prime ideals has a significant place in the theory of rings, and it is used to characterize certain classes of rings. For years, there have been many studies and generalizations on this issue. See, for example, [3, 6, 9, 10, 11, 12, 13]. Badawi generalized the concept of prime ideals in [3]. We recall from [3] that a proper ideal I of a commutative ring R is said to be a 2-absorbing ideal if whenever $abc \in I$ for $a, b, c \in R$, then $ab \in I$ or $ac \in I$ or $bc \in I$ (also see [6]). In 2019, Hamed and Malek [10] introduced the notion of an S-prime ideal, i.e. let $S \subseteq R$ be a multiplicative set and I an ideal of R disjoint from S. We say that I is S-prime if there exists an $s \in S$ such that for all $a, b \in R$ with $ab \in I$, we have $sa \in I$ or

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$sb \in I$ (also see [12]). In 2020, Ulucak, Tekir and Koc [13] introduced the notion of an S-2-absorbing submodules, i.e. let $S \subseteq R$ be a multiplicative set and P a submodule of an R-module M with $S \cap (P :_R M) = \emptyset$. We say that P is an S-2-absorbing submodule if there exists an element $s \in S$ and whenever $abm \in P$ for some $a, b \in R$ and $m \in M$, then $sab \in (P :_R M)$ or $sam \in P$ or $sbm \in P$. In 2021, Naji [11] introduced the notion of an S-2-absorbing primary submodules, i.e. let $S \subseteq R$ be a multiplicative set and P a submodule of an R-module M with $S \cap (P :_R M) = \emptyset$. We say that P is an S-2-absorbing primary submodule if there exists an element $s \in S$ and whenever $abm \in P$ for some $a, b \in R$ and $m \in M$, then $sab \in \sqrt{(P :_R M)}$ or $sam \in P$ or $sbm \in P$.

Let $\pounds$ be a bounded distributive lattice. We say that a subset $S \subseteq \pounds$ is *join closed* if $0 \in S$ and $s_1 \vee s_2 \in S$ for all $s_1, s_2 \in S$ (if $\mathbf{p}$ is a prime filter of $\pounds$, then $\pounds \smallsetminus \mathbf{p}$ is a join closed subset of $\pounds$). Among many results in this paper, the first, preliminaries section contains elementary observations needed later on. Section 3 is dedicated to the investigate the some basic properties of S-2-absorbing filters. At first, we give the definition of S-2-absorbing filters (Definition 3.1) and provide an example (Example 3.2) of an S-2-absorbing filter of $\pounds$ that is not a 2-absorbing filter. It is shown (Theorem 3.1) that if S is a join closed subset of $\pounds$, then the intersection of two S-prime filters is an S-2-absorbing filter. Also, we give three other characterizations of S-2-absorbing filters (see Lemma 3.1, Lemma 3.2, Proposition 3.2 and Theorem 3.2). We continue in Section 4 by investigation the stability of S-2-absorbing filters in various lattice-theoretic constructions. Indeed, we investigate the behavior of S-2-absorbing filters under homomorphism, in factor lattices, S-Noetherian lattices, and in cartesian products of lattices (see Theorem 4.2, Theorem 4.3, Theorem 4.4, Theorem 4.5, Theorem 4.7, and Theorem 4.8).

2. Preliminaries

Let us recall some notions and notations. By a lattice we mean a *poset* $(\pounds, \leqslant)$ in which every couple elements x, y has a g.l.b. (called the *meet* of x and y, and written $x \wedge y$) and a l.u.b. (called the *join* of x and y, and written $x \vee y$). A lattice $\pounds$ is *complete* when each of its subsets X has a l.u.b. and a g.l.b. in $\pounds$. Setting $X = \pounds$, we see that any non-void complete lattice contains a least element 0 and greatest element 1 (in this case, we say that $\pounds$ is a lattice with 0 and 1). A lattice $\pounds$ is called a *distributive* lattice if $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all a, b, c in $\pounds$ (equivalently, $\pounds$ is distributive if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all a, b, c in $\pounds$). A non-empty subset F of a lattice $\pounds$ is called a *filter*, if for $a \in F$, $b \in \pounds$, $a \leqslant b$ implies $b \in F$, and $x \wedge y \in F$ for all $x, y \in F$ (so if $\pounds$ is a lattice with 1, then $1 \in F$ and $\{1\}$ is a filter of $\pounds$). A proper filter F of $\pounds$ is called *prime* if $x \vee y \in F$, then $x \in F$ or $y \in F$. A proper filter F of $\pounds$ is said to be *maximal* if G is a filter in $\pounds$ with $F \subsetneqq G$, then $G = \pounds$. The intersection of all filters containing a given subset A of $\pounds$ is the *filter generated by* it, is denoted by $T(A)$. A filter F is called *finitely generated* if there is a finite subset A of F such that $F = T(A)$. A proper filter F of a lattice $\pounds$ is called a 2-*absorbing* filter if whenever $a, b, c \in \pounds$ and $a \vee b \vee c \in F$, then $a \vee b \in F$ or $a \vee c \in F$ or $b \vee c \in F$. Let $\mathbf{p}$ be a filter of $\pounds$ and S a join closed subset of $\pounds$ disjoint with S. We say that $\mathbf{p}$ is an *S-prime* filter of $\pounds$ if there is an element $s \in S$ such that for all $x, y \in \pounds$ if $x \vee y \in \mathbf{p}$, then $x \vee s \in \mathbf{p}$ or $y \vee s \in \mathbf{p}$.

A lattice $\pounds$ with 1 is called *$\pounds$-domain* if $a \vee b = 1$ $(a, b \in \pounds)$, then $a = 1$ or $b = 1$ (so $\pounds$ is $\pounds$-domain if and only if $\{1\}$ is a prime filter of $\pounds$). If $x \in \pounds$, then a *complement* of x in $\pounds$ is an element $y \in \pounds$ such that $x \vee y = 1$ and $x \wedge y = 0$. The lattice $\pounds$ is *complemented* if every element of $\pounds$ has a complement in $\pounds$. If $\pounds$ and $\pounds'$ are lattices, then a *lattice homomorphism* $f : \pounds \to \pounds'$ is a map from $\pounds$ to $\pounds'$ satisfying $f(x \vee y) = f(x) \vee f(y)$ and $f(x \wedge y) = f(x) \wedge f(y)$ for $x, y \in \pounds$. First we need the following lemmas proved in [5, 6, 7, 8, 9].

Lemma 2.1. *Let $\pounds$ be a lattice.*

(1) A non-empty subset F of $\pounds$ is a filter of $\pounds$ if and only if $x \vee z \in F$ and $x \wedge y \in F$ for all $x, y \in F$, $z \in \pounds$. Moreover, since $x = x \vee (x \wedge y)$, $y = y \vee (x \wedge y)$ and F is a filter, $x \wedge y \in F$ gives $x, y \in F$ for all $x, y \in \pounds$.

(2) Let A be an arbitrary non-empty subset of $\pounds$. Then

$$T(A) = \{x \in \pounds : \ a_1 \wedge a_2 \wedge \cdots \wedge a_n \leqslant x \text{ for some } a_i \in A \ (1 \leqslant i \leqslant n)\}.$$

Moreover, if F is a filter and A is a subset of $\pounds$ with $A \subseteq F$, then $T(A) \subseteq F$, $T(F) = F$ and $T(T(A)) = T(A)$

(3) If $\{F_i\}_{i \in \Delta}$ is a chain of filters of $\pounds$, then $\bigcup_{i \in \Delta} F_i$ is a filter of $\pounds$.

Lemma 2.2. *Let F, G be filters of $\pounds$ and $x \in \pounds$. The following hold:*

(1) $F \vee G = \{a \vee b : a \in F, b \in G\}$ and $x \vee F = \{a \vee y : y \in F\}$ are filters of $\pounds$ with $F \vee G = F \cap G$.

(2) If $\pounds$ is distributive, then $F \wedge G = \{a \wedge b : a \in F, b \in G\}$ is a filter of $\pounds$ with $F, G \subseteq F \wedge G$

(3) If $\pounds$ is distributive, F, G are filters of $\pounds$ and $y \in \pounds$, then $(G :_{\pounds} F) = \{x \in \pounds : x \vee F \subseteq G\}$ and $(F :_{\pounds} T(\{y\})) = (F :_L y) = \{a \in \pounds : a \vee y \in F\}$ are filters of $\pounds$.

(4) If $\pounds$ is distributive, G, F_1, F_2 are filters of $\pounds$, then $G \vee (F_1 \wedge F_2) = (G \vee F_1) \wedge (G \vee F_2)$.

Assume that $(\pounds_1, \leqslant_1), (\pounds_2, \leqslant_2), \cdots, (\pounds_n, \leqslant_n)$ are lattices $(n \geqslant 2)$ and let $\pounds = \pounds_1 \times \pounds_2 \times \cdots \times \pounds_n$. We set up a partial order $\leqslant_c$ on $\pounds$ as follows: for each $x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in \pounds$, we write $x \leqslant_c y$ if and only if $x_i \leqslant_i y_i$ for each $i \in \{1, 2, \cdots, n\}$. The following notation below will be kept in this paper: It is straightforward to check that $(\pounds, \leqslant_c)$ is a lattice with $x \vee_c y = (x_1 \vee y_1, x_2 \vee y_2, \cdots, x_n \vee y_n)$ and $x \wedge_c y = (x_1 \wedge y_1, \cdots, x_n \wedge y_n)$. In this case, we say that $\pounds$ is a *decomposable lattice.*

Quotient lattices are determined by equivalence relations rather than by ideals as in the ring case. If F is a filter of a lattice $(\pounds, \leqslant)$, we define a relation on $\pounds$, given by $x \sim y$ if and only if there exist $a, b \in F$ satisfying $x \wedge a = y \wedge b$. Then $\sim$ is an equivalence relation on $\pounds$, and we denote the equivalence class of a by $a \wedge F$ and these collection of all equivalence classes by $\pounds/F$. We set up a partial order $\leqslant_Q$ on $\pounds/F$ as follows: for each $a \wedge F, b \wedge F \in \pounds/F$, we write $a \wedge F \leqslant_Q b \wedge F$ if and only if $a \leqslant b$. The following notation below will be used in this paper: It is straightforward to check that $(\pounds/F, \leqslant_Q)$ is a lattice with $(a \wedge F) \vee_Q (b \wedge F) = (a \vee b) \wedge F$ and

$(a \wedge F) \wedge_Q (b \wedge F) = (a \wedge b) \wedge F$ for all elements $a \wedge F, b \wedge F \in \pounds/F$. Note that $e \wedge F = F = 1 \wedge F$ if and only if $e \in F$ (see [8, Remark 4.2 and Lemma 4.3]).

3. Characterization of S-2-absorbing filters

In this section, we collect some basic properties concerning S-2-absorbing filters. We remind the reader with the following definition.

DEFINITION 3.1. *Let* $\mathbf{p}$ *be a filter of* $\pounds$ *and* S *a join closed subset of* $\pounds$. *A filter* $\mathbf{p}$ *is said to be* S*-2-absorbing if* $\mathbf{p} \cap S = \emptyset$ *and there exists a fixed* $s \in S$ *such that for any* $x, y, z \in \pounds$ *with* $x \vee y \vee z \in \mathbf{p}$, *then* $s \vee x \vee y \in \mathbf{p}$ *or* $s \vee x \vee z \in \mathbf{p}$ *or* $s \vee y \vee z \in \mathbf{p}$.

EXAMPLE 3.1. *(1) If* $S = \{0\}$, *then the* 2*-absorbing and the* S*-2-absorbing filters of* $\pounds$ *are the same.*

(2) If $\mathbf{p}$ *is a* 2*-absorbing filter of* $\pounds$ *disjoint with* S, *then* $\mathbf{p}$ *is an* S*-2-absorbing filter.*

(3) Let$\pounds = \{0, a, b, c, 1\}$ *be a lattice with the relations* $0 \leqslant a \leqslant c \leqslant 1$, $0 \leqslant b \leqslant c \leqslant 1$, $a \vee b = c$ *and* $a \wedge b = 0$. *An inspection will show that the nontrivial filters (i.e. different from* $\pounds$ *and* $\{1\}$*) of* $\pounds$ *are* $\mathbf{p_1} = \{1, c\}$, $\mathbf{p_2} = \{1, c, a\}$ *and* $\mathbf{p_3} = \{1, c, b\}$. *Set* $S = \{0, a\}$. *Then* S *is a join closed subset of* $\pounds$ *with* $S \cap \mathbf{p_1} = \emptyset$. *Since* $\mathbf{p_2} \cap \mathbf{p_3} = \mathbf{p_1}$, *we conclude that* $\mathbf{p_1}$ *is a* 2*-absorbing filter by [6, Theorem 2.8]; hence* $\mathbf{p_1}$ *is* S*-2-absorbing by (2).*

EXAMPLE 3.2. *Let* $\pounds_1 = \{0, a, b, c, d, 1\}$ *be a lattice with the relations* $0 \leqslant a \leqslant d \leqslant 1$, $0 \leqslant b \leqslant d \leqslant 1$, $0 \leqslant c \leqslant 1$ *and* $a \wedge b = a \wedge c = d \wedge c = c \wedge b = 0$. *Suppose that* $\pounds = \pounds_1 \times \pounds_1$, $\mathbf{p} = \{b, d, 1\} \times \{1\}$ *and* $S = \{0, c\} \times \{0, c\}$; *so* $\mathbf{p}$ *is a filter of* $\pounds$ *with* $\mathbf{p} \cap S = \emptyset$. *Then* $\mathbf{p}$ *is an* S*-2-absorbing filter. Indeed, let* $(a_1, b_1) \vee_c (a_2, b_2) \vee_c (a_3, b_3) \in \mathbf{p}$ *for some* $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in \pounds$. *Then an inspection will show that* $(c, c) \in S$ *and* $(a_1, b_1) \vee_c (a_2, b_2) \vee_c (c, c) \in \mathbf{p}$ *or* $(a_1, b_1) \vee_c (a_3, b_3) \vee_c (c, c) \in \mathbf{p}$ *or* $(a_3, b_3) \vee_c (a_2, b_2) \vee_c (c, c) \in \mathbf{p}$, *as needed.*

On the other hand, $\mathbf{p}$ *is not a* 2*-absorbing filter since* $(b, 0) \vee (c, d) \vee (0, c) = (1, 1) \in \mathbf{p}$ *but neither* $(b, 0) \vee (c, d) = (1, d) \in \mathbf{p}$ *nor* $(b, 0) \vee (0, c) = (b, c) \in \mathbf{p}$ *nor* $(c, d) \vee (0, c) = (c, 1) \in \mathbf{p}$. *Thus an* S*-2-absorbing filter need not be a* 2*-absorbing filter.*

EXAMPLE 3.3. *Let* $S' \subseteq S$ *be join closed subsets of* $\pounds$ *and* $\mathbf{p}$ *a filter of* $\pounds$ *disjoint with* S. *It is clear that if* $\mathbf{p}$ *is an* S'*-2-absorbing filter of* $\pounds$, *then* $\mathbf{p}$ *is an* S*-2-absorbing filter. However, the converse is not true in general. Indeed, assume that* $\pounds$ *is the lattice as in Example 3.2 and let* $S' = \{(0, 0)\} \subseteq S = \{0, c\} \times \{0, c\}$. *Then* $\mathbf{p} = \{b, d, 1\} \times \{1\}$ *is an* S*-2-absorbing filter of* $\pounds$ *but not an* S'*-2-absorbing filter of* $\pounds$.

PROPOSITION 3.1. *Let* S, S' *be join closed subsets of* $\pounds$. *The following hold:*

(1) Every S*-prime filter is an* S*-2-absorbing filter;*

(2) If $S' \subseteq S$ *such that for any* $s \in S$, *there exists* $t \in S$ *satisfying* $s \vee t \in S'$. *If* $\mathbf{p}$ *is an* S*-2-absorbing filter of* $\pounds$, *then* $\mathbf{p}$ *is an* S'*-2-absorbing filter of* $\pounds$.

PROOF. (1) It is clear.

(2) Let $x, y, z \in \pounds$ such that $x \vee y \vee z \in \mathbf{p}$. Then there exists $s \in S$ such that $s \vee x \vee y \in \mathbf{p}$ or $s \vee x \vee z \in \mathbf{p}$ or $s \vee y \vee z \in \mathbf{p}$. By the hypothesis, there is $t \in S$ such that $s \vee t \in S'$ and then $s \vee t \vee x \vee y \in \mathbf{p}$ or $s \vee t \vee x \vee z \in \mathbf{p}$ or $s \vee t \vee y \vee z \in \mathbf{p}$, as $\mathbf{p}$ is a filter. This shows that $\mathbf{p}$ is S'-2-absorbing. □

Compare the next theorem with Proposition 2 in [13].

THEOREM 3.1. *If S is a join closed subset of $\pounds$, then the intersection of two S-prime filters is an S-2-absorbing filter.*

PROOF. Let $\mathbf{p_1}, \mathbf{p_2}$ be two S-prime filters of $\pounds$ and $\mathbf{p} = \mathbf{p_1} \cap \mathbf{p_2}$. Suppose that $a \vee b \vee c \in \mathbf{p}$ for some $a, b, c \in \pounds$. Since $\mathbf{p_1}$ is an S-prime filter and $a \vee b \vee c \in \mathbf{p_1}$, there exists $t_1 \in S$ such that $t_1 \vee a \in \mathbf{p_1}$ or $t_1 \vee b \vee c \in \mathbf{p_1}$. If $t_1 \vee b \vee c \in \mathbf{p_1}$, then $\mathbf{p_1}$ is an S-prime gives there exists $t_1' \in S$ such that either $t_1' \vee b \in \mathbf{p_1}$ or $t_1' \vee t_1 \vee c \in \mathbf{p_1}$. Set $s_1 = t_1 \vee t_1' \in S$. Then either $s_1 \vee b \in \mathbf{p_1}$ or $s_1 \vee c \in \mathbf{p_1}$. Similarly, since $\mathbf{p_2}$ is an S-prime filter and $a \vee b \vee c \in \mathbf{p_2}$, we conclude that there exists $s_2 \in S$ such that $s_2 \vee a \in \mathbf{p_2}$ or $s_2 \vee b \in \mathbf{p_2}$ or $s_2 \vee c \in \mathbf{p_2}$. Without loss of generality, we can assume that $s_1 \vee a \in \mathbf{p_1}$ and $s_2 \vee c \in \mathbf{p_2}$. Now we put $s = s_1 \vee s_2$. This shows that $s \vee a \vee c \in \mathbf{p}$ and so $\mathbf{p}$ is an S-2-absorbing filter of $\pounds$. □

LEMMA 3.1. *Let $\mathbf{p}$ be a filter of $\pounds$ and S a join closed subset of $\pounds$ disjoint with $\mathbf{p}$. The following assertions are equivalent:*

(1) $\mathbf{p}$ is an S-2-absorbing filter of $\pounds$;

(2) There exists an $s \in S$ such that whenever $(a \vee b) \vee F \subseteq \mathbf{p}$ for some filter F of $\pounds$ and $a, b \in \pounds$ implies either $s \vee a \vee b \in \mathbf{p}$ or $(s \vee b) \vee F \subseteq \mathbf{p}$ or $(s \vee a) \vee F \subseteq \mathbf{p}$.

PROOF. (1) $\Rightarrow$ (2) By the hypothesis, there exists an $s \in S$ such that $x \vee y \vee z \in \mathbf{p}$ for some $x, y, z \in \pounds$ implies $s \vee x \vee y \in \mathbf{p}$ or $s \vee x \vee z \in \mathbf{p}$ or $s \vee y \vee z \in \mathbf{p}$. Let $(a \vee b) \vee F \subseteq \mathbf{p}$ for some $a, b \in \pounds$ and a filter F of $\pounds$. Let $s \vee a \vee b \notin \mathbf{p}$ and $(s \vee a) \vee F \nsubseteq \mathbf{p}$. So there exists $f \in F$ such that $s \vee a \vee f \notin \mathbf{p}$. Since $a \vee b \vee f \in \mathbf{p}$ and $\mathbf{p}$ is S-2-absorbing, we get $s \vee b \vee f \in \mathbf{p}$. We show that $(s \vee b) \vee F \subseteq \mathbf{p}$. Let $e \in F$. Then $(e \wedge f) \vee a \vee b \in \mathbf{p}$ and hence either $s \vee (f \wedge e) \vee b \in \mathbf{p}$ or $s(f \wedge e) \vee a \in \mathbf{p}$. If $s \vee (f \wedge e) \vee b = (s \vee b \vee f) \wedge (s \vee b \vee e) \in \mathbf{p}$, then $s \vee b \vee e \in \mathbf{p}$ by Lemma 2.1. If $s \vee (f \wedge e) \vee a = (s \vee a \vee f) \wedge (s \vee a \vee e) \in \mathbf{p}$ gives $s \vee a \vee f \in \mathbf{p}$ by Lemma 2.1 which is a contradiction; so $s \vee b \vee e \in \mathbf{p}$. This shows that $(s \vee b) \vee F \subseteq \mathbf{p}$.

(2) $\Rightarrow$ (1) If $a \vee b \vee c \in \mathbf{p}$ for some $a, b, c \in \pounds$, then $(a \vee b) \vee T(\{c\}) \subseteq \mathbf{p}$ gives there exists an $s \in S$ such that either $s \vee a \vee b \in \mathbf{p}$ or $s \vee b \vee c \in (s \vee b) \vee T(\{c\}) \subseteq \mathbf{p}$ or $s \vee a \vee c \in (s \vee a) \vee T(\{c\}) \subseteq \mathbf{p}$ by (2), as needed. □

LEMMA 3.2. *Let $\mathbf{p}$ be a filter of $\pounds$ and S a join closed subset of $\pounds$ disjoint with $\mathbf{p}$. The following assertions are equivalent:*

(1) $\mathbf{p}$ is an S-2-absorbing filter of $\pounds$;

(2) There exists an $s \in S$ such that $a \vee (F \vee G) \subseteq \mathbf{p}$ for some filters F, G of $\pounds$ and $a \in \pounds$, then either $(s \vee a) \vee F \subseteq \mathbf{p}$ or $(s \vee a) \vee G \subseteq \mathbf{p}$ or $s \vee (F \vee G) \subseteq \mathbf{p}$.

PROOF. (1) $\Rightarrow$ (2) Let $\mathbf{p}$ be an S-2-absorbing filter of $\pounds$. Then we assume that $s \in S$ satisfies S-2-absorbing condition. Let $a \vee (F \vee G) \subseteq \mathbf{p}$ for some filters F, G

of $\pounds$ and $a \in \pounds$ and suppose that $(s \vee a) \vee F \nsubseteq \mathbf{p}$ and $(s \vee a) \vee G \nsubseteq \mathbf{p}$. We want to show that $s \vee (F \vee G) \subseteq \mathbf{p}$. Let $f \in F$ and $g \in G$. There exists $b \in F \setminus \mathbf{p}$ such that $s \vee a \vee b \notin \mathbf{p}$. Since $(a \vee b) \vee G \subseteq \mathbf{p}$, we conclude that $(s \vee b) \vee G \subseteq \mathbf{p}$ by Lemma 3.1 and so $s \vee (F \setminus \mathbf{p}) \vee G \subseteq \mathbf{p}$. Similarly, there exists $c \in G \setminus \mathbf{p}$ such that $(s \vee c) \vee F \subseteq \mathbf{p}$ and $s \vee (G \setminus \mathbf{p}) \vee F \subseteq \mathbf{p}$. Thus we have $s \vee b \vee c \in \mathbf{p}$, $s \vee b \vee g \in \mathbf{p}$ and $s \vee f \vee c \in \mathbf{p}$. As $b \wedge f \in F$ and $g \wedge c \in G$, we conclude that $(b \wedge f) \vee (g \wedge c) \vee a \in \mathbf{p}$. Hence, $s \vee (b \wedge f) \vee a \in \mathbf{p}$ or $s \vee (g \wedge c) \vee a \in \mathbf{p}$ or $s \vee (b \wedge f) \vee (g \wedge c) \in \mathbf{p}$. If $s \vee (b \wedge f) \vee a = (s \vee a \vee b) \wedge (s \vee a \vee f) \in \mathbf{p}$, then $s \vee a \vee b \in \mathbf{p}$ by Lemma 2.1, a contradiction. Similarly, $s \vee (c \wedge g) \vee a \notin \mathbf{p}$. So $s \vee (b \wedge f) \vee (g \wedge c) = (s \vee c \vee f) \wedge (s \vee c \vee b) \wedge (s \vee g \vee f) \wedge (s \vee g \vee b) \in \mathbf{p}$. This shows that $s \vee f \vee g \in \mathbf{p}$ by Lemma 2.1. Therefore, $s \vee (F \vee G) \subseteq \mathbf{P}$.

(2) $\Rightarrow$ (1) If $a \vee b \vee c \in \mathbf{p}$ for some $a, b, c \in \pounds$, then $a \vee (T(\{b\}) \vee T(\{c\})) \subseteq \mathbf{p}$ gives there exists an $s \in S$ such that $s \vee a \vee b \in (s \vee a) \vee T(\{b\}) \subseteq \mathbf{p}$ or $s \vee a \vee c \in (s \vee a) \vee T(\{c\}) \subseteq \mathbf{p}$ or $s \vee b \vee c \in s \vee (T(\{b\}) \vee T(\{c\})) \subseteq \mathbf{p}$ by (2), as required. □

PROPOSITION 3.2. *Let* $\mathbf{p}$ *be a filter of* $\pounds$ *and* S *a join closed subset of* $\pounds$ *disjoint with* $\mathbf{p}$. *The following assertions are equivalent:*

(1) $\mathbf{p}$ *is an* S*-2-absorbing filter of* $\pounds$*;*

(2) There exists a fixed $s \in S$ *such that whenever* $F \vee G \vee K \subseteq \mathbf{p}$ *for some filters* F, G, K *of* $\pounds$*, then either* $s \vee (F \vee G) \subseteq \mathbf{p}$ *or* $s \vee (F \vee K) \subseteq \mathbf{p}$ *or* $s \vee (G \vee K) \subseteq \mathbf{p}$.

PROOF. (1) $\Rightarrow$ (2) Let $\mathbf{p}$ be an S-2-absorbing filter of $\pounds$ and assume that $s \in S$ satisfies S-2-absorbing condition. Suppose that $F \vee G \vee K \subseteq \mathbf{p}$ for some filters F, G, K of $\pounds$ and $s \vee (F \vee G) \nsubseteq \mathbf{p}$. Then for each $x \in K$, $x \vee (F \vee G) \subseteq \mathbf{p}$ gives either $(s \vee x) \vee F \subseteq \mathbf{p}$ or $(s \vee x) \vee G \subseteq \mathbf{p}$ by Lemma 3.2. If for every $x \in K$, $(s \vee x) \vee F \subseteq \mathbf{p}$, then $s \vee (F \vee K) \subseteq \mathbf{p}$. Similarly, if for all $x \in K$, $(s \vee x) \vee G \subseteq \mathbf{p}$, we have $s \vee (G \vee K) \subseteq \mathbf{p}$. We are going to show that either $s \vee (F \vee K) \subseteq \mathbf{p}$ or $s \vee (G \vee K) \subseteq \mathbf{p}$. Assume on the contrary, that $s \vee (F \vee K) \nsubseteq \mathbf{p}$ and $s \vee (G \vee K) \nsubseteq \mathbf{p}$. So there exist $k_1, k_2 \in K$ such that $(s \vee k_1) \vee F \nsubseteq \mathbf{p}$ and $(s \vee k_2) \vee G \nsubseteq \mathbf{p}$. Therefore, $(s \vee k_2) \vee F \subseteq \mathbf{p}$ and $(s \vee k_1) \vee G \subseteq \mathbf{p}$. Since $(k_1 \wedge k_2) \vee (F \vee G) \subseteq \mathbf{p}$, we conclude that $s \vee (k_1 \wedge k_2) \vee F \subseteq \mathbf{p}$ or $s \vee (k_1 \wedge k_2) \vee G \subseteq \mathbf{p}$ by Lemma 3.2. If $s \vee (k_1 \wedge k_2) \vee F \subseteq \mathbf{p}$, then for all $f \in F$, $s \vee (k_1 \wedge k_2) \vee f = (s \vee k_1 \vee f) \wedge (s \vee k_2 \vee f) \in \mathbf{p}$ which implies that $s \vee k_1 \vee f \in \mathbf{p}$ by Lemma 2.1; hence $(s \vee k_1) \vee F \subseteq \mathbf{p}$ which is impossible. Similarly, by $s \vee (k_1 \wedge k_2) \vee G \subseteq \mathbf{p}$, we get a contradiction. Therefore, $s \vee (F \vee K) \subseteq \mathbf{p}$ or $s \vee (G \vee K) \subseteq \mathbf{p}$.

(2) $\Rightarrow$ (1) If $a \vee b \vee c \in \mathbf{p}$ for some $a, b, c \in \pounds$, then $T(\{a\}) \vee (T(\{b\}) \vee T(\{c\})) \subseteq \mathbf{p}$ gives there exists an $s \in S$ such that $s \vee a \vee b \in s \vee (T(\{a\}) \vee T(\{b\})) \subseteq \mathbf{p}$ or $s \vee a \vee c \in s \vee (T(\{a\})) \vee T(\{c\})) \subseteq \mathbf{p}$ or $s \vee b \vee c \in s \vee (T(\{b\}) \vee T(\{c\})) \subseteq \mathbf{p}$ by (2), as required. □

We next give three other characterizations of S-2-absorbing filters. Compare the next theorem with Theorem 1 in [13].

THEOREM 3.2. *Let* $\mathbf{p}$ *be a filter of* $\pounds$ *and* S *a join closed subset of* $\pounds$ *disjoint with* $\mathbf{p}$. *The following assertions are equivalent:*

(1) $\mathbf{p}$ *is an* S*-2-absorbing filter of* $\pounds$*;*

(2) There exists an $s \in S$ such that whenever $(a \vee b) \vee F \subseteq \mathbf{p}$ for some filter F of $£$ and $a, b \in £$ implies either $s \vee a \vee b \in \mathbf{p}$ or $(s \vee b) \vee F \subseteq \mathbf{p}$ or $(s \vee a) \vee F \subseteq \mathbf{p}$.

(3) (2) There exists an $s \in S$ such that $a \vee (F \vee G) \subseteq \mathbf{p}$ for some filters F, G of $£$ and $a \in £$, then either $(s \vee a) \vee F \subseteq \mathbf{p}$ or $(s \vee a) \vee G \subseteq \mathbf{p}$ or $s \vee (F \vee G) \subseteq \mathbf{p}$.

(4) There exists an $s \in S$ such that whenever $F \vee G \vee K \subseteq \mathbf{p}$ for some filters F, G, K of $£$, then either $s \vee (F \vee G) \subseteq \mathbf{p}$ or $s \vee (F \vee K) \subseteq \mathbf{p}$ or $s \vee (G \vee K) \subseteq \mathbf{p}$.

Proof. This is a direct consequence Lemma 3.1, Lemma 3.2 and Proposition 3.2. □

4. Further results

We continue in this section by investigation the stability of S-2-absorbing filters in various lattice-theoretic constructions.

Proposition 4.1. *Let $\mathbf{p}$ be a filter of $£$, $e \in £$ and S a join closed subset of $£$ with $(\mathbf{p} :_{£} e) \cap S = \emptyset$. If $\mathbf{p}$ is an S-2-absorbing filter of $£$, then $(\mathbf{p} :_{£} e)$ is a S-2-absorbing filter of $£$.*

Proof. Let P be an S-2-absorbing filter of $£$. Suppose that $s \in S$ satisfies the S-2-absorbing condition. Let $a \vee b \vee c \in (\mathbf{p} :_{£} e)$ for some $a, b, c \in £$. Set $F = T(\{a\})$, $G = T(\{b\})$ and $K = T(\{c \vee e\})$. Then $F \vee G \vee K \subseteq \mathbf{p}$ gives $s \vee a \vee b \in s \vee (F \vee G) \subseteq \mathbf{p}$ (so $s \vee a \vee b \vee e \in \mathbf{p}$) or $s \vee a \vee c \vee e \in s \vee (F \vee K) \subseteq \mathbf{p}$ or $s \vee b \vee c \vee e \in s \vee (G \vee K) \subseteq \mathbf{p}$ by Theorem 3.10; so $s \vee a \vee b \in (\mathbf{p} :_{£} e)$ or $s \vee a \vee c \in (\mathbf{p} :_{£} e)$ or $s \vee b \vee c \in (\mathbf{p} :_{£} e)$. Hence, $(\mathbf{p} :_{£} e)$ is an S-2-absorbing filter of $£$. □

Compare the next theorem with Theorem 4 in [13].

Theorem 4.1. *Let $\mathbf{p}$ be a filter of $£$ and S a join closed subset of $£$ with $\mathbf{p} \cap S = \emptyset$. The following assertions are equivalent:*

(1) $\mathbf{p}$ is an S-2-absorbing filter of $£$;

(2) $(\mathbf{p} :_{£} s)$ is a 2-absorbing filter for some $s \in S$.

Proof. (1) $\Rightarrow$ (2) Let P be an S-2-absorbing filter of $£$. Then we keep in mind that there exists a fixed $s \in S$ that satisfies the S-2-absorbing condition. Since $\mathbf{p} \cap S = \emptyset$, we conclude that $(\mathbf{p} :_{£} s) \cap S = \emptyset$. Let $x \vee y \vee z \in (\mathbf{p} :_{£} s)$ for some $x, y, z \in £$. Then by (1), $s \vee x \vee y \in \mathbf{p}$ or $s \vee x \vee (s \vee z) = s \vee x \vee z \in \mathbf{p}$ or $s \vee y \vee (s \vee z) = s \vee y \vee z \in \mathbf{p}$ which gives $x \vee y \in (\mathbf{p} :_{£} s)$ or $x \vee z \in (\mathbf{p} :_{£} s)$ or $y \vee z \in (\mathbf{p} :_{£} s)$. Hence, $(\mathbf{p} :_{£} s)$ is a 2-absorbing filter of $£$.

(2) $\Rightarrow$ (1) Let $(\mathbf{p} :_{£} s)$ be a 2-absorbing filter for some $s \in S$ and $x \vee y \vee z \in \mathbf{p}$ (so $s \vee x \vee y \vee z \in \mathbf{p}$) for some $x, y, z \in £$ which implies that $x \vee y \vee z \in (\mathbf{p} :_{£} s)$. Then $(\mathbf{p} :_{£} s)$ is 2-absorbing gives $x \vee y \in (\mathbf{p} :_{£} s)$ or $x \vee z \in (\mathbf{p} :_{£} s)$ or $y \vee z \in (\mathbf{p} :_{£} s)$, as required. □

Proposition 4.2. *Let $\mathbf{p}$ be a filter of $£$ and S a join closed subset of $£$ disjoint with $\mathbf{p}$. The following hold:*

(1) Let $\mathbf{q}$ be a filter of $£$ such that $\mathbf{q} \cap S \neq \emptyset$. If $\mathbf{p}$ is an S-2-absorbing filter, then $\mathbf{p} \vee \mathbf{q}$ is an S-2-absorbing filter of $£$;

(2) Let $\pounds \subseteq \pounds'$ an extension of lattices. If $\mathbf{p}$ *is an* S*-2-absorbing filter of* $\pounds'$*, then* $\mathbf{p} \vee \pounds$ *is an* S*-2-absorbing filter of* $\pounds$*;*

Proof. (1) Clearly, $S \cap (\mathbf{p} \vee \mathbf{q}) = \emptyset$. Let $t \in S \cap \mathbf{q}$ and $x \vee y \vee z \in \mathbf{p} \vee \mathbf{q} \subseteq \mathbf{p}$ for some $x, y, z \in \pounds$. Then there is an element $s \in S$ such that $s \vee x \vee y \in \mathbf{p}$ or $s \vee y \vee z \in \mathbf{p}$ or $s \vee x \vee z \in \mathbf{p}$ which gives $s \vee t \vee x \vee y \in \mathbf{p} \vee \mathbf{q}$ or $s \vee t \vee y \vee z \in \mathbf{p} \vee \mathbf{q}$ or $s \vee t \vee x \vee z \in \mathbf{p} \vee \mathbf{q}$, where $s \vee t \in S$, i.e. (1) holds.

(2) Since $\pounds \cap S \neq \emptyset$, then (1) shows that (2) holds. □

Let $\pounds$ and $\pounds'$ be two lattices and $f : \pounds \to \pounds'$ be a lattice homomorphism such that $f(1) = 1$. Then it is easy to see that $\mathrm{Ker}(f) = \{x \in \pounds : f(x) = 1\}$ is a filter of $\pounds$. Compare the next theorem with Proposition 4 in [13].

Theorem 4.2. *Let* $f : \pounds \to \pounds'$ *be a lattice homomorphism such that* $f(1) = 1$ *and* $f(a) \neq 1$ *for all* $1 \neq a \in \pounds$ *and* S *is a join closed subset of* $\pounds$*. The following hold:*

(1) $f(S)$ *is a join closed subset of* $\pounds'$ *and if* $\mathbf{q}$ *is an* $f(S)$*-2-absorbing filter of* $\pounds'$*, then* $\mathbf{p} = f^{-1}(\mathbf{q})$ *is an* S*-2-absorbing filter of* $\pounds$*.*

(2) If $\pounds$ *is a complemented lattice,* f *is onto and* $\mathbf{p}$ *is an* S*-2-absorbing filter of* $\pounds$ *containing* $\mathrm{Ker}(f)$ *with* $f(S) \cap f(\mathbf{p}) = \emptyset$*, then* $f(\mathbf{p})$ *is an* $f(S)$*-2-absorbing filter of* $\pounds'$*.*

Proof. (1) Clearly, $f(S)$ is a join closed subset of $\pounds'$ and $\mathbf{p}$ is a filter of $\pounds$. By assumption, there exists $s \in S$ such that for all $x, y, z \in \pounds'$ if $x \vee y \vee z \in \mathbf{q}$, then $f(s) \vee x \vee y \in \mathbf{q}$ or $f(s) \vee y \vee z \in \mathbf{q}$ or $f(s) \vee x \vee z \in \mathbf{q}$. It is clear that $\mathbf{p} \cap S = \emptyset$. Let $a, b, c \in \pounds$ such that $a \vee b \vee c \in \mathbf{p}$; so $f(a \vee b \vee c) = f(a) \vee f(b) \vee f(c) \in \mathbf{q}$ which gives $f(s) \vee f(a) \vee f(b) = f(s \vee a \vee b) \in \mathbf{q}$ or $f(s) \vee f(b) \vee f(c) = f(s \vee b \vee c) \in \mathbf{q}$ or $f(s) \vee f(a) \vee f(c) = f(s \vee a \vee c) \in \mathbf{q}$. This implies that $s \vee a \vee b \in \mathbf{p}$ or $s \vee a \vee c \in \mathbf{p}$ or $s \vee b \vee c \in \mathbf{q}$, as required.

(2) It is easy to see that $f(\mathbf{p})$ is a filter of $\pounds'$. Suppose that $x \vee y \vee z \in f(\mathbf{p})$ for some $x, y, z \in \pounds'$. Then there exist $a, b, c \in \pounds$ such that $x = f(a)$, $y = f(b)$ and $z = f(c)$. Therefore $f(a \vee b \vee c) = f(a) \vee f(b) \vee f(c) \in f(\mathbf{p})$; so $f(a \vee b \vee c) = f(d)$ for some $d \in \mathbf{p}$. Since $\pounds$ is complemented, there exists $e \in \pounds$ such that $e \vee d = 1$ and $d \wedge e = 0$. Set $v = a \vee b \vee c$ (so $v \vee d \in \mathbf{p}$, as $\mathbf{p}$ is a filter). Then $f(v \vee e) = f(v) \vee f(e) = f(d) \vee f(e) = f(1) = 1$; hence $v \vee e \in \mathrm{Ker}(f) \subseteq \mathbf{p}$. Now $\mathbf{p}$ is a filter gives $(v \vee d) \wedge (v \vee e) = v \in \mathbf{p}$. Therefore there is an element $s \in S$ such that $s \vee a \vee b \in \mathbf{p}$ or $s \vee a \vee c \in \mathbf{p}$ or $s \vee b \vee c \in \mathbf{p}$, and so $f(s) \vee x \vee y \in f(\mathbf{p})$ or $f(s) \vee x \vee z \in f(\mathbf{p}')$ or $f(s) \vee y \vee z \in \mathbf{p}$. Hence $f(\mathbf{p})$ is a $f(S)$-2-absorbing filter of $\pounds'$. □

An element x of $\pounds$ is called identity join of a lattice $\pounds$, if there exists $1 \neq y \in \pounds$ such that $x \vee y = 1$. The set of all identity joins of a lattice $\pounds$ is denoted by $\mathrm{I}(\pounds)$. Let $\mathbf{p}$ be a filter of $\pounds$ and S a join closed subset of $\pounds$ disjoint with $\mathbf{p}$. It is clear that $S_Q = \{s \wedge \mathbf{p} : s \in S\}$ is a join closed subset of $\pounds/\mathbf{p}$. The next result determines the class of lattices for which their S-2-absorbing filters and 2-absorbing filters are the same.

Proposition 4.3. *Let* $\mathbf{p}$ *be a filter of* $\pounds$ *and* S *a join closed subset of* $\pounds$ *disjoint with* $\mathbf{p}$. *If* $\mathrm{I}(\pounds/\mathbf{p}) \cap S_Q = \emptyset$, *then the* 2*-absorbing and the* S*-2-absorbing filters coincide.*

Proof. It suffices to show that $\mathbf{p} = (\mathbf{p} :_{\pounds} s)$ for all $s \in S$ by Theorem 4.1. Since the inclusion $P \subseteq (P :_{\pounds} s)$ is clear, we will prove the reverse inclusion. Let $s \in S$ and $x \in (\mathbf{p} :_{\pounds} s)$. Then $s \vee x \in \mathbf{p}$ gives $(s \wedge \mathbf{p}) \vee_Q (x \wedge \mathbf{p}) = (s \vee x) \wedge \mathbf{p} = 1 \wedge \mathbf{p}$ by [8, Lemma 4.3]. Since $\mathrm{I}(\frac{\pounds}{\mathbf{p}}) \cap S_Q = \emptyset$, we conclude that $x \wedge \mathbf{p} = 1 \wedge \mathbf{p}$; so $x \wedge p_1 = 1 \wedge p_2 = p_2$ for some $p_1, p_2 \in \mathbf{p}$. This implies that $x \in \mathbf{p}$ by Lemma 2.1, as needed. □

Theorem 4.3. *Let* $\mathbf{q}$ *be a filter of* $\pounds$ *and* S *a join closed subset of* $\pounds$ *with* $S \cap \mathbf{q} = \emptyset$. *The following hold:*

(1) If $\mathbf{p}$ *is a proper* S*-2-absorbing filter of* $\pounds$ *containing* $\mathbf{q}$, *then* $\mathbf{p}/\mathbf{q}$ *is an* S_Q*-2-absorbing filter of* $\pounds/\mathbf{q}$;

(2) If $\mathbf{p}$ *is a proper filter of* $\pounds$ *containing* $\mathbf{q}$ *such that* $(\mathbf{p}/\mathbf{q}) \cap S_Q = \emptyset$, *then* $\mathbf{p}$ *is an* S*-2-absorbing filter of* $\pounds$ *if and only if* $\mathbf{p}/\mathbf{q}$ *is an* S_Q*-2-absorbing filter of* $\pounds/\mathbf{q}$.

Proof. (1) By assumption, there is an element $s \in S$ such that for all $x, y, z \in \pounds$, if $x \vee y \vee z \in \mathbf{p}$, then $s \vee x \vee y \in \mathbf{p}$ or $s \vee x \vee z \in \mathbf{p}$ or $s \vee y \vee z \in \mathbf{p}$. Let $a \wedge \mathbf{q}, b \wedge \mathbf{q}, c \wedge \mathbf{q} \in \pounds/\mathbf{q}$ such that $(a \wedge \mathbf{q}) \vee_Q (b \wedge \mathbf{q}) \vee_Q (c \wedge \mathbf{q}) = (a \vee b \vee c) \wedge \mathbf{q} \in \mathbf{p}/\mathbf{q}$ which implies $a \vee b \vee c \in \mathbf{p}$ by [8, Lemma 4.3]; hence $s \vee a \vee b \in \mathbf{p}$ or $s \vee a \vee c \in \mathbf{p}$ or $s \vee b \vee c \in \mathbf{p}$. Therefore $(s \wedge \mathbf{q}) \vee_Q (a \wedge \mathbf{q}) \vee_Q (b \wedge \mathbf{q}) \in \mathbf{p}/\mathbf{q}$ or $(s \wedge \mathbf{q}) \vee_Q (a \wedge \mathbf{q}) \vee_Q (c \wedge \mathbf{q}) \in \mathbf{p}/\mathbf{q}$ or $(s \wedge \mathbf{q}) \vee_Q (b \wedge \mathbf{q}) \vee_Q (c \wedge \mathbf{q}) \in \mathbf{p}/\mathbf{q}$. Thus $\mathbf{p}/\mathbf{q}$ is an S_Q-2-absorbing filter of $\pounds/\mathbf{q}$.

(2) One side follows from (1). To see the other side, suppose that $\mathbf{p} \cap S \neq \emptyset$. Then $(\mathbf{p}/\mathbf{q}) \cap S_Q \neq \emptyset$ which is impossible. Hence $S \cap \mathbf{p} = \emptyset$. Since $\mathbf{p}/\mathbf{q}$ is an S_Q-2-absorbing filter, we conclude that there exists $s \in S$ such that for all $x \wedge \mathbf{q}, y \wedge \mathbf{q}, z \wedge \mathbf{q} \in \mathbf{p}/\mathbf{q}$ with $(x \wedge \mathbf{q}) \vee_Q (y \wedge \mathbf{q}) \vee_Q (z \wedge \mathbf{q}) \in \mathbf{p}/\mathbf{q}$, we obtain $(s \wedge \mathbf{q}) \vee_Q (x \wedge \mathbf{q}) \vee_Q (y \wedge \mathbf{q}) \in \mathbf{p}/\mathbf{q}$ or $(s \wedge \mathbf{q}) \vee_Q (x \wedge \mathbf{q}) \vee_Q (z \wedge \mathbf{q}) \in \mathbf{p}/\mathbf{q}$ or $(s \wedge \mathbf{q}) \vee_Q (y \wedge \mathbf{q}) \vee_Q (z \wedge \mathbf{q}) \in \mathbf{p}/\mathbf{q}$. Now, let $a, b, c \in \pounds$ such that $a \vee b \vee c \in \mathbf{p}$. Then $(a \wedge \mathbf{q}) \vee_Q (b \wedge \mathbf{q}) \vee_Q (c \wedge \mathbf{q}) \in \mathbf{p}/\mathbf{q}$ gives $(s \wedge \mathbf{q}) \vee_Q (a \wedge \mathbf{q}) \vee_Q (b \wedge \mathbf{q}) \in \mathbf{p}/\mathbf{q}$ or $(s \wedge \mathbf{q}) \vee_Q (a \wedge \mathbf{q}) \vee_Q (c \wedge \mathbf{q}) \in \mathbf{p}/\mathbf{q}$ or $(s \wedge \mathbf{q}) \vee_Q (c \wedge \mathbf{q}) \vee_Q (b \wedge \mathbf{q}) \in \mathbf{p}/\mathbf{q}$; hence $s \vee a \vee b \in \mathbf{p}$ or $s \vee a \vee c \in \mathbf{p}$ or $s \vee b \vee c \in \mathbf{p}/\mathbf{q}$, as needed. □

Classically, in the lattice $\pounds$ every proper filter is contained in a maximal filter, its S-version is the following result.

Theorem 4.4. *If* S *is a join closed subset of* $\pounds$, *then each proper filter of* $\pounds$ *disjoint with* S *is contained in an* S*-2-absorbing filter of* $\pounds$.

Proof. Let F be a proper filter of $\pounds$ with $F \cap S = \emptyset$ and put Ω the set of filters containing F disjoint with S. Since $F \in \Omega$, $\Omega \neq \emptyset$. Moreover, $(\Omega, \subseteq)$ is a partial order. It is easy to see that Ω is closed under taking unions of chains and so Ω has at least one maximal element by Zorn's Lemma, say $\mathbf{p}$. Since $S \cap \mathbf{p} = \emptyset$ and $0 \in S$, we see that $0 \notin \mathbf{p}$ and $\mathbf{p} \neq \pounds$. It remains to show that $\mathbf{p}$ is a 2-absorbing filter by Example 3.1 (2). Now let $a \vee b, a \vee c, b \vee c \notin \mathbf{p}$; we must show that $a \vee b \vee c \notin \mathbf{p}$ for some elements $a, b, c \in \pounds$. Since $a \vee b \notin \mathbf{p}$, we have $F \subseteq \mathbf{p} \subsetneqq \mathbf{p} \wedge T(\{a \vee b\})$. By maximality of $\mathbf{p}$ in Ω, we must have $S \cap (\mathbf{p} \wedge T(\{a \vee b\})) \neq \emptyset$, and so there exist $s_1 \in S$, $t_1 \in \pounds$ and $p_1 \in \mathbf{p}$ such that $s_1 = p_1 \wedge (a \vee b \vee t_1)$. Similarly, there exist $s_2, s_3 \in S$, $t_2, t_3 \in \pounds$ and $p_2, p_3 \in \mathbf{p}$ such that $s_2 = p_2 \wedge (a \vee c \vee t_2)$ and $s_3 = p_3 \wedge (b \vee c \vee t_3)$. Put $u = a \vee b \vee t_1$, $v = a \vee c \vee t_2$ and $w = b \vee c \vee t_3$. Then $s_1 \vee s_2 \vee s_3 = (p_1 \wedge u) \vee (p_2 \wedge v) \vee (p_3 \wedge w) =$

$$((p_1 \vee p_2) \wedge (p_2 \vee u) \wedge (p_1 \vee v) \wedge (u \vee v)) \vee (p_3 \wedge w) =$$

$((p_1 \vee p_2) \wedge (p_2 \vee u) \wedge (p_1 \vee v) \wedge (a \vee b \vee c \vee t_1 \vee t_2)) \vee (p_3 \wedge w)$. Since $s_1 \vee s_2 \vee s_3 \in S$ and $(p_1 \vee p_2) \wedge (p_2 \vee u) \wedge (p_1 \vee v) \in \mathbf{p}$ (as $\mathbf{p}$ is a filter), we conclude that $a \vee b \vee c \notin \mathbf{p}$ since $S \cap \mathbf{p} = \emptyset$. Thus $\mathbf{p}$ is a 2-absorbing filter of $\pounds$. □

Definition 4.1. *Let S be a join closed subset of $\pounds$ and $\mathbf{p}$ a filter of $\pounds$ disjoint with S. Then $\mathbf{p}$ is said to be an S-maximal filter if there exists a fixed $s \in S$ and whenever $\mathbf{p} \subseteq \mathbf{q}$ for some filter $\mathbf{q}$ of $\pounds$, then either $s \vee \mathbf{q} \subseteq \mathbf{p}$ or $\mathbf{q} \cap S \neq \emptyset$.*

Classically, in the lattice $\pounds$ every maximal filter is a 2-absorbing filter, its S-version is the following result.

Theorem 4.5. *If S is a join closed subset of $\pounds$, then every S-maximal filter of $\pounds$ is an S-2-absorbing filter.*

Proof. Let $\mathbf{p}$ be an S-maximal filter. So if there exists a fixed $s \in S$, $\mathbf{p} \subseteq \mathbf{q}$ for some filter $\mathbf{q}$ of $\pounds$ implies that $s \vee \mathbf{q} \subseteq \mathbf{p}$ or $\mathbf{q} \cap S \neq \emptyset$. Now, we will show that $\mathbf{p}$ is an S-2-absorbing filter. Let $x \vee y \vee z \in \mathbf{p}$ for some $x, y, z \in \pounds$. It is enough to show that $s \vee x \vee y \in \mathbf{p}$ or $s \vee x \vee z \in \mathbf{p}$ or $s \vee y \vee z \in \mathbf{p}$. On the contrary, assume that $s \vee x \vee y \notin \mathbf{p}$, $s \vee x \vee z \notin \mathbf{p}$ and $s \vee y \vee z \notin \mathbf{p}$. This gives $\mathbf{p} \subsetneqq \mathbf{p} \wedge T(\{x \vee y\})$, $\mathbf{p} \subsetneqq \mathbf{p} \wedge T(\{x \vee z\})$ and $\mathbf{p} \subsetneqq \mathbf{p} \wedge T(\{y \vee z\})$. Since $\mathbf{p}$ is S-maximal, we conclude that $s \vee (\mathbf{p} \wedge T(\{x \vee y\})) \subseteq \mathbf{p}$ or $(\mathbf{p} \wedge T(\{x \vee y\})) \cap S \neq \emptyset$. If $s \vee (\mathbf{p} \wedge T(\{x \vee y\})) \subseteq \mathbf{p}$, then $s \vee x \vee y = s \vee (1 \wedge (0 \vee x \vee y)) \in \mathbf{p}$ which is impossible. So $(\mathbf{p} \wedge T(\{x \vee y\})) \cap S \neq \emptyset$. Likewise, $(\mathbf{p} \wedge T(\{x \vee z\})) \cap S \neq \emptyset$ and $(\mathbf{p} \wedge T(\{y \vee z\})) \cap S \neq \emptyset$. Then there exist $s_1, s_2, s_3 \in S$ such that $s_1 = p_1 \wedge (a \vee x \vee y)$, $s_2 = p_2 \wedge (b \vee x \vee z)$ and $s_3 = p_3 \wedge (c \vee y \vee z)$ for some $p_1, p_2, p_3 \in \mathbf{p}$ and $a, b, c \in \pounds$. Put $u = a \vee x \vee y$, $v = b \vee x \vee z$ and $w = c \vee y \vee z$ (so $u \vee v, u \vee w, v \vee w \in \mathbf{p}$). Then $s_1 \vee s_2 \vee s_3 = (p_1 \wedge u) \vee (p_2 \wedge v) \vee (p_3 \wedge w) = ((p_1 \vee p_2) \wedge (p_2 \vee u) \wedge (p_1 \vee v) \wedge (a \vee b \vee c \vee a \vee b)) \vee (p_3 \wedge w) \in \mathbf{p} \cap S$, as $\mathbf{p}$ is a filter, a contradiction. Thus $\mathbf{p}$ is an S-2-absorbing filter of $\pounds$. □

Definition 4.2. *Let S be a join closed subset of $\pounds$. We say that a filter F of $\pounds$ is S-finite if $s \vee F \subseteq G \subseteq F$ for some finitely generated filter G of $\pounds$ and some $s \in S$. We say that $\pounds$ is S-Noetherian if each filter of $\pounds$ is S-finite.*

Proposition 4.4. *If S is a join closed subset of $\pounds$ and F a filter of $\pounds$ which is maximal among all non-S-finite filters of $\pounds$, then F is a 2-absorbing filter of $\pounds$.*

Proof. Since $s \vee \pounds \subseteq T(\{s\}) \subseteq \pounds$ for every $s \in S \cap \pounds$, $\pounds$ is S-finite. If F is not 2-absorbing, then there exist $x, y, z \in \pounds$ such that $x \vee y, x \vee z, y \vee z \notin F$ but $x \vee y \vee z \in F$. Since $F \subsetneqq F \wedge T(\{x \vee y\})$, we conclude that $F \wedge T(\{x \vee y\})$ is S-finite by maximality of F; hence $s \vee (F \wedge T(\{x \vee y\}) \subseteq$

$$T(\{f_1 \wedge (x \vee y \vee a_1), \cdots, f_n \wedge (x \vee y \vee a_n)\})$$

for some $s \in S$, $f_1, \cdots, f_n \in F$ and $a_1, \cdots, a_n \in \pounds$. Since $F \subsetneqq (F :_{\pounds} x \vee y)$, we have $(F :_{\pounds} x \vee y)$ is S-finite, so $t \vee (F :_{\pounds} x \vee y) \subseteq T(\{b_1, \cdots, b_k\})$ for some $t \in S$ and $b_1, \cdots, b_n \in (F :_{\pounds} a)$. Now let $f \in F$ (so $f \vee s \in s \vee (F \wedge T(\{x \vee y\}))$. Then $s \vee f = (s \vee f) \vee \wedge_{i=1}^{n}(f_i \wedge (x \vee y \vee a_i)) =$

$$\wedge_{i=1}^{n}(s \vee f \vee f_i) \wedge (\wedge_{i=1}^{n}(s \vee f \vee x \vee y \vee a_i)),$$

so $u = \wedge_{i=1}^{n}(s \vee f \vee a_i) \in (F :_{\pounds} x \vee y)$ (so $t \vee u \in (F :_{\pounds} x \vee y)$) which gives

$$t \vee u = (t \vee u) \vee (\wedge_{i=1}^{k} b_i) = \wedge_{i=1}^{k}(t \vee u \vee b_i).$$

Therefore $s \vee f \vee t = \wedge_{i=1}^{n}(s \vee f \vee f_i \vee t) \wedge (\wedge_{i=1}^{n}(s \vee f \vee t \vee a_i \vee x \vee y)) =$

$\wedge_{i=1}^{n}(s \vee f \vee f_i \vee t) \wedge (x \vee y \vee t \vee u) = \wedge_{i=1}^{n}(s \vee f \vee f_i \vee t) \wedge (\wedge_{i=1}^{k}(x \vee y \vee b_i \vee t \vee u))$.

So $(s \vee t) \vee F \subseteq T(A) \subseteq F$, where $A = \{f_1 \vee t, \cdots, f_n \vee t, x \vee y \vee b_1, \cdots, x \vee y \vee b_k\} \subseteq F$; hence F is S-finite, a contradiction. Thus F is a 2-absorbing filter of $\pounds$. □

Compare the next theorem with Proposition 4 in [1].

Theorem 4.6. *If S is a join closed subset of $\pounds$, then $\pounds$ is S-Noetherian if and only if every 2-absorbing filter of $\pounds$ (disjoint from S) is S-finite.*

Proof. Suppose that for each 2-absorbing filter of $\pounds$ (disjoint from S) is S-finite. Assume that $\pounds$ is not S-Noetherian and look for a contradiction. Then the set Ω of all non-S-finite filters of $\pounds$ is inductively ordered under inclusion. By Zorn's Lemma, choose $\mathbf{p}$ maximal in Ω. Then Proposition 4.4 implies that $\mathbf{p}$ is a 2-absorbing filter. If $\mathbf{p} \cap S \neq \emptyset$, then $s \vee \mathbf{p} \subseteq T(\{s\}) \subseteq \mathbf{p}$ for every $s \in \mathbf{p} \cap S$ gives $\mathbf{p}$ is S-finite, a contradiction. Thus $\mathbf{p} \cap S = \emptyset$. Now, by the hypothesis, $\mathbf{p}$ is S-finite which is impossible since $\mathbf{p} \in \Omega$. Thus $\pounds$ is S-Noetherian. The other implication is clear. □

We obtain the following S-version of Cohen's Theorem [4].

Theorem 4.7. *Let S be a join closed subset of $\pounds$. The following assertions are equivalent:*

(1) $\pounds$ is S-Noetherian;

(2) Every S-2-absorbing filter of $\pounds$ is S-finite;

(3) Every 2-absorbing filter of $\pounds$ is S-finite.

Proof. The implication (1) ⇒ (2) is clear. To see the implication (2) ⇒ (3), let $\mathbf{q}$ be a 2-absorbing filter of $\pounds$. If $\mathbf{q} \cap S \neq \emptyset$, then $s \vee \mathbf{q} \subseteq T(\{s\}) \subseteq \mathbf{q}$ for every $s \in \mathbf{q} \cap S$ gives $\mathbf{q}$ is S-finite. If $\mathbf{q} \cap S = \emptyset$, then $\mathbf{q}$ is an S-2-absorbing filter of $\pounds$; so by (2), $\mathbf{q}$ is S-finite.

(3) ⇒ (1) Follows from Theorem 4.6. □

Proposition 4.5. *Let $\pounds = \pounds_1 \times \pounds_2$ be a decomposable lattice and $S = S_1 \times S_2$, where S_i is a join closed subset of $\pounds_i$. Suppose that $\mathbf{p} = \mathbf{p_1} \times \mathbf{p_2}$ is a filter of $\pounds$. The following statements are equivalent:*

(1) $\mathbf{p}$ is an S-2-absorbing filter of $\pounds$;

(2) $\mathbf{p_1}$ is an S_1-2-absorbing filter of $\pounds_1$ and $\mathbf{p_2} \cap S_2 \neq \emptyset$ or $\mathbf{p_2}$ is an S_2-2-absorbing filter of $\pounds_2$ and $\mathbf{p_1} \cap S_1 \neq \emptyset$ or $\mathbf{p_1}$ is an S_1-prime filter of $\pounds_1$ and $\mathbf{p_2}$ is an S_2-prime filter of $\pounds_2$.

Proof. (1) $\Rightarrow$ (2) Suppose that $\mathbf{p}$ is an S-2-absorbing filter of $\pounds$. Then we keep in mind that there exists a fixed $s = (s_1, s_2) \in S$ that satisfies the S-2-absorbing condition. Since $\mathbf{p} \cap S = \emptyset$, we have either $\mathbf{p_1} \cap S_1 = \emptyset$ or $\mathbf{p_2} \cap S_2 = \emptyset$. If $\mathbf{p_1} \cap S_1 \neq \emptyset$, we will show that $\mathbf{p_2}$ is an S_2-2-absorbing filter of $\pounds_2$. Let $x \vee y \vee z \in \mathbf{p_2}$ for some $x, y, z \in \pounds_2$. Then $(1, x) \vee_c (1, y) \vee_c (1, z) = (1, x \vee y \vee z) \in \mathbf{p}$ gives $s \vee_c (1, x) \vee_c (1, y) = (1, s_2 \vee x \vee y) \in \mathbf{p}$ or $s \vee_c (1, x) \vee_c (1, z) = (1, s_2 \vee x \vee z) \in \mathbf{p}$ or $s \vee_c (1, y) \vee_c (1, z) = (1, s_2 \vee y \vee z) \in \mathbf{p}$. This shows that $s_2 \vee x \vee y \in \mathbf{p_2}$ or $s_2 \vee x \vee z \in \mathbf{p_2}$ or $s_2 \vee y \vee z \in \mathbf{p_2}$. Hence, $\mathbf{p_2}$ is an S_2-2-absorbing filter of $\pounds_2$. Similarly, if $S_2 \cap \mathbf{p_2} \neq \emptyset$, then $\mathbf{p_1}$ is an S_1-2-absorbing filter of $\pounds_1$. Now assume that $S_1 \cap \mathbf{p_1} = \emptyset$ and $S_2 \cap \mathbf{p_2} = \emptyset$. We will show that $\mathbf{p_1}$ is an S_1-prime filter of $\pounds_1$ and $\mathbf{p_2}$ is an S_2-prime filter of $\pounds_2$. Suppose that $\mathbf{p_1}$ is not an S_1-prime filter of $\pounds_1$. Then there exist $a, b \in \pounds_1$ such that $a \vee b \in \mathbf{p_1}$ but $s_1 \vee a \notin \mathbf{p_1}$ and $s_1 \vee b \notin \mathbf{p_1}$. Since $S_2 \cap \mathbf{p_2} = \emptyset$, we conclude that $s_2 \notin \mathbf{p_2}$. Then $(a, 0) \vee_c (0, 1) \vee_c (b, s_2) = (a \vee b, 1) \in \mathbf{p}$ gives $s \vee_c (a, 0) \vee_c (0, 1) = (s_1 \vee a, 1) \in \mathbf{p}$ or $s \vee_c (a, 0) \vee_c (b, s_2) = (s_1 \vee a \vee b, s_2) \in \mathbf{p}$ or $s \vee_c (0, 1) \vee_c (b, s_2) = (s_1 \vee b, 1) \in \mathbf{p}$; so $s_1 \vee a \in \mathbf{p_1}$ or $s_2 \in \mathbf{p_2}$ or $s_1 \vee b \in \mathbf{p}$ which is a contradiction. Therefor, $\mathbf{p_1}$ is an S_1-prime filter of $\pounds_1$. Similarly, $\mathbf{p_2}$ is an S_2-prime filter of $\pounds_2$.

(2) $\Rightarrow$ (1) Let $\mathbf{p_1} \cap S_1 \neq \emptyset$ and $\mathbf{p_2}$ be an S_2-2-absorbing filter of $\pounds_2$. At first, note that $\mathbf{p} \cap S = \emptyset$. Let $(a, x) \vee_c (b, y) \vee_c (c, z) = (a \vee b \vee c, x \vee y \vee z) \in \mathbf{p}$ for some $(a, x), (b, y), (c, z) \in \pounds$. Since $\mathbf{p_1} \cap S_1 \neq \emptyset$, there exists $s_1 \in S_1$ such that $s_1 \vee u \in \mathbf{p_1}$ for all $u \in \pounds_1$. Also, there exists $s_2 \in S_2$ satisfying $\mathbf{p_2}$ to be an S_2-2-absorbing filter of $\pounds_2$. Now, put $s = (s_1, s_2) \in S$. Since $\mathbf{p_2}$ is an S_2-2-absorbing filter and $x \vee y \vee z \in \mathbf{p_2}$, we conclude that $s_2 \vee x \vee y \in \mathbf{p_2}$ or $s_2 \vee x \vee z \in \mathbf{p_2}$ or $s_2 \vee y \vee z \in \mathbf{p_2}$. This shows that $s \vee_c (a, x) \vee_c (b, y) \in \mathbf{p}$ or $s \vee_c (a, x) \vee_c (c, z) \in \mathbf{p}$ or $s \vee_c (c, z) \vee_c (b, y) \in \mathbf{p}$. Hence, $\mathbf{p}$ is an S-2-absorbing filter of $\pounds$. If $\mathbf{p_2} \cap S_2 \neq \emptyset$ and $\mathbf{p_1}$ is an S_1-2-absorbing filter of $\pounds_1$, similar argument shows that $\mathbf{p}$ is an S-2-absorbing filter. Now, suppose that for each $i = 1, 2$, $\mathbf{p_i}$ is an S_i-prime filter of $\pounds_i$. Let $(a, x) \vee_c (b, y) \vee_c (c, z) = (a \vee b \vee c, x \vee y \vee z) \in \mathbf{p}$ for some $(a, x), (b, y), (c, z) \in \pounds$. since $\mathbf{p_1}$ is an S_1-prime filter and $a \vee b \vee c \in \mathbf{p_1}$, there exists a fixed $s_1 \in S_1$ such that $s_1 \vee a \in \mathbf{p_1}$ or $s_1 \vee b \in \mathbf{p_1}$ or $s_1 \vee c \in \mathbf{p_1}$. Similarly, there exists $s_2 \in S_2$ such that $s_2 \vee x \in \mathbf{p_2}$ or $s_2 \vee y \in \mathbf{p_2}$ or $s_2 \vee z \in \mathbf{p_2}$. Put $s = (s_1, s_2) \in S$. Without loss of generality, we may assume that $s_1 \vee a \in \mathbf{p_1}$ and $s_2 \vee z \in \mathbf{p_2}$. Then $s \vee_c (a, x) \vee_c (c, z) \in \mathbf{p}$. Therefore, $\mathbf{p}$ is an S-2-absorbing filter of $\pounds$. □

Corollary 4.1. *Let $\pounds = \pounds_1 \times \pounds_2$ be a decomposable lattice and $S = S_1 \times S_2$, where S_i is a join closed subset of $\pounds_i$. Suppose that $\mathbf{p} = \mathbf{p_1} \times \mathbf{p_2}$ is a filter of $\pounds$. The following statements are equivalent:*

(1) $\mathbf{p}$ is an S-prime filter of $\pounds$;

(2) $\mathbf{p_1}$ is an S_1-prime filter of $\pounds_1$ and $\mathbf{p_2} \cap S_2 \neq \emptyset$ or $\mathbf{p_2}$ is an S_2-prime filter of $\pounds_2$ and $\mathbf{p_1} \cap S_1 \neq \emptyset$.

Proof. This is a direct consequence of Proposition 3.1 (1) and Proposition 4.5. □

Corollary 4.2. *Let $\pounds = \pounds_1 \times \cdots \times \pounds_n$ be a decomposable lattice and $S = S_1 \times \cdots \times S_n$, where S_i is a join closed subset of $\pounds_i$. Suppose that $\mathbf{p} = \mathbf{p_1} \times \cdots \times \mathbf{p_n}$ is a filter of $\pounds$. The following statements are equivalent:*

(1) $\mathbf{p}$ is an S-prime filter of $\pounds$;

(2) $\mathbf{p_i}$ is an S_i-prime filter of $\pounds_i$ for some $i \in \{1, \cdots, n\}$ and $\mathbf{p_j} \cap S_j \neq \emptyset$ for all $j \in \{1, \cdots, n\} \setminus \{i\}$.

Proof. We use induction on n. For $n = 1$, the result is true. If $n = 2$, then (1) and (2) are equivalent by Corollary 4.1. Assume that (1) and (2) are equivalent when $k < n$. Set $\mathbf{p'} = \mathbf{p_1} \times \cdots \times \mathbf{p_{n-1}}$, $S' = S_1 \times \cdots \times S_{n-1}$ and $\pounds' = \pounds_1 \times \cdots \times \pounds_{n-1}$. Then by Corollary 4.1, $\mathbf{p} = \mathbf{p'} \times \mathbf{p_n}$ is an S-prime filter of $\pounds$ if and only if $\mathbf{p'}$ is an S'-prime filter of $\pounds'$ and $\mathbf{p_n} \cap S_n \neq \emptyset$ or $\mathbf{p'} \cap S' \neq \emptyset$ and $\mathbf{p_n}$ is a S_n-prime filter of $\pounds_n$. Now the assertion follows from the induction hypothesis. □

Compare the next theorem with Theorem 3 in [13].

Theorem 4.8. *Let $\pounds = \pounds_1 \times \cdots \times \pounds_n$ be a decomposable lattice and $S = S_1 \times \cdots \times S_n$, where S_i is a join closed subset of $\pounds_i$. Suppose that $\mathbf{p} = \mathbf{p_1} \times \cdots \times \mathbf{p_n}$ is a filter of $\pounds$. The following statements are equivalent:*

(1) $\mathbf{p}$ is an S-2-absorbing filter of $\pounds$;

(2) $\mathbf{p_k}$ is an S_k-2-absorbing filter of $\pounds_k$ for some $k \in \{1, \cdots, n\}$ and $\mathbf{p_j} \cap S_j \neq \emptyset$ for all $j \in \{1, \cdots, n\} \setminus \{k\}$ or $\mathbf{p}_{k_1}$ is an S_{k_1}-prime filter of $\pounds_{k_1}$ and $\mathbf{p}_{k_2}$ is an S_{k_2}-prime filter of $\pounds_{k_1}$ for some $1 \leqslant k_1 \neq k_2 \leqslant n$ and $\mathbf{p_j} \cap S_j \neq \emptyset$ for each $j \in \{1, \cdots, n\} \setminus \{k_1, k_2\}$.

Proof. We use induction on n. For $n = 1$, the result is true. If $n = 2$, then (1) and (2) are equivalent by Proposition 4.5. Suppose that (1) and (2) are equivalent when $k < n$. Set $\mathbf{p'} = \mathbf{p_1} \times \cdots \times \mathbf{p_{n-1}}$, $S' = S_1 \times \cdots \times S_{n-1}$ and $\pounds' = \pounds_1 \times \cdots \times \pounds_{n-1}$. Then by Proposition 4.3, $\mathbf{p} = \mathbf{p'} \times \mathbf{p_n}$ is an S-2-absorbing filter of $\pounds$ if and only if $\mathbf{p'} \cap S' \neq \emptyset$ and $\mathbf{p_n}$ is a S_n-2-absorbing filter of $\pounds_n$ or $\mathbf{p'}$ is an S'-2-absorbing filter of $\pounds'$ and $\mathbf{p_n} \cap S_n \neq \emptyset$ or $\mathbf{p'}$ is an S'-prime filter of $\pounds'$ and $\mathbf{p_n}$ is an S_n-prime filter of $\pounds_n$. Now the assertion follows from the induction hypothesis and Corollary 4.2. □

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