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METATHEOREM AND A SEMIGROUP THAT IS A SEMILATTICE OF GROUPS

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Abstract. Characterizations of a semigroup that is a semilattice of left groups, semilattice of right groups and semilattice of groups in terms of fuzzy left ideals, fuzzy right ideals, fuzzy ideals, fuzzy bi-ideal ideals, fuzzy interior ideals, and fuzzy quasi-ideals are provided by using Tom Head's metatheorem. The classes of different types of fuzzy ideals are shown to be closed under projection. Tom Head's metatheorem is also used to provide alternative proofs of several results pertaining to these fuzzy ideals which doesn't involve calculations. Interplay of regularity and semilattice is established by using these fuzzy ideals.

1. Introduction

In the exploration of semigroups, the theory of ideals, akin to other algebraic structures, plays a crucial role. Steinfeld introduced the concept of quasi-ideals in rings and semigroups in his articles [**16**] and [**17**]. Iseki [**4**] further developed this notion for semirings without a zero element, studying significant characterizations of semirings using quasi-ideals.

The notion of bi-ideals, generalization of quasi-ideals, was introduced by Lajos and Szasz [**10**] in associative rings, and later by Good and Hughes [**2**] in semigroups. Steinfeld [**16**] showed that quasi-ideals are a specific case of bi-ideals. Lajos [**11**] extended the concept of bi-ideals to the family of semigroups. The concept of interior ideals in semigroups was introduced by Szasz [**18**] and Kuroki [**6**–**8**] introduced the notions of fuzzy bi-ideal and fuzzy interior ideal in semigroups. In 1995, Tom

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Head [**3**] proposed his metatheorem, offering a framework for studying the intrinsic behavior of fuzzy algebraic structures.

The primary objective of this metatheorem was to derive fuzzy versions of classical outcomes. In essence, metatheorem establishes a straightforward conceptual framework for transferring results from crisp settings to fuzzy contexts, whenever such a transition is feasible. Subsequently, this metatheorem approach was adopted by [**1**, **5**, **14**] in the context of fuzzy algebra. The evident clarity and conciseness of our proofs, in contrast to the pre-existing proofs for certain outcomes, definitively validate the metatheorem approach as a potent tool in investigating fuzzy algebraic structures. This investigation firmly establishes the metatheorem approach as an unquestionably effective instrument for a comprehensive exploration of fuzzy algebraic structures.

This research paper aims to characterize semigroups that serves as a semilattice of left groups, semilattice of right groups, and semilattice of groups, using the notions of fuzzy left ideals, fuzzy right ideals, fuzzy ideals, fuzzy bi-ideals, fuzzy interior ideals, and fuzzy quasi-ideals, utilizing Tom Head's metatheorem. Notably, the proofs presented in this paper exhibit lucidity and brevity when compared to existing proofs, underscoring the metatheorem approach as an effective tool for studying fuzzy algebraic structures. Lastly, the interplay between regularity and semilattice is explored using different type of fuzzy ideals in semigroups.

2. Preliminaries

We recall a *fuzzy set* f on a non-set X defined by Zadeh [**19**] as a mapping $f : X \to [0, 1]$.

Definition 2.1 ([**6**]). *Let S be a semigroup. A fuzzy set f of S is called a* fuzzy subsemigroup *of S if $f(xy) \geqslant \min\{f(x), f(y)\}$ $\forall\, x, y \in S$.*

Definition 2.2 ([**15**]). *A fuzzy set f of S a* fuzzy left *(*right*) ideal if $f(xy) \geqslant f(y)$ $(f(xy) \geqslant f(x))$ $\forall\, x, y \in S$.*

A fuzzy set f of S is called a *fuzzy ideal* if f is both a fuzzy left and a fuzzy right ideal of S.

Definition 2.3 ([**6**]). *A fuzzy subsemigroup f of a semigroup S is called a* fuzzy bi-ideal *of S if $f(xyz) \geqslant \min\{f(x), f(z)\}$ $\forall\, x, y, z \in S$.*

Definition 2.4 ([**7**]). *A fuzzy subsemigroup f of a semigroup S is called a* fuzzy interior ideal *of S if $f(xyz) \geqslant f(y)$ $\forall\, x, y, z \in S$.*

Definition 2.5 ([**14**]). *A fuzzy subsemigroup f of a semigroup S is called a* fuzzy quasi-ideal *of S if $f(z) \geqslant \min\{\sup_{z=xy} f(x), \sup_{z=xy} f(y)\}$ $\forall\, z \in S$.*

Let S be a semigroup. Let $P(S)$, $C(S)$ and $F(S)$ denote respectively be the set of all subsets of S, the set of all characteristic functions of S and the set of all fuzzy sets of S. It is well known that the mapping $\mathrm{Chi} : P(S) \to C(S)$ defined by $\mathrm{Chi}(A) = \chi_A$ is a bijection. $\mathrm{Chi} : P(S) \to C(S)$ provides an isomorphism of complete lattices.

Definition 2.6 ([**3**]). *Let S be a semigroup and $J = [0, 1)$. The representation function* Rep : $F(S) \to C(S)^J$ *is defined by*

$$\operatorname{Rep}(f)(r)(x) = \begin{cases} 1, & \text{if } f(x) > r, \\ 0, & \text{if } f(x) \leqslant r, \end{cases}$$

where $f \in F(X)$ and $r \in J$.

Proposition 2.1 ([**3**]). Rep *is an injective function and commutes with infs of finite sets of fuzzy sets and with sups of arbitrary sets of fuzzy sets.*

Proposition 2.2 ([**3**]). *For each $f \in F(S)$,* Rep(f) *is an order reversing function from J onto $C(S)$ and* Rep *is an order isomorphism of $F(S)$ onto $I(S)$, where $I(S)$ be the image of the Rep function.*

We define a binary operation $* : F(S) \times F(S) \to F(S)$ as follows:

$$(f_1 * f_2)(x) = \begin{cases} \sup\limits_{x = x_1 * x_2} [\min\{f_1(x_1), f_2(x_2)\}] & \\ 0, & \text{if } x \text{ not expressed as } x = x_1 * x_2 \end{cases}$$

The above binary operation $*$ on $F(S)$ is called by Tom Head [**3**], the convolutional extension of the binary operation $*$ on S. Interestingly the convolutional extension on $F(S)$ coincides with the product of fuzzy sets introduced by Liu [**13**] in a semigroup or in any other algebraic system.

Proposition 2.3 ([**3**]). *For the binary operation $*$ in a semigroup S, $C(S)$ is closed with respect to the convolutional extension of $*$ to $F(S)$. Moreover, the bijection* Chi : $P(S) \to C(S)$ *commutes with the $*$ operation on $P(S)$ and $C(S)$ i.e., for $A, B \in P(S)$,* $\operatorname{Chi}(A * B) = \operatorname{Chi}A * \operatorname{Chi}B$.

Proposition 2.4 ([**3**]). *For the binary operation $*$ on a semigroup S, the representation function* Rep : $F(S) \to C(S)^J$ *commutes with the convolutional extension of $*$ i.e.,* $\operatorname{Rep}(f_1 * f_2) = \operatorname{Rep}(f_1) * \operatorname{Rep}(f_2)$ *for $f_1, f_2 \in F(S)$.*

Definition 2.7 ([**3**]). *Let $\mathcal{C}$ be a class of fuzzy sets in a semigroup S. We say that $\mathcal{C}$ is closed under projection if for each $f \in \mathcal{C}$ and $r \in J$,* $\operatorname{Rep}(f)(r) \in \mathcal{C}$.

Proposition 2.5 ([**3**]). *Let C, D be the classes of crisp subsets of a semigroup S and $\mathcal{C}$, $\mathcal{D}$ be their corresponding fuzzy classes which are closed under projection. Then,*

(1) *$\mathcal{C} \subseteq \mathcal{D}$ if and only if $C \subseteq D$.*
(2) *$\mathcal{C} = \mathcal{D}$ if and only if $C = D$.*

Metatheorem 2.1 ([**3**]). Let S be a semigroup with binary operation $*$ on S. Let $F(S)$ be provided with the operations inf, sup and $*$. Let $D(v_1, v_2, \ldots, v_m)$ and $E(v_1, v_2, \ldots, v_m)$ betwo expressions on $P(S)$ over the variables set $\{v_1, v_2, \ldots, v_m\}$ and operations set $\{\inf, \sup, *\}$. Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m$ be the classes of fuzzy sets of S that are closed under projection. Then the inequality

$$D(f_1, f_2, \ldots, f_m) \quad \text{REL} \quad E(f_1, f_2, \ldots, f_m)$$

holds for all fuzzy sets f_1 in $\mathcal{C}_1, \ldots, f_m$ in $\mathcal{C}_m$ if and only if it holds for all crisp sets f_1 in $\mathcal{C}_1, \ldots, f_m$ in $\mathcal{C}_m$ where REL is anyone of the three relations $\leqslant$, $=$ or $\geqslant$.

3. Metatheorem and semigroup that is a semilattice of groups

Metatheorem serves as a crucial tool that allows theorems containing references to crisp substructures to be reformulated to refer to fuzzy substructures. The main tool used for establishing the metatheorem is the concept of representation function $\mathrm{Rep} : F(X) \to C(X)^J$, J being the interval $[0, 1[$.

THEOREM 3.1 ([**14**]). *The classes $\mathcal{C}_l$, $\mathcal{C}_r$, $\mathcal{C}_i$, $\mathcal{C}_b$, $\mathcal{C}_q$ and $\mathcal{C}_s$ of all fuzzy left ideals, fuzzy right ideals, fuzzy ideals, fuzzy bi-ideals and fuzzy quasi-ideals of a semigroups S are closed under projection.*

THEOREM 3.2. *The class $\mathcal{C}_{in}$ of all fuzzy interior ideals is closed under projection.*

PROOF. Let $f \in \mathcal{C}_{in}$. Therefore f is a fuzzy subsemigroup of S and $f(xyz) \geqslant f(y)$ for all $x, y, z \in S$. Firstly we show that the class, $\mathcal{C}_{ss}$ of all fuzzy subsemigroups of a semigroup S is closed under projection. Let $f \in \mathcal{C}_{ss}$. Therefore, $f(xy) \geqslant \min\{f(x), f(y)\}$ $\forall$ $x, y \in S$. We claim that $\mathrm{Rep}(f)(r)(xy) \geqslant \min\{\mathrm{Rep}(f)(r)(x), \mathrm{Rep}(f)(r)(y)\}$ $\forall$ $x, y \in S$ and $\forall$ $r \in J$. Let $x, y \in S$ and $r \in J$. Suppose that $\min\{\mathrm{Rep}(f)(r)(x), \mathrm{Rep}(f)(r)(y)\} = 1$. Then, $\mathrm{Rep}(f)(r)(x) = 1 = \mathrm{Rep}(f)(r)(y)$. This implies that $f(x) > r < f(y)$. Therefore, $f(xy) \geqslant \min\{f(x), f(y)\} > r$. Thus, $\mathrm{Rep}(f)(r)(xy) = 1$. If $\min\{\mathrm{Rep}(f)(r)(x), \mathrm{Rep}(f)(r)(y)\} = 0$, then the inequality holds trivially. Therefore, the class of all fuzzy subsemigroups of S is closed under projection.

Next to show that $\mathrm{Rep}(f)(r)(xyz) \geqslant \mathrm{Rep}(f)(r)(y)$ $\forall$ $x, y, z \in S$ and $\forall$ $r \in J$. Let $x, y, z \in S$ and $r \in J$. Suppose that $\mathrm{Rep}(f)(r)(y) = 1$. This implies that $f(y) > r$. Therefore, $f(xyz) \geqslant f(y) > r$. Thus, $\mathrm{Rep}(f)(r)(xyz) = 1$. If $\mathrm{Rep}(f)(r)(y) = 0$, then the inequality holds trivially. Therefore, the class of all fuzzy subsemigroups of S is closed under projection. □

THEOREM 3.3. *Every fuzzy interior ideal of a semigroup S that is a semilatice of left groups is a fuzzy left ideal of S.*

PROOF. By Theorem 3.2 and 3.1, both the classes $\mathcal{C}_{in}$ and $\mathcal{C}_l$ of fuzzy interior ideals and fuzzy left ideals of S are closed under projection respectively. Therefore, by Proposition 2.5, $\mathcal{C}_{in} \subseteq \mathcal{C}_l$ if and only if $C_{in} \subseteq C_l$, where C_{in} and C_l are classes of all crisp interior ideals and crisp left ideals of S respectively. By Theorem 4.2.3 of [**9**], every interior ideal of a semigroup that is a semilattice of left groups is a left ideal and Chi is the isomorphism from $P(S)$ to $C(S)$. Therefore, by Proposition 2.5, $C_{in} \subseteq C_l$. Hence $\mathcal{C}_{in} \subseteq \mathcal{C}_l$. □

THEOREM 3.4. *Every fuzzy bi-ideal of a semigroup S that is a semilatice of groups is a fuzzy two sided ideal of S.*

PROOF. By Theorem 3.1, both the classes $\mathcal{C}_b$ and $\mathcal{C}_i$ of fuzzy bi-ideals and fuzzy two sided ideals are closed under projection. Therefore, by Proposition 2.5, $\mathcal{C}_b \subseteq \mathcal{C}_i$

if and only if $C_b \subseteq C_i$, where C_b and C_i are classes of all crisp bi-ideals and crisp two sided deals of S respectively. The later proposition follows as every bi-ideal of a semigroup that is a semilatice of groups is a two sided ideal by Lemma 8.2 of [**8**] and Chi is an isomorphism from $P(S)$ to $C(S)$. □

THEOREM 3.5. *The following statements are equivalent in a semigroup S:*
(1) *S is a semilattice of right groups.*
(2) $f \cap g = g \circ f \ \forall\ f \in \mathcal{C}_q$ *and* $\forall\ g \in \mathcal{C}_r(\mathcal{C}_i, \mathcal{C}_{in})$.
(3) $f \cap g = g \circ f \ \forall\ f \in \mathcal{C}_b$ *and* $\forall\ g \in \mathcal{C}_{in}$.
(4) $f \cap g = g \circ f \ \forall\ f \in \mathcal{C}_l(\mathcal{C}_r)$ *and* $g \in \mathcal{C}_{in}$.

PROOF. We establish that S is a semilattice of right groups if and only if $f \cap g = g \circ f \ \forall\ f \in \mathcal{C}_q$ and $\forall\ g \in \mathcal{C}_r$. Consider the $(\inf, \sup, o)$-algebra $F(S)$. Let $D(u, v) = u \cap v$ and $E(u, v) = vu$ be two expressions over the set of variables $\{u, v\}$ and set of operations $\{\inf, \sup, \cdot\}$ on $P(S)$. By Theorem 3.1, both the classes $\mathcal{C}_r$ and $\mathcal{C}_q$ of fuzzy right ideals and fuzzy quasi-ideals of S are closed under projection. Therefore, by metatheorem, $D(f, g) = E(f, g) \ \forall\ f \in \mathcal{C}_q$ and $\forall\ g \in \mathcal{C}_r$ if and only if $D(f, g) = E(f, g) \ \forall\ f \in C_q$ and $\forall\ g \in C_r$, where C_q and C_r are classes of all crisp quasi-ideals and crisp right ideals of S.

By Theorem 35 of [**12**], a semigroup S is a semilattice of right groups if and only if $A \cap B = BA$ for every quasi-ideal A and every right ideal B of S. Since $P(S)$ is isomorphic to $C(S)$ under the isomorphism Chi, $A \cap B = BA$ if and only if $\chi_{A \cap B} = \chi_{BA}$. Moreover, Chi commutes with finite intersection and product of sets, therefore, S is a semilattice of right groups if and only if $\chi_A \cap \chi_B = \chi_B \chi_A$ for every quasi-ideal A and every right ideal B of S. That is S is a semilattice of right groups if and only if $D(f, g) = E(f, g) \ \forall\ f \in C_q$ and $\forall\ g \in C_r$. Hence S is a semilattice of right groups if and only if $D(f, g) = E(f, g) \ \forall\ f \in \mathcal{C}_q$ and $\forall\ g \in \mathcal{C}_r$. □

The results of the following theorem that characterize a semigroup that is a semilattice of left groups can be proved similarly.

THEOREM 3.6. *The following statements are equivalent in a semigroup S:*
(1) *S is a semilattice of left groups.*
(2) $f \cap g = f \circ g \ \forall\ f \in \mathcal{C}_q$ *and* $\forall\ g \in \mathcal{C}_l(\mathcal{C}_i)$.
(3) $f \cap g = f \circ g \ \forall\ f \in \mathcal{C}_b(\mathcal{C}_q)$ *and* $\forall\ g \in \mathcal{C}_{in}$
(4) $f \cap g = f \circ g \ \forall\ f \in \mathcal{C}_l(\mathcal{C}_r)$ *and* $\forall\ g \in \mathcal{C}_{in}$

THEOREM 3.7. *A semigroup S is a semilattice of right groups if and only if* $f \cap g \cap h = f \circ h \circ g \ \forall\ f \in \mathcal{C}_i, \forall\ g \in \mathcal{C}_l$ *and* $\forall\ h \in \mathcal{C}_r$.

PROOF. Consider the $(\inf, \sup, o)$-algebra $F(S)$. Let $D(u, v, w) = u \cap v \cap w$ and $E(u, v, w) = uwv$ be two expressions over the set of variables $\{u, v, w\}$ and set of operations $\{\inf, \sup, \cdot\}$ on $P(S)$. By Theorem 3.1, the classes $\mathcal{C}_i$, $\mathcal{C}_l$ and $\mathcal{C}_r$ of all fuzzy two sided ideals, all fuzzy left ideals and all fuzzy right ideals of S respectively are closed under projection. Therefore, by metatheorem, $D(f, g, h) = E(f, g, h) \ \forall\ f \in \mathcal{C}_i, \forall\ g \in \mathcal{C}_l$ and $\forall\ h \in \mathcal{C}_r$ if and only if $D(f, g, h) = E(f, g, h) \ \forall\ f \in C_i, \forall\ g \in C_l$ and $\forall\ h \in C_r$, where C_i, C_l and C_r are classes of all crisp two sided ideals, crisp left ideals and crisp right ideals of S.

By Theorem 80 of [**12**], a semigroup S is a semilattice of right groups if and only if $A \cap B \cap C = ACB$ for every two sided ideal A, every left ideal B and every right ideal C of S. Since $P(S)$ is isomorphic to $C(S)$ under the isomorphism Chi, $A \cap B \cap C = ACB$ if and only if $\chi_{A\cap B\cap C} = \chi_{ACB}$. Moreover, Chi commutes with finite intersection and product of sets, therefore, S is a semilattice of left groups if and only if $\chi_A \cap \chi_B \cap \chi_C = \chi_A\chi_C\chi_B$ for every two sided ideal A, every left ideal B and every right ideal C of S. That is, S is a semilattice of right groups if and only if $D(f,g,h) = E(f,g,h)$ $\forall$ $f \in C_i$, $\forall$ $g \in C_l$ and $\forall$ $h \in C_r$. Hence S is a semilattice of right groups if and only if $D(f,g,h) = E(f,g,h)$ $\forall$ $f \in \mathcal{C}_i$, $\forall$ $g \in \mathcal{C}_l$ and $\forall$ $h \in \mathcal{C}_r$. □

The results of the following theorem that characterize a semigroup that is a semilattice of left groups can be proved similarly.

THEOREM 3.8. *The following statements are equivalent in a semigroup S:*

(1) *S is a semilattice of left groups.*
(2) $f \cap h \cap g = f \circ h \circ g$ $\forall$ $f \in \mathcal{C}_b(\mathcal{C}_q)$, $\forall$ $g \in \mathcal{C}_i$ *and* $\forall$ $h \in \mathcal{C}_l$.

THEOREM 3.9. *The following statements are equivalent in a semigroup S:*

(1) *S is a semilattice of right groups.*
(2) $f \cap g = S \circ g \circ f$ $\forall$ $f \in \mathcal{C}_l$ *and* $\forall$ $g \in \mathcal{C}_r$.

PROOF. Consider the $(\inf, \sup, o)$-algebra $F(S)$. Let $D(u,v) = u \cap v$ and $E(u,v) = Svu$ be two expressions over the set of variables $\{u, v\}$ and set of operations $\{\inf, \sup, \cdot\}$ on $P(S)$. By Theorem 3.1, both the classes $\mathcal{C}_i$ and $\mathcal{C}_r$ of fuzzy left ideals and fuzzy right ideals of S are closed under projection. Therefore, by metatheorem, $D(f,g) = E(f,g)$ $\forall$ $f \in \mathcal{C}_l$ and $\forall$ $g \in \mathcal{C}_r$ if and only if $D(f,g) = E(f,g)$ $\forall$ $f \in C_l$ and $\forall$ $g \in C_r$, where C_l and C_r are classes of all crisp left ideals and crisp right ideals of S.

By Theorem 80 of [**12**], a semigroup S is a semilattice of right groups if and only if $A \cap B = SBA$ for every left ideal A and every right ideal B of S. Since $P(S)$ is isomorphic to $C(S)$ under the isomorphism Chi, $A \cap B = ASB$ if and only if $\chi_{A\cap B} = \chi_{SBA}$. Moreover Chi commutes with finite intersection and product of sets, therefore, S is a semilattice of right groups if and only if $\chi_A \cap \chi_B = S\chi_B\chi_A$ for every left ideal A and every right ideal B of S. That is, S is a semilattice of right groups if and only if $D(f,g) = E(f,g)$ $\forall$ $f \in C_l$ and $\forall$ $g \in C_r$. Hence S is a semilattice of right groups if and only if $D(f,g) = E(f,g)$ $\forall$ $f \in \mathcal{C}_l$ and $\forall$ $g \in \mathcal{C}_r$. □

Similarly we can prove the following theorems:

THEOREM 3.10. *The following statements are equivalent in a regular semigroup S:*

(1) *S is a semilattice of left groups.*
(2) $f \cap g = f \circ S \circ g$ $\forall$ $f \in \mathcal{C}_q$ *and* $\forall$ $g \in \mathcal{C}_l$.
(3) $f \cap g = f \circ S \circ g$ $\forall$ $f \in \mathcal{C}_b$ *and* $\forall$ $g \in \mathcal{C}_l$.

THEOREM 3.11. *The following statements are equivalent in a regular semigroup S:*

(1) *S is a semilattice of right groups.*
(2) $f \cap g = g \circ S \circ f \ \forall\ f \in \mathcal{C}_q$ *and* $\forall\ g \in \mathcal{C}_r$.
(3) $f \cap g = g \circ S \circ f.\ \forall\ f \in \mathcal{C}_b$ *and* $\forall\ g \in \mathcal{C}_r$.

THEOREM 3.12. *The following statements are equivalent in a semigroup S:*
(1) *S is a semilattice of left(right) groups.*
(2) $f \cap g = f \circ g \circ f \ \forall\ f \in \mathcal{C}_q$ *and* $\forall\ f \in \mathcal{C}_i(\mathcal{C}_r)$.
(3) $f \cap g = f \circ g \circ f \ \forall\ f \in \mathcal{C}_b$ *and* $\forall\ f \in \mathcal{C}_l(\mathcal{C}_r)$.

THEOREM 3.13. *A semigroup S is semi-simple if and only if set of all fuzzy interior ideals is a semilattice under the multiplication of fuzzy subsets.*

PROOF. It is equivalent to show that $f \circ g = g \circ f$ and $f \circ f = f$ for all fuzzy interior ideals f, g of S. Consider the $(\inf, \sup, o)$-algebra $F(S)$. Define the expressions $D(v_1, v_2) = v_1v_2$, $E(v_1, v_2) = v_2v_1$, $D'(v_1) = v_1^2$, $E'(v_1) = v_1$ over the set of variables (v_1, v_2) and the set of operations $\{\inf, \sup, \cdot\}$. By Theorem 3.2, $\mathcal{C}_{in}$, the class of all fuzzy interior ideal of S is closed under projection. Therefore, by metatheorem, $D(f, g) = E(f, g)$ and $D'(f) = E'(f)$ holds for all f, g in $\mathcal{C}_{in}$ if and only if it holds for all crisp f, g in C_{in}, where C_{in} is the class of all crisp interior ideals of S.

By Theorem 3.4.9 of [**9**], A semigroup S is semi-simple if and only if set of all interior ideals is a semilattice under the multiplication of subsets. The later proposition follows as $P(S)$ is isomorphism to $C(S)$ under the isomorphism Chi and Chi commutes with finite intersection and product of sets. □

THEOREM 3.14. *A semigroup S is a semillatice of groups if and only if set of all fuzzy quasi-ideals is a semilattice under the multiplication of fuzzy subsets.*

PROOF. By Theorem 3.1, $\mathcal{C}_q$, the class of all fuzzy quasi-ideal of S is closed under projection. By Theorem 4.3.3 of [**9**], S is a semilattice of groups if and only if $AB = BA$ and $A^2 = A$ for every pair of quasi-ideals A, B of S. Therefore, the result follows similar to Theorem 3.13 in view of metatheorem, $P(S)$ is isomorphism to $C(S)$ under the isomorphism Chi and Chi commutes with finite intersection and product of sets. □

THEOREM 3.15. *A semigroup S is a semilattice of groups if and only if the product of any two fuzzy bi-ideals of S is equal to their intersection.*

PROOF. Consider the $(\inf, \sup, o)$-algebra $F(S)$. Let $D(u, v) = uv$ and $E(u, v) = u \cap v$ be two expressions over the set of variables $\{u, v\}$ and set of operations $\{\inf, \sup, \cdot\}$. By Theorem 3.1, the class, $\mathcal{C}_b$ of fuzzy bi-ideals of S is closed under projection. Therefore, by metatheorem, $D(f, g) = E(f, g) \ \forall\ f, g \in \mathcal{C}_b$ if and only if $D(f, g) = E(f, g) \ \forall\ f, g \in C_b$, where C_b is the class of all crisp bi-ideals of S.

By Theorem 13 of [**12**], a semigroup S is a semilattice of groups if and only if $AB = A \cap B$ for every pair of bi-ideals A, B of S. Since $P(S)$ is isomorphic to $C(S)$ under the isomorphism Chi, $AB = A \cap B$ if and only if $\chi_{AB} = \chi_{A \cap B}$. Moreover, Chi commutes with finite intersection and product of sets, therefore, S is a semilattice of groups if and only if $\chi_A \chi_B = \chi_A \cap \chi_B$ for every pair of bi-ideals A, B in S. That is, S is a semilattice of groups if and only if $D(f,g) = E(f,g)\ \forall\ f,g \in C_b$. Hence S is a semilattice of groups if and only if $D(f,g) = E(f,g)\ \forall\ f,g \in \mathcal{C}_b$. □

The above result can be extended to finite cases in the following way:

THEOREM 3.16. *A semigroup S is a semilattice of groups if and only if* $\bigcap_{i=1}^{k} f_i = \prod_{i=1}^{k} f_i$ *holds for every k fuzzy bi-ideals of S (where k is an integer > 1).*

PROOF. Define the expressions $D(u_1,u_2,\ldots,u_k) = u_1u_2\ldots u_k = \prod_{i=1}^{k} u_i$ and $E(u_1,u_2,\ldots,,u_k) = \bigcap_{i=1}^{k} u_i$ on $P(S)$. Since the class $\mathcal{C}_b$ of all fuzzy bi-ideals is closed under projection, therefore by metatheorem, $D(f_1,f_2,\ldots,f_k) = E(f_1,f_2,\ldots,f_k)$ for all fuzzy ideal $f_1,f_2,\ldots,f_k$ in $\mathcal{C}_b$ if and only if $D(f_1,f_2,\ldots,f_k) = E(f_1,f_2,\ldots,f_k)$ for all crisp $f_1,f_2,\ldots,f_k$ in C_b.

By Theorem 45 of [**12**], A semigroup S is a semilattice of groups if and only if $\bigcap_{i=1}^{k} A_i = \prod_{i=1}^{k} A_i$ holds for every k bi-ideals of S *(*where k is integer $k > 1$*)*. Proceeding as in Theorem 4.15, the later proposition follows as Chi commutes with finite intersection and product of sets and Chi is an isomorphism from $P(S)$ to $C(S)$. □

THEOREM 3.17. *A semigroup S is a semilattice of groups if and only if*

(1) *$f \cap g = f \circ S^n \circ g$ for every fuzzy left ideal f and every fuzzy right ideal g of*
(2) *$S($ where n is a fixed positive integer $)$.*

PROOF. Consider the $(\inf,\sup,o)$-algebra $F(S)$. Let $D(u,v) = u \cap v$ and $E(u,v) = uS^nv$ be two expressions over the set of variables $\{u,v\}$ and set of operations $\{\inf,\sup,\cdot\}$ on $P(S)$. By Theorem 3.1, both the classes, $\mathcal{C}_l$ and $\mathcal{C}_r$ of fuzzy left ideals and fuzzy right ideals of S are closed under projection. Therefore, by metatheorem, $D(f,g) = E(f,g)\ \forall\ f \in \mathcal{C}_l$ and $\forall\ g \in \mathcal{C}_r$ if and only if $D(f,g) = E(f,g)\ \forall\ f \in C_l$ and $\forall\ g \in C_r$, where C_l and C_r are classes of all crisp left ideals and all crisp right ideals of S.

By Theorem 63 of [**12**], a semigroup S is a semilattice of groups if and only if $A \cap B = AS^nB$ for every left ideal A and every right ideal B of S. Since $P(S)$ is isomorphic to $C(S)$ under the isomorphism Chi, $A \cap B = AS^nB$ if and only if $\chi_{A\cap B} = \chi_{AS^nB}$. Moreover, Chi commutes with the finite intersection and product of fuzzy sets, therefore, S is a semilattice of groups if and only if $\chi_A \cap \chi_B = \chi_A\chi_{S^n}\chi_B = \chi_A S^n \chi_B$ for every left ideal A and every right ideal B in S. That is, S is a semilattice of groups if and only if $D(f,g) = E(f,g)\ \forall\ f \in C_l$ and $\forall\ g \in C_r$. Hence S is a semilattice of groups if and only if $D(f,g) = E(f,g)\ \forall\ f \in \mathcal{C}_l$ and $\forall$ $g \in \mathcal{C}_r$ □

Following theorem that characterize a semilattice of groups in terms of fuzzy ideals bi-ideal and fuzzy left ideal of S is an extension of Theorem 64 of [**12**] to the fuzzy setting and can be proved similarly.

THEOREM 3.18. *A semigroup S is a semilattice of left groups if and only if $f \cap g = f \circ S^n \circ g$ for every fuzzy bi-ideal f and every fuzzy left ideal g of S (where n is a fixed positive integer).*

THEOREM 3.19. *The following statements are equivalent in a semigroup S:*

(1) *S is a semilattice of groups.*
(2) $f \cap g = g \circ f \circ g \ \forall\ f \in \mathcal{C}_q$ *and* $\forall\ f \in \mathcal{C}_i(\mathcal{C}_{in})$.
(3) $f \cap g = g \circ f \circ g \ \forall\ f \in \mathcal{C}_b$ *and* $\forall\ f \in \mathcal{C}_i$.
(4) $f \cap g = g \circ f \circ g \ \forall\ f \in \mathcal{C}_b$ *and* $\forall\ f \in \mathcal{C}_{in}$.

PROOF. We establish $(\mathbf{1}) \Leftrightarrow (\mathbf{3})$. Consider the $(\inf, \sup, o)$-algebra $F(S)$. Let $D(u, v) = u \cap v$ and $E(u, v) = vuv$ be two expressions over the set of variables $\{u, v\}$ and set of operations $\{\inf, \sup, \cdot\}$. By Theorem 3.1, both the classes $\mathcal{C}_b$ and $\mathcal{C}_i$ of fuzzy bi-ideals and fuzzy two sided ideals of S are closed under projection. Therefore, by metatheorem, $D(f, g) = E(f, g) \ \forall\ f \in \mathcal{C}_b$ and $\forall\ g \in \mathcal{C}_i$ if and only if $D(f, g) = E(f, g) \ \forall\ f \in C_b$ and $\forall\ g \in C_i$, where C_i and C_b are classes of all crisp two sided ideals and crisp bi-ideals of S.

By Theorem 82 of [**12**], a semigroup S is a semilattice of groups if and only if if $A \cap B = BAB$ for every bi-ideal A and every two sided B of S. Since $P(S)$ is isomorphic to $C(S)$ under the isomorphism Chi, $A \cap B = BAB$ if and only if $\chi_{A \cap B} = \chi_{BAB}$. Moreover, Chi commutes with the finite intersection and product of fuzzy sets, therefore, S is a semilattice of groups if and only if $\chi_A \cap \chi_B = \chi_B \chi_A \chi_B$for every bi-ideal A and every two sided ideal B of S. That is S is a semilattice of groups if and only if $D(f, g) = E(f, g) \ \forall\ f \in C_b$ and $\forall\ g \in C_i$. Hence S is a semilattice of groups if and only if $D(f, g) = E(f, g) \ \forall\ f \in \mathcal{C}_b$ and $\forall\ g \in \mathcal{C}_i$. □

THEOREM 3.20. *The following statements are equivalent in a semigroup S:*

(1) *S is a semilattice of groups.*
(2) $f \cap g = f \circ g \circ f \ \forall\ f \in \mathcal{C}_i$ *and* $\forall\ g \in \mathcal{C}_l$.
(3) $f \cap g = f \circ g \circ f \ \forall\ f \in \mathcal{C}_i$ *and* $\forall\ g \in \mathcal{C}_q$.

THEOREM 3.21. *The following statements are equivalent in a semigroup S:*

(1) *S is a semilattice of groups.*
(2) $f \circ g = g \cap f \ \forall\ f \in \mathcal{C}_i$ *and* $\forall\ g \in \mathcal{C}_b$.
(3) $f \circ g = g \cap f \ \forall\ f \in \mathcal{C}_i$ *and* $\forall\ g \in \mathcal{C}_q$.
(4) $f \circ g = g \cap f \ \forall\ f \in \mathcal{C}_i$ *and* $\forall\ g \in \mathcal{C}_l$.

PROOF. We establish $(\mathbf{1}) \Leftrightarrow (\mathbf{3})$. Consider the $(\inf, \sup, o)$-algebra $F(S)$. Let $D(u, v) = u \cap v$ and $E(u, v) = vu$ be two expressions over the set of variables $\{u, v\}$ and set of operations $\{\inf, \sup, \cdot\}$ on $P(S)$. By Theorem 3.1, both the classes, $\mathcal{C}_i$ and $\mathcal{C}_q$ of fuzzy two sided ideals and fuzzy quasi-ideals of S are closed under projection. Therefore, by metatheorem, $D(f, g) = E(f, g) \ \forall\ f \in \mathcal{C}_i$ and $\forall\ g \in \mathcal{C}_q$ if and only if $D(f, g) = E(f, g) \ \forall\ f \in C_i$ and $\forall\ g \in C_q$, where C_i and C_q are the classes of all crisp two sided ideals and crisp quasi-ideals of S.

By Theorem 35 of [**12**], a semigroup S is a semilattice of groups if and only if $A \cap B = BA$ for every two sided ideal A and every quasi-ideal B of S. Since $P(S)$ is isomorphic to $C(S)$ under the isomorphism Chi, $AB = A \cap B$ if and only if $\chi_{A\cap B} = \chi_{BA}$. Moreover, Chi commutes with the finite intersection and product of fuzzy sets, therefore, S is a semilattice of groups if and only if $\chi_A \cap \chi_B = \chi_B \chi_A$ for every two sided ideal A and every quasi-ideal B of S. That is S is a semilattice of groups if and only if $D(f, g) = E(f, g)$ for every crisp f in C_i and every crisp g in C_q. Hence S is a semilattice of groups if and only if $D(f, g) = E(f, g)$ for every f in $\mathcal{C}_i$ and every g in $\mathcal{C}_q$. □

Following theorem that characterize a semilattice of groups in terms of fuzzy ideals of S is an extension of Theorem 82 of [**12**] and Lemma 3.1 of [**8**] to the fuzzy setting and can be proved similarly.

THEOREM 3.22. *The following statements are equivalent in a semigroup S:*

(1) *S is a semilattice of groups.*
(2) $f \cap g = f \circ g \ \forall \ f \in \mathcal{C}_b$ *and* $\forall \ g \in \mathcal{C}_l(\mathcal{C}_i)$.
(3) $f \cap g = f \circ g \ \forall \ f \in \mathcal{C}_l$ *and* $\forall \ g \in \mathcal{C}_b(\mathcal{C}_q)$.
(4) $f \cap g = f \circ g \ \forall \ f \in \mathcal{C}_l$ *and* $\forall \ g \in \mathcal{C}_r$.
(5) $f \cap g = f \circ g \ \forall \ f \in \mathcal{C}_b(\mathcal{C}_q)$ *and* $\forall \ g \in \mathcal{C}_r$.
(6) $f \cap g = f \circ g \ \forall f, g \in \mathcal{C}_b(\mathcal{C}_q)$.
(7) $f \cap g = f \circ g \ \forall \ f \in \mathcal{C}_b(\mathcal{C}_q)$ *and* $\forall \ g \in \mathcal{C}_q(\mathcal{C}_b)$.

Finally interplay of regularity and semilattice is established by utilizing metatheorem on fuzzy ideals of a semigroup S. Following theorem that characterize the semilattice of groups and regularity in terms of fuzzy ideals of S, is an extension of Theorem 82 of [**2**] to the fuzzy setting.

THEOREM 3.23. *The following statements are equivalent in a semigroup S:*

(1) *S is regular.*
(2) *The set of all bi-ideals of S is a semilattice under the multiplication of subsets.*
(3) *The set of all fuzzy bi-ideals of S is a semilattice under the multiplication of fuzzy subsets.*
(4) *S is a semilattice of groups.*

PROOF. (1)⇔(2) by Theorem 57 of [**12**] and (2)⇔(4) by Lemma 8.4 of [**8**]. Now we establish (1)⇔(3). It is equivalent to show that $f \circ g = g \circ f$ and $f \circ f = f \ \forall \ f, g \in \mathcal{C}_b$. Consider the $\{\inf, \sup, o\}$-algebra $F(S)$. Define the expressions $D(v_1, v_2) = v_1 v_2$, $E(v_1, v_2) = v_2 v_1$, $D'(v) = v^2$ and $E'(v) = v$ over the set of variables $\{v_1, v_2, v\}$ and the set of operations $\{\inf, \sup, \cdot\}$ on $P(S)$. By Theorem 3.1, $\mathcal{C}_b$, the class of all fuzzy bi-ideal of S is closed under projection. Therefore, by metatheorem, $D(f, g) = E(f, g)$ and $D'(f) = E'(f)$ holds $\forall \ f, g$ in $\mathcal{C}_b$ if and only if it holds $\forall \ f, g \in \ C_b$, where C_b is the class of all crisp bi-ideals of S.

The later proposition follows as (1)⇔(2), $P(S)$ is isomorphism to $C(S)$ under the isomorphism Chi and Chi commutes with finite intersection and product of sets. □

Similarly we can establish the following theorem:

THEOREM 3.24. *The following statements are equivalent in a semigroup S:*

(1) *S is a semilattice of groups.*
(2) *Set of all bi-ideals (left ideal) is a commutative regular semigroup under the multiplication of subsets.*
(3) *Set of all fuzzy bi-ideals (left ideal) is a commutative regular semigroup under the multiplication of fuzzy subsets.*

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