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PELL NUMBERS THAT CAN BE WRITTEN AS THE SUM OF TWO MERSENNE NUMBERS

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Abstract. This study reports an investigation of the Pell (or Mersenne) numbers that can be written in terms of the summation of two random Mersenne (or Pell) numbers within the framework of linear forms in logarithms of algebraic numbers by using Matveev's theorem and Dujella-Pethö reduction lemma. More precisely, all the solutions to the Diophantine equations $P_k = M_m + M_n$ and $M_k = P_m + P_n$ are presented herein. Additionally, the maple codes used in the calculations made throughout the article are also shared.

1. Introduction

Let $\{P_n\}_{n\geqslant 0}$ and $\{M_n\}_{n\geqslant 0}$ be the Pell and Mersenne numbers given by the recursive formulas $P_n = 2P_{n-1} + P_{n-2}$ with $(P_0, P_1) = (0, 1)$ and $M_n = 3M_{n-1} - 2M_{n-2}$ with $(M_0, M_1) = (0, 1)$, respectively. Moreover, these can also be generated with the Binet-like formulas in the following:

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \text{ and } M_n = 2^n - 1, \tag{1.1}$$

for all non-negative integers, where $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$. Scientists have studied Pell and Mersenne numbers, because they have fascinating and surprising applications in many branches of science, especially mathematics and geometry. More precise samples of Pell and Mersenne numbers can be found by consulting the fundamental reference in [**15**].

In recent years, there has been an increase in the study of several Diophantine equations involving integer sequences such as Fibonacci, Lucas, Pell, Pell-Lucas,

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or Jacobsthal. In particular, the case where the products or sums of integer sequences are equal to another integer sequence has been investigated. Ddamulira et al. investigated Fibonacci or Pell numbers that can be written in terms of the product of two Pell or Fibonacci numbers in [**17**]. In [**20**] Alekseyev researched the intersection terms of Fibonacci, Pell, Lucas, and Pell-Lucas numbers. In [**14**], Bensella and Behloul considered the Leonardo numbers that coincide with the Jacobsthal numbers. In [**11**] Gaber examined the terms that make the sum of two Jacobsthal numbers a balancing number or a balancing-Lucas number and the intersections of these terms. In [**12**] the same author also studied coincidence terms of Pell, Pell -Lucas numbers, and sums of two Jacobsthal numbers. In [**9**] and [**7**], Erduvan and Keskin find all Fibonacci numbers that are products of two Jacobsthal numbers and Fibonacci numbers that are products of two balancing numbers, respectively. In [**18**], Marques and Togbé proved that the sum of powers greater than two of two consecutive Fibonacci numbers cannot be written as a Fibonacci number. It is possible to increase the number of example papers. It is possible to increase the number of example papers. To reduce the size of the paper, more literature surveys are neglected, but readers can investigate the documents listed in references [**3**], [**4**], [**5**], [**8**], [**10**], [**13**], [**16**], [**21**], [**22**], [**23**].

Mersenne numbers that can be expressed as the product of two random Pell numbers were studied by Alan and Alan in [**1**]. Also, Bravo et al. examined powers of two, which can be written as the sum of three Pell numbers in [**2**]. However, according to the current literature, the Mersenne or Pell numbers that can be expressed as the sum of two random Pell or Mersenne numbers have yet to be investigated. Motivated by this, we make an effort to address this issue. In this paper, we consider the Diophantine equations

$$P_k = M_m + M_n \tag{1.2}$$

and

$$M_k = P_m + P_n, \tag{1.3}$$

where $k \geqslant 1$ and $1 \leqslant m \leqslant n$.

In this paper, although Eq. (1.3) appears to be a specific case of the study in [**2**] (i.e., $l = 1$ for P_l), we investigate the common solutions of Eqs. (1.2) and (1.3). Eq. (1.2) has not been previously studied, making this study original. Therefore, the solution of Eq. (1.2) will be given in detail.

2. Preliminaries

This section of the paper provides fundamental definitions, results, and notations from algebraic number theory.

The following lemma is found in many articles and books on number theory.

LEMMA 2.1. *For all* $n \geqslant 1$,

$$\gamma^{n-2} \leqslant P_n \leqslant \gamma^{n-1}. \tag{2.1}$$

PROOF. The proof is clear due to the Binet's formula of P_n in Eq. (1.1). □

Further, the following lemma related to the Mersenne numbers has been given.

LEMMA 2.2. *For all* $n \geqslant 1$,

$$2^{n-1} \leqslant M_n < 2^n. \tag{2.2}$$

PROOF. The proof is clear due to the Binet-like formula of M_n in Eq. (1.1). □

The following lemma will be used in the proof process.

LEMMA 2.3 (Ddamulira et al. [**17**]). *For all* $z \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, $|z| < 2\left|e^z - 1\right|$ *is satisfied.*

Let ξ be an algebraic number of degree t and

$$b_0 x^t + b_1 x^{t-1} + \ldots + b_t = \sum_{i=0}^{t} b_i x^{t-i}$$

be its minimal polynomial in $\mathbb{Z}[x]$ where the b_i's are relatively prime integers and $b_0 > 0$. The logarithmic height of ξ is denoted by $h(\xi)$ and defined by

$$h(\xi) = \frac{1}{t}\left(\log b_0 + \sum_{i=1}^{t} \log\left(\max\left\{\left|\xi^{(i)}\right|, 1\right\}\right)\right), \tag{2.3}$$

where $\xi^{(i)}$'s are the conjugates of ξ.

There are also various features related to logarithmic height mentioned in the references. These features are as follows:

$$h(\xi_1 + \xi_2) \leqslant h(\xi_1) + h(\xi_2) + \log 2, \tag{2.4}$$

$$h\left(\xi_1 \xi_2^{\pm 1}\right) \leqslant h(\xi_1) + h(\xi_2), \tag{2.5}$$

$$h(\xi^r) = |r|\, h(\xi). \tag{2.6}$$

Let $\xi_1, \xi_2, \ldots, \xi_r$ be nonzero real algebraic numbers in a number field $\mathbb{L}$ of degree D, and let $s_1, s_2, \ldots, s_r$ be nonzero rational numbers. Also

$$\Lambda = {\xi_1}^{s_1} {\xi_2}^{s_2} \ldots {\xi_r}^{s_r} - 1 \text{ and } B \geqslant \max\{|s_1|, |s_2|, \ldots, |s_r|\}.$$

Let $A_1, A_2, \ldots, A_r$ be the positive real numbers such that

$$A_j \geqslant \max\{Dh(\xi_j), |\log \xi_j|, 0.16\} \text{ for all } j = 1, 2, \ldots, r. \tag{2.7}$$

Based on the notations mentioned above, an important theorem established by Matveev [**19**], will be presented as follows:

THEOREM 2.1 (Matveev's theorem [**19**]). *If* $\Lambda \neq 0$ *and* $\mathbb{L}$ *is real algebraic number field of degree* D*, then,*

$$\log(|\Lambda|) > -1.4 \times 30^{r+3} \times r^{4.5} \times D^2 \times (1 + \log D) \times (1 + \log B) \times A_1 \times A_2 \times ... \times A_r.$$

To reduce the bounds from applying Theorem 2.1, the following lemma was developed by Dujella and Pethö [**6**].

Lemma 2.4 (Dujella and Pethö [**6**]). *Let M be a positive integer, p/q be a convergent of the continued fraction of the irrational number τ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\varepsilon =: \|\mu q\| - M\|\tau q\|$, where $\|\cdot\|$ is the distance from the nearest integer. If $\varepsilon > 0$, then there is no integer solution (k, n, ψ) of inequality*

$$0 < k\tau - n + \mu < AB^{-\psi}$$

with

$$k \leqslant M \text{ and } \psi \geqslant \frac{\log(Aq/\varepsilon)}{\log B}.$$

3. Main results

The fundamental result of the paper is given below.

Theorem 3.1. *Let k, m, and n be a positive integer. Then,*

- *Equation (1.2) is satisfied only for the triples of*

$$(k, m, n) \in \{(2, 1, 1), (6, 3, 6)\} \tag{3.1}$$

- *Eq. (1.3) holds only for the triples of*

$$(k, m, n) \in \{(2, 1, 2), (3, 2, 3), (5, 2, 5)\}. \tag{3.2}$$

Proof. Here, we will focus only on Eq. (1.2) but are making a similar process for Eq. (1.3) in the background of the paper. Assume that $n > m$. Then, considering Lemmas 2.1 and 2.2, we can write

$$\gamma^{k-2} \leqslant P_k = M_n + M_m < 2^n + 2^m < 2^{n+m} \tag{3.3}$$

and

$$2^{k-1} \leqslant M_k = P_n + P_m \leqslant \gamma^{n-1} + \gamma^{m-1} < \gamma^{n+m}. \tag{3.4}$$

From Eq. (3.3), we conclude that

$$(k-2)\log\gamma \leqslant (n+m)\log 2 \Rightarrow k \leqslant 2 + (n+m)\frac{\log 2}{\log\gamma} < 2n+2$$

which satisfies $k < 2n + 2$. Applying the Binet's formulas in Eq. (1.1) to Eq. (1.2) yields

$$P_k = M_n + M_m \Rightarrow \frac{\gamma^k - \delta^k}{2\sqrt{2}} = (2^n - 1) + (2^m - 1) \tag{3.5}$$

and from this, we get

$$\gamma^k - \sqrt{2}\cdot 2^{n+1} = \sqrt{2}\cdot 2^{m+1} - 4\sqrt{2} + \delta^k.$$

Dividing both sides of the last equation by $2\sqrt{2}\cdot 2^n$ and taking absolute values we get

$$\left|\frac{\gamma^k 2^{-n}}{2\sqrt{2}} - 1\right| = \left|\frac{1}{2^{n-m}} - \frac{2}{2^n} + \frac{\delta^k}{\sqrt{2}\cdot 2^{n+1}}\right| < \frac{1}{2^{n-m}} + \frac{2}{2^n} + \frac{|\delta|^k}{\sqrt{2}\cdot 2^{n+1}} < \frac{4}{2^{n-m}}.$$

As a result, we have

$$|\Lambda_1| < \frac{4}{2^{n-m}}, \quad \Lambda_1 := \gamma^k \cdot 2^{-n} \cdot \left(2\sqrt{2}\right)^{-1} - 1. \tag{3.6}$$

According to Theorem 2.1, we get $r = 3$, $\xi_1 = \gamma$, $\xi_2 = 2$, $\xi_3 = 2\sqrt{2}$, $s_1 = k$, $s_2 = -n$, and $s_3 = -1$. Because of $\xi_1, \xi_2, \xi_3 \in \mathbb{Q}\left(\sqrt{2}\right)$, we should consider $\mathbb{L} = \mathbb{Q}\left(\sqrt{2}\right)$ of degree $D = 2$. It is clear that $\Lambda_1 \neq 0$. If $\Lambda_1 = 0$, then we get $\frac{\gamma^k}{\sqrt{2}} = 2^{n+1}$. If computing the second power of both sides of the equation $\frac{\gamma^k}{\sqrt{2}} = 2^{n+1}$, we obtain an integer in the left-hand side, which is impossible. So, $\Lambda_1 \neq 0$. From Eqs. (2.3) and (2.7), we can compute

$$h(\xi_1) = \frac{1}{2}\log\gamma,\ h(\xi_2) = \log 2,\ h(\xi_3) = \frac{3}{2}\log 2,$$

$$A_1 = \log\gamma,\ A_2 = 2\log 2, \text{ and } A_3 = 3\log 2.$$

Besides, for $B = 2n+2$, $B \geqslant \max\{k, |-n|, |-1|\}$, since $k < 2n+2$. As a result, based on Theorem 2.1, with certain mathematical simplifications, we obtain

$$\log(\Lambda_1) > -2.5 \times 10^{12}\left(1 + \log(2n+2)\right) \tag{3.7}$$

and from inequality (3.6),

$$\log(\Lambda_1) < \log 4 - (n-m)\log 2. \tag{3.8}$$

From inequalities (3.7) and (3.8), we get that

$$-m\log 2 < 2.5 \times 10^{12}\left(1 + \log(2n+2)\right) + \log 4 - n\log 2. \tag{3.9}$$

By the way, if rearranging the Eq. (1.2) as

$$\begin{aligned} P_k = M_m + M_n &\Rightarrow \frac{\gamma^k - \delta^k}{2\sqrt{2}} = (2^n - 1) + (2^m - 1) \\ &\Rightarrow \frac{\gamma^k}{2\sqrt{2}} - 2^n\left(1 + 2^{m-n}\right) = -2 + \frac{\delta^k}{2\sqrt{2}}. \end{aligned}$$

Taking absolute values after dividing both sides of the last equation by $2^n\left(1 + 2^{m-n}\right)$, we get

$$\left|\frac{\gamma^k}{2\sqrt{2}\cdot 2^n\left(1 + 2^{m-n}\right)} - 1\right| < \frac{4}{2^m}$$

and

$$|\Lambda_2| < \frac{4}{2^m}, \quad \Lambda_2 := \gamma^k 2^{-n}\frac{1}{2\sqrt{2}\left(1 + 2^{m-n}\right)} - 1. \tag{3.10}$$

To apply Matveev's theorem into Eq. (3.10), we can consider that case where $r = 3$, $\xi_1 = \gamma$, $\xi_2 = 2$, $\xi_3 = 2\sqrt{2}\left(1 + 2^{m-n}\right)$, $s_1 = k$, $s_2 = -n$, and $s_3 = -1$. Since $\xi_1, \xi_2, \xi_3 \in \mathbb{Q}\left(\sqrt{2}\right)$, we can take $\mathbb{L} = \mathbb{Q}\left(\sqrt{2}\right)$ of degree $D = 2$. As can be seen, since $\frac{\gamma^k}{\sqrt{2}} = 2(2^n + 2^m)$ is never satisfied, $\Lambda_2 \neq 0$. Besides, if we take $B = 2n+2$, then $B \geqslant \max\{k, |-n|, |-1|\}$, since $k < 2n+2$. In this case, we can compute the followings:

$$h(\xi_1) = \frac{1}{2}\log\gamma,\ h(\xi_2) = \log 2, A_1 = \log\gamma,\ \text{and}\ A_2 = 2\log 2.$$

From (2.4), (2.5), (2.6), and (2.7) we get

$$h(\xi_3) \leqslant \frac{5}{2}\ \log 2 + (n-m)\log 2$$

and

$$A_3 =\ 5\ \log 2 + 2(n-m)\log 2 = D\cdot(\frac{5}{2}\ \log 2 + (n-m)\log 2) \geqslant D\cdot h(\xi_3).$$

In this case, according to Matveev's theorem, we can write

$$\log(\Lambda_2) > -1.2\times 10^{12}\times(1+\log(2n+2))\times(5\ \log 2 + 2(n-m)\log 2)\,. \tag{3.11}$$

From the right-hand side of the inequality (3.10) we get

$$\log(\Lambda_2) < \log 4 - m\log 2. \tag{3.12}$$

Considering the inequalities (3.9), (3.11), and (3.12), we deduce that

$$n < 4\times 10^{28}. \tag{3.13}$$

By the same token, after making the same mathematical consideration for Eq. (1.3), we can attain the following definition and results:

$$|\Lambda_3| < \frac{8}{\gamma^{n-m}},\quad \Lambda_3 := 2^k\cdot\gamma^{-n}\cdot 2\sqrt{2} - 1,\ k < 3n+1,$$

$$|\Lambda_4| < \frac{7}{\gamma^m},\ \Lambda_4 := 2^k\gamma^{-n}\frac{2\sqrt{2}}{1+\gamma^{m-n}} - 1,\ n < 4\times 10^{28}.$$

Thus, we can summarize the results mentioned above with a lemma as follows:

Lemma 3.1. *All the possible solutions of Eqs.* (1.2) *and* (1.3) *are over the ranges* $k < 3n+1$, $1 \leqslant m \leqslant n$, *and* $n < 4\times 10^{28}$.

As can be seen, we have determined a finite number of solutions to our problems, even though it has pretty extensive borders. To limit these wide bounds, we will utilize the Dujella-Pethö reduction lemma.

We first consider the notation

$$\Gamma_1 := k\log\gamma - n\log 2 - \log 2\sqrt{2}.$$

Then, we get

$$|\Lambda_1| = |\exp(\Gamma_1) - 1| < \frac{4}{2^{n-m}}.$$

Also, from Lemma 2.3, we can write

$$|\Gamma_1| = \left|k\log\gamma - n\log 2 - \log 2\sqrt{2}\right| < \frac{8}{2^{n-m}}$$

for $n - m > 3$. So we get

$$0 < \left|k\frac{\log\gamma}{\log 2} - n + \frac{\log\left(1/2\sqrt{2}\right)}{\log 2}\right| \leqslant \frac{8}{2^{n-m}\log 2} < \frac{12}{2^{n-m}}.$$

Then, accordingly Lemma 2.4, for $M = 1.3 \times 10^{29}$ $(M > 3n+1 > k)$ and $\tau = \frac{\log\gamma}{\log 2}$, $60th$ convergent of the continued fraction expansion of τ is

$$\frac{p_{60}}{q_{60}} = \frac{143901431815323899257844117213518}{113169799061744850180040631725663}$$

and so $6M < q_{60} = 113169799061744850180040631725663$. Therefore we have

$$\varepsilon = \|\mu q_{60}\| - M\|\tau q_{60}\|, \ \varepsilon > 0.49, \ \mu = \frac{\log\left(1/2\sqrt{2}\right)}{\log 2}.$$

We calculated the above process of finding ϵ with the help of Maple© and we present the Maple Codes 1 below.

Maple Codes 1.

```
>DNI:= proc( x ) # DNI: denote the distance from x to the nearest integer
  if abs(frac(x))=0.5 then evalf[1](abs(frac(x))) else
  if abs(frac(x))<0.5 then evalf[30](abs(frac(x))) else
  if abs(frac(x))>0.5 then evalf[30](1- abs(frac(x))) else 0
  end if; end if; end if; end proc:
>M := 1.3*10ˆ29;
  MU := ln(1/(2*sqrt(2)))/ln(2);
  TAU := ln(1+sqrt(2))/ln(2);
  q := 113169799061744850180040631725663
>Epsilon(M, MU, TAU, q)
```

So, taking $A := 12$, $B := 2$, and $\psi := n - m$ into account, since from Lemma 2.4 the inequality

$$n - m > 111 > \frac{\log\left(Aq_{60}/\varepsilon\right)}{\log B}$$

has no solution, we deduce that $n - m \leqslant 111$.

Now we consider the notation

$$\Gamma_2 := k\log\gamma - n\log 2 + \log\left(\frac{1}{2\sqrt{2}\left(1+2^{m-n}\right)}\right)$$

and

$$|\Lambda_2| = |\exp\left(\Gamma_2\right) - 1| < \frac{4}{2^m}.$$

Accordingly, in the above result of Lemma 2.3, we get

$$0 < \left|k\frac{\log\gamma}{\log 2} - n + \frac{\log\left(1/2\sqrt{2}\left(1+2^{m-n}\right)\right)}{\log 2}\right| < \frac{8}{2^m\log 2} < \frac{12}{2^m},$$

for $m > 3$. Based on the Lemma 2.4 for $M = 1.3 \times 10^{29}$ $(M > 3n+1 > k)$ and $\tau = \frac{\log\gamma}{\log 2}$, $72th$ convergent of the continued fraction expansion of τ is

$$\frac{p_{72}}{q_{72}} = \frac{1066517035645177499285949748538976838 7}{838751339005367467598651129112109249 8}$$

and $6M < q_{72} = 83875133900536746759865112911210924 98$. As a result, with the help of Maple, as shown above, for $m > 3$, we have

$$\varepsilon = \|\mu q_{72}\| - M\,\|\tau q_{72}\|\,,\ \varepsilon > 0.009,\ \mu = \frac{\log\left(1/2\sqrt{2}\,(1+2^{m-n})\right)}{\log 2}.$$

Finally, taking $A := 12$, $B := 2$, and $\psi := m$ into account, we obtain that $m \leqslant 133$. so, $n \leqslant 244$ and $k < 733$.

Applying the same methodologies to Λ_3 and Λ_4, we get that $m \leqslant 95$, $n \leqslant 177$ and $k < 532$. Structuring an iterative algorithm in Maple© for Eqs. (1.2) and (1.3) over the range $m \leqslant 133$ and $n \leqslant 244$ shall prove the validity of Theorem 3.1.

We calculated the process of finding all solutions of Eqs. (1.2) and (1.3) with the help of Maple© and we present the Maple Codes 2 below only Equation (1.2).

Maple Codes 2.

```
>M := proc (a) 2^a-1 end proc
>P := proc (n) option remember;
 if n <= 1 then n else 2*procname(n-1)+procname(n-2)
 end if end proc
>for a to 10^3 do
 for b to 10^3 do
 for c to 10^3 do
 if P(a) = M(b)+M(c) then print(P[a] = P(a), M[b] = M(b), M[c] = M(c))
 else end if end do end do end do
```

□

It is possible to reach the following conclusion from Theorem 3.1 .

Corollary 3.1. *No Pell number can be twice the Mersenne number, and no Mersenne number can be twice the Pell number.*

Proof. We know that if $P_k = 2^m - 1$, then there are no positive integer solutions for $k > 1$, see in [**2**]. For this and the case where $m = n$ in Eqs. (1.2) and (1.3), the result follows. □

References

[1] M. Alan and K. S. Alan, *Mersenne numbers which are products of two Pell numbers*, Boletín de la Sociedad Matemática Mexicana, **28**(2) (2022), 38.

[2] J. J. Bravo, B. Faye, and F. Luca, *Powers of Two as Sums of Three Pell Numbers*, Taiwanese Journal of Mathematics, **21**(4) (2017), 739-751.

[3] T. P. Chalebgwa and M. Ddamulira, *Padovan Numbers which are Palindromic Concatenations of Two Distinct Repdigits*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, **115** (2021), 108.

[4] A.P. Chaves and D. Marques, *A Diophantine Equation Related to the Sum of Squares of Consecutive k-Generalized Fibonacci Numbers*, Fibonacci Quarterly, **52**(1) (2014), 70-74.

[5] A.P. Chaves and D. Marques, *A Diophantine Equation Related to the Sum of Powers of Two Consecutive Generalized Fibonacci Numbers*, Journal of Number Theory, **156** (2015), 1-14.

[6] A. Dujella and A. Pethö, *A Generalization of a Theorem of Baker and Davenport*, The Quarterly Journal of Mathematics, **49**(195) (1998), 291-306.

[7] F. Erduvan and R. Keskin, *Fibonacci Numbers which are Products of Two Balancing Numbers*, Annales Mathematicae et Informaticae, **50** (2019), 57-70.
[8] F. Erduvan and R. Keskin, *Repdigits as Products of Two Fibonacci or Lucas Numbers*, Proceedings-Mathematical Sciences, **130**(28) (2020), 1-14.
[9] F. Erduvan and R. Keskin, *Fibonacci Numbers which are Products of Two Jacobsthal Numbers*, Tbilisi Mathematical Journal, **14**(2) (2021), 105-116.
[10] B. Faye, F. Luca, R. S. Eddine, and A. Togbé, *The Diophantine equations* $P_n^x + P_{n+1}^y = P_m^x$ *or* $P_n^y + P_{n+1}^x = P_m^x$, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A, Matemáticas, **117**(2) (2023).
[11] A. Gaber, *Intersections of Balancing, Lucas-Balancing and Sums of Two Jacobsthal Numbers*, hal-03823108, Hal Open Science, (2022).
[12] A. Gaber, *Intersections of Pell, Pell-Lucas Numbers and Sums of Two Jacobsthal Numbers*, Punjab University Journal of Mathematics, **55**(5-6) (2023), 241-252.
[13] C. A. Gómez, J. C. Gómez, and F. Luca, *On the Exponential Diophantine Equation* $F_{n+1}^x - F_{n-1}^x = F_m^y$, Taiwanese Journal of Mathematics, **26**(4) (2022), 685-712.
[14] H. Bensella and D. Behloul, *Common Terms of Leonardo and Jacobsthal Numbers*, Rendiconti del Circolo Matematico di Palermo Series 2, **73** (2024), 259-265.
[15] T. Koshy, *Pell and Pell-Lucas numbers with applications*, Springer, New York, 2014.
[16] F. Luca and A. Togbé, *On the x-Coordinates of Pell Equations which are Fibonacci Numbers*, Mathematica Scandinavica, **122**(1) (2018), 18-30.
[17] M. Ddamulira, F. Luca, and M. Rakotomalala, *Fibonacci Numbers which are Products of Two Pell Numbers*, Fibonacci Quarterly, **54**(1) (2016), 11-18.
[18] D. Marques and A. Togbé, *On the Sum of Powers of Two Consecutive Fibonacci Numbers*, Proc. Japan Acad. Ser. A Math. Sci., **86**(10) (2010), 174-176.
[19] E. M. Matveev, *An Explicit Lower Bound for a Homogeneous Rational Linear form in the Logarithms of Algebraic Numbers. II*, Izvestiya: Mathematics, **64**(6) (2000), 1217.
[20] M. A. Alekseyev, *On the Intersections of Fibonacci, Pell, and Lucas Numbers*, Integers, **11** (2011), 239-259.
[21] P. Pongsriiam, *Fibonacci and Lucas Numbers Associated with Brocard-Ramanujan Equation*, Commun. Korean Math. Soc., **32**(3) (2017), 511-522.
[22] M. K. Sahukar and G. K. Panda, *Diophantine Equations with Balancing-Like Sequences Associated to Brocard-Ramanujan-Type Problem*, Glasnik matematički, **54**(2) (2019), 255–270.
[23] L. Szalay, *Diophantine Equations with Binary Recurrences Associated to the Brocard–Ramanujan Problem*, Portugaliae Mathematica, **69**(3) (2012), 213-220.

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