BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS/BULLETIN
Bull. Int. Math. Virtual Inst., **14**(1)(2024), 157–168
DOI: 10.7251/BIMVI2401157D

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

ON THE GEOMETRY OF CONTACT PSEUDO-SLANT SUBMANIFOLDS OF PARA β-KENMOTSU MANIFOLDS

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Abstract. The aim of the present paper is to define and study contact pseudo-slant submanifolds of para β- Kenmotsu manifolds. We investigate the geometry of leaves which arise the definition of contact pseudo-slant submanifolds of para β- Kenmotsu manifolds and obtaine integrability conditions of distributions. We also consider parallel conditions of projections on study contact pseudo-slant submanifolds of a para β- Kenmotsu manifold.

1. Introduction

Complex and contact geometry have many aplication in mathematics and physics. Many geometric properties that ocur in complex structures were examined on contact structures. Moreover, important results were obtained regarding the geometric properties of the contact structures themselves. Para complex manifold is defined a 2n-dimensional differentiable manifold with endomorphism $J^2 = I$ such that 1-eigen distribution. Similary, a para contact manifold is defined (2n+1)-dimensional differentiable manifold with $\varphi^2 = I + \eta \otimes \xi$, where φ is (1,1)-type tensor field, η and ξ is contact form and characteristic vector field, respectively.

In 1985, Kaneyuki and Williams defined and studied para-contact manifolds [**9**]. After Zamkovoy investigated some properties of an almost para-contact metric

2020 *Mathematics Subject Classification.* Primary 53C15; Secondary 53C14.
Key words and phrases. Para β-Kenmotsu manifold, Contact pseudo-slant submanifolds .
Communicated by Dusko Bogdanic.

manifolds and their subclasses [**18**]. A Para-Kenmotsu manifold is a class of para-contact manifold which were defined by Sinha and Sai Prasad [**16**] in 1995. After [**14**], Olszak introduced para β-Kenmotsu manifold.

Slant submanifolds are known to generalize invariant and anti-invariant submanifolds, many geometrs have expressed an interest in this research. Chen [**4**], [**5**] started this research on complex manifolds. Lotta [**13**]pioneered slant immersions in a almost contact metric manifold. Carriezo defined a new class of submanifolds known as hemi-slant submanifolds (Also known as anti-slant or pseudo-slant submanifolds) [**1**], [**2**], [**3**]. The contact version of a pseudo-slant submanifold in a Sasakian manifold was then defined and studied by V. A. Khan and M. A. Khan. [**10**] [**11**]. Later many geometers such as [**12**], [**6**] studied pseudo-slant and Hemi slant submanifolds on various manifolds. Recently, S. Dirik studied contact pseudo-slant submanifold on various manifolds see [**7**] [**8**]. Also, R.Sarı and S.Dirik on Generic Submanifods of Para β-Kenmotsu Manifold studied [**15**], [**17**].

In the light of the above studies, our article, the following is how this paper is structured: Section 2 includes some fundamental formulas and definitons of para β-Kenmotsu manifold and it is submanifolds. Section 3 we review some definitions and proves some basic results on the contact pseudo-slant submanifolds of para β-Kenmotsu manifold. Also, the final section looks at the totally umbilical contact pseudo-slant in para β-Kenmotsu manifolds.

2. Preliminaries

Let $\overline{M}$ be a $(2n+1)$-dimensional differentiable manifold endowed with a quadruplet (φ, ξ, η, g), where φ is $(1,1)-$tensor field, ξ is a vector field, η is a $1-$form, and g is a pseudo-Riemannian metric such that

$$\varphi^2 X = \mu(X - \eta(X)\xi), \qquad \eta(\xi) = 1 \tag{2.1}$$

$$g(\varphi X, \varphi Y) = -\mu(g(X,Y) - \epsilon\eta(X)\eta(Y)) \tag{2.2}$$

for all $X, Y \in \Gamma(TM)$, where $\mu, \epsilon = \pm 1$. In addition, we have

$$\varphi\xi = 0, \quad \eta o\varphi = 0, \quad \eta(X) = \epsilon g(X, \xi). \tag{2.3}$$

The manifold $\overline{M}$ will be called almost para contact metric, and the quadruplet (φ, ξ, η, g) will be called the almost para contact metric structure on $\overline{M}$.

When $\mu = 1$, then the manifold $\overline{M}$ is an almost contact metric manifold. In this case the metric g is assumed to be pseudo-Riemannian in general, including Riemannian. Thus, if "$\varepsilon = 1$, the signature of g is equal to $2p$, where $0 \leqslant p \leqslant n$ and if " $\varepsilon = 1$, the signature of g is equal to $2p + 1$, where $0 \leqslant p \leqslant n$. When $\mu = 1$, then the manifold $\overline{M}$ is an almost paracontact metric manifold. In this case, the metric g is pseudo-Riemannian, and its signature is equal to n when " $\varepsilon = 1$, or $n+1$ when " $\varepsilon = -1$. One notes that in this case, the eigenspaces of the linear operator φ corresponding to the eigenvalues 1 and -1 are both n dimensional at every point of the manifold [**14**].

Then a 2-form Φ is defined by $\Phi(X,\ Y) = g(X, \varphi Y)$, for any $X, Y \in \Gamma(TM)$, called the fundamental 2$-$form. Moreover, a almost para contact metric manifold is normal if $[\varphi,\varphi] - 2d\eta \otimes \xi = 0$. Where $[\varphi,\varphi]$ is denoting the Nijenhuis tensor field associated to φ [**14**]. A normal almost para contact metric manifold is called para contact metric manifold.

Definition 2.1 ([**17**]). *Let $\overline{M}$ be an almost para contact metric manifold of dimension* $(2n+1)$*, with* $(\varphi,\xi,\eta,\mathrm{g})$*. $\overline{M}$ is said to be an almost para β-Kenmotsu manifold if 1-form η are closed and* $\mathrm{d}\Phi = 2\beta\eta\wedge\Phi$*. A normal almost para β-Kenmotsu manifold* M *is called a para β-Kenmotsu manifold.*

If $\overline{M}$ is also normal then we call $\overline{M}$ is called a para β-Kenmotsu manifold. The following theorem gives us the neccesary and sufficient condition for $\overline{M}$ to be para β-Kenmotsu manifold.

Theorem 2.1 ([**17**]). *Let* $(\overline{\mathrm{M}},\varphi,\xi,\eta,g)$ *be a para contact metric manifold.* $\overline{\mathrm{M}}$ *is a para β-Kenmotsu manifold if and only if*

$$(\overline{\nabla}_X\varphi)Y = \beta\{g\left(\varphi X, Y\right)\xi - \eta(Y)\varphi X\} \tag{2.4}$$

for all $X, Y \in \Gamma(T\overline{M})$.

Corollary 2.1 ([**17**]). *Let* $\overline{\mathrm{M}}$ *be (2n+1)-dimensional a para β-Kenmotsu manifold with structure (φ, ξ, η, g). Then we have*

$$\overline{\nabla}_X\xi = \beta\varphi^2 X \tag{2.5}$$

for all $X, Y \in \Gamma(T\overline{M})$.

3. Submanıfolds of para β-Kenmotsu manifold

Let $\overline{M}$ be a $(2n+1)-$dimensional $\beta-$ Kenmotsu manifold. M be a $n-$ dimensional submanifold of $\overline{M}$. Then Gauss and Weingarten formulas are given

$$\overline{\nabla}_X Y = \nabla_{\mathrm{X}}\mathrm{Y} + \sigma\left(\mathrm{X},\mathrm{Y}\right) \tag{3.1}$$

$$\overline{\nabla}_X V = -\mathrm{A}_{\mathrm{V}}\mathrm{X} + \nabla^{\perp}_{\mathrm{X}}\mathrm{V} \tag{3.2}$$

for any $X, Y \in \Gamma\left(TM\right)$ and $V \in \Gamma(TM)^{\perp}$. Where σ is the second fundamental from of M, $\nabla^{\perp}$ is the connection in the normal bundle and A_{V} is the Weingarten endomorphism associated with V. Shape operator A and *t*he second fundamental form σ related by

$$g\left(\sigma\left(X,Y\right),V\right)=g(\mathrm{A_V}X,Y). \tag{3.3}$$

On the other hand, the mean curvature tensor H is defined by $H=\frac{1}{m}\sum_{i=1}^{m}\sigma(e_i,e_i)$ where $\{e_1,\ldots,e_m\}$ is a local orthonormal basis of TM. A submanifold M of contact metric manifold is said to totally umbilical if

$$\sigma(X,Y)=g(X,Y)H \tag{3.4}$$

for all $X,Y\in\mathit{\Gamma}(TM)$. A submanifold M is said to be totally geodezik if $\sigma=0$ and M is said to be minimall if $H=0$.

For every tangent vector field X on M we can write

$$\varphi X=PX+FX. \tag{3.5}$$

Where PX (resp. FX) denotes the tangential (resp. normal) component of φX. Moreover for every normal vector field V we can state

$$\varphi V=pV+fV. \tag{3.6}$$

Where pV (resp. fV) denotes the tangential (resp. normal) component of φV.

For any $X,Y\in\Gamma\left(TM\right)$, we have

$$g\left(\varphi X,Y\right)=-g\left(X,\varphi Y\right).$$

From (3.5), we can see $g\left(PX+FX,Y\right)=-g\left(X,PY+FY\right)$
so we obtain

$$g\left(PX,Y\right)=-g\left(X,PY\right).$$

On the other hand, for any $X\in\Gamma\left(TM\right)$ and $V\in\Gamma(TM)^{\perp}$ we have

$$g\left(\varphi X,V\right)=-g\left(X,\varphi V\right).$$

From (3.5) and (3.6) , we write

$$g\left(PX+FX,V\right)=-g\left(X,pV+fV\right).$$

Thus ve have

$$g\left(FX,V\right)=-g\left(X,pV\right).$$

Finally,for any $W,\ V\in\mathit{\Gamma}(TM)^{\perp}$ we can state

$$g\left(\varphi W,V\right)=-g\left(W,\varphi V\right)$$

from (3.6), we write

$$g\left(pW+fW,V\right)=-g\left(W,pV+fV\right).$$

So we obtain

$$g\left(fW,V\right)=-g\left(W,fV\right).$$

We can summarize all these results by the following proposition.

PROPOSITION 3.1. *Let* $(\overline{\mathrm{M}},\varphi,\xi,\eta,\mathrm{g})$ *be a para contact metric manifold. Then we have*

$$g\left(PX,Y\right)=-g\left(X,PY\right),$$
$$g\left(fW,V\right)=-g\left(W,fV\right),$$
$$g\left(FX,V\right)=-g\left(X,pV\right),$$

for any $X,Y\in\Gamma\left(TM\right)$ *and for* $W,\ V\in\Gamma(TM)^{\perp}$.

Suppose that $\xi\in\Gamma\left(TM\right)$. Then we have

$$\varphi\xi=P\xi+F\xi=0.$$

Since $T\overline{M}=TM\oplus TM^{\perp}$, it is obvius that $P\xi=F\xi=0$.
On the other hand ,we have

$$\eta(\varphi X)=g\left(\varphi X,\xi\right)=\ g\left(PX+FX,\xi\right)=g\left(PX,\xi\right)+\ g\left(FX,\xi\right)=\eta\left(PX\right)+\eta\left(FX\right)=0.$$

And thus we get $\eta oP=\eta oF=0$.

After following similar steps we have

$\varphi^2X=\varphi\left(PX+FX\right)=\varphi\left(PX\right)+\varphi\left(FX\right)=P^2X+FPX+BFX+CFX=X-\eta\left(X\right)\xi.$

Since $P^2X+BFX\in\Gamma\left(TM\right)$ and $FPX+CFX\in\Gamma(TM)^{\perp}$ we get

$P^2+BF=I-\eta\otimes\xi$ and $FP+fF=0$.

Similar to

$\varphi^2V=\varphi\left(pV+fV\right)=\ \varphi\left(pV\right)+\varphi\left(fV\right)=f^2V+pfV+PpV+FpV=V,$

since $f^2+Fp\in\Gamma(TM)^{\perp}$ and $pf+Pp\in\Gamma\left(TM\right)$ we get

$f^2+Fp=I$ and $pf+Pp=0$.

We can summarize all these results as folloving.

PROPOSITION 3.2. *Let* $(\overline{\text{M}},\varphi,\xi,\eta,\text{g})$ *be a para contact metric manifold. Then we have*

$$P\xi = F\xi = 0 \ \textit{and} \ \eta o P = \eta o F = 0,$$
$$P^2 + pF = I + \eta\otimes\xi \quad \textit{and} \ FP + fF = 0,$$
$$f^2 + FB = I \ \textit{and} \ pf + Pp = 0.$$

Also defined are the covariant derivatives of the tensor fields P, F, p, and f

$$(\nabla_X P)\,Y = \nabla_X P)Y - P\nabla_X Y \tag{3.7}$$

$$(\nabla_X F)\,Y = \nabla^{\perp}_{\text{X}}\text{FY} - F\nabla_X Y \tag{3.8}$$

$$(\nabla_X B)\,V = \nabla_X BV - B\nabla^{\perp}_{\text{X}}\text{V} \tag{3.9}$$

$$(\nabla_X C)\,V = \nabla^{\perp}_{\text{X}}\text{CV} - C\nabla^{\perp}_{\text{X}}\text{V}. \tag{3.10}$$

The covariant derivative of φ can be defined by

$$(\overline{\nabla}_X\varphi)Y = \overline{\nabla}_X\varphi Y - \varphi\overline{\nabla}_X Y \tag{3.11}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$. Where ∇ is the Riemannian connection on $\overline{\text{M}}$.

Now, for later use, we establish a result for a submanifold para β-Kenmotsu manifold.

PROPOSITION 3.3. *Let* M *be submanifold of para* β*-Kenmotsu manifold* $\overline{\text{M}}$. *Then we have*

$$(\nabla_X P)\,Y = \text{A}_{FY}\text{X} + \text{p}\sigma\,(\text{X},\text{Y}) + \beta g\,(PX,Y)\,\xi - \beta\eta\,(Y)\,PX \tag{3.12}$$

$$(\nabla_X F)\,Y = f\sigma\,(\text{X},\text{Y}) - \sigma\,(X,PY) - \beta\eta\,(Y)\,FX \tag{3.13}$$

$$(\nabla_X p)\,V = A_{fV}X - PA_V X - \beta g(FX,V)\xi \tag{3.14}$$

$$(\nabla_X f)V = -\sigma(pV,X) - FA_V X \tag{3.15}$$

for all $X, Y \in \Gamma\,(TM)$ *and* $V \in \Gamma(T^{\perp}M)$.

Using (2.5), (3.1), (3.2), and (3.3), we have that is tangent to M.

$$\nabla_X\xi = \beta(X - \eta\,(X)\,\xi) \tag{3.16}$$

$$\sigma(\text{X},\xi) = 0 \tag{3.17}$$

for all $X \in \Gamma(TM)$.

Let us now same definetions of classes submanifolds.

(1) If F is identically zero in (3.5), then the submanifold is invariant.
(2) If P is identically zero in (3.5), then the submanifold is anti-invariant,
(3) If there is a constant angle $\theta(x) \in \left[0, \frac{\pi}{2}\right]$ between $\varphi X$and TM for all nonzero vector X tangent to M at x, the manifold is called slant.

A proper slant submanifold is one that is not invariant or anti-invariant. i. e. As a result, the following theorem characterized slant submanifolds of almost paracontact metric manifolds;

Theorem 3.1. *[**1**] Let M be a slant submanifolds of an almost paracontact metric manifold $\overline{M}$ such that $\in \Gamma$(TM), then, M is a slant if and only if a constant $\lambda \in$ [0,1] exists such that*

$$P^2{=}\lambda(\mathrm{I}{-}\eta{\otimes}\xi) \tag{3.18}$$

furthermore, in this situation, if θ is the slant angle of M. Then it satisfies $\lambda = \cos^2\theta$.

Corollary 3.1. *[**1**]. Let M be a slant submanifolds of an almost paracontact metric manifold $\overline{M}$. Then for all $X,\ Y\ \in\ \Gamma(TM)$ we have*

$$g\left(PX, PY\right) = -\cos^2\theta \left\{g\left(X, Y\right) - \epsilon\eta(X)\eta(Y)\right\} \tag{3.19}$$

$$g(FX, FY) = -\sin^2\theta \left\{g\left(X, Y\right) - \epsilon\eta(X)\eta(Y)\right\}. \tag{3.20}$$

4. Contact pseudo-slant submanifolds of a para β-Kenmotsu manifold

In this section, In a para β-Kenmotsu manifold, necessary and sufficient conditions are given for a submanifold to be a contact pseudo-slant submanifold.

Let M be a slant submanifold of an almost paracontact metric manifold $\overline{M}$. M is said to be pseudo-slant of $\overline{M}$ if there exit two orthogonal distributions D_θ and $\mathrm{D}^\perp$ on M such that: [**10**].

(1) TM has the orthogonal direct decomposition
$$TM \ = \ D^\perp \oplus \ D_\theta, \quad \xi \in D_\theta$$
(2) The distribution D_θ is slant with slant angle, that is, the slant angle between of D_θ and $\varphi\mathrm{D}_\theta$is a constant.
(3) The distribution $\mathrm{D}^\perp$ is an anti-invariant, That is,
$$\varphi\mathrm{D}^\perp \subset \mathrm{T}^\perp\mathrm{M}.$$

Let d_1 and d_2 be dimensional of distributions D and D_θ respectively. Then

(1) If $d_2 = 0$ then, M is an anti-invariant submanifold.
(2) If d_1= 0 and $\theta = 0$ then, M is an invariant submanifold.
(3) If d_1= 0 and $\theta \in \left(0,\ \frac{\pi}{2}\right)$then, M proper slant submanifold.
(4) If $\theta =\ \frac{\pi}{2}$ then, M is an anti-invariant submanifold.

(5) If $d_1\ d_2 \neq 0$ and $\theta \in \left(0,\ \frac{\pi}{2}\right)$ then, M is a proper pseudo-slant submanifold.
(6) If $d_1\ d_2 \neq 0$ and $\theta = 0$ then, M is a semi-invariant submanifold.

From the definitions, we can see that a slant submanifold is a generalization of invariant (if $\theta =\ 0$)and anti-invariant $\left(\text{if } \theta =\ \frac{\pi}{2}\right)$ submanifolds.

If the orthogonal complementary of φTM in $\mathrm{T}^{\perp}M$ is denoted by V, then the normal bundle $\mathrm{T}^{\perp}M$ can be decombosed as follows.

$$T^{\perp}M = FD_{\theta} \oplus FD^{\perp} \oplus \nu, \quad FD_{\theta} \perp FD^{\perp}.$$

Definition 4.1. *A contact pseudo slant submanidold M of para β -Kenmotsu manifold $\overline{M}$ is said to be mixed-geodesic submanifold if $\sigma(X,Y) = 0$ for all $X \in \Gamma(D_{\theta}), Y \in \Gamma(D^{\perp})$.*

Theorem 4.1. *Let M be proper contact pseudo slant submanidold para β-Kenmotsu manifold $\overline{M}$. M is either an anti-invariant or a mixed geodesic if p is parallel.*

Proof. For all $X \in \mathit{\Gamma}(D_{\theta}), Y \in \mathit{\Gamma}(D^{\perp})$, from (3.13) and (3.14)
p parallel if and only if F parallel, thus $\nabla F = 0$.
This implies

$$f\sigma(X,Y) - \sigma(X,PY) - \beta\eta(Y)FX = 0.$$

Replacing X by PX in the above equation, we get

$$f\sigma(PX,Y) - \sigma(PX,PY) = 0$$

for $Y \in \Gamma(\mathrm{D}^{\perp})$, $PY = 0$. Hence

$$f\sigma(PX,Y) = 0.$$

Replacing X by PX in the above equation, we have

$$f\sigma(P^2X,Y) = f\cos^2\theta\sigma(X,Y) = 0.$$

Hence we have either $\sigma(X,Y) = 0$ (M is mixed geodesic) or $\theta =\ \frac{\pi}{2}$ (M is anti-invariant). □

Theorem 4.2. *Let M be totally umbilical proper contact pseudo slant submanidold of a para β-Kenmotsu manifold $\overline{M}$. If p is parallel, then M is either minimal or anti-invariant submanifold.*

Proof. For all $X \in \mathit{\Gamma}(D_{\theta}), Y \in \mathit{\Gamma}(D^{\perp})$ from (3.13) and (3.14), we have
p parallel if and only if F parallel, so $\nabla F = 0$.

This implies

$$f\sigma(X,Y) - \sigma(X,PY) - \beta\eta(Y)FX = 0.$$

Replacing X by PX in the above equation, we get

$$f\sigma(PX,Y) - \sigma(PX,PY) = 0$$

for $Y\in \Gamma(D^{\perp})$, $PY=0$. Hence

$$f\sigma(PX,Y)=0.$$

Since M is totally umbilical, from (3.4)

$$fg(PX,Y)H=0$$

replacing X by PX in the above equation, we have

$$fg(P^2X,Y)H=f\cos^2\theta g(X,Y)H=0.$$

Hence we have either $\theta=\frac{\pi}{2}$ (M is anti-invariant) or $H=0$ (M is minimal).

□

THEOREM 4.3. *Let M be contact pseudo slant submanidold of a para β-Kenmotsu manifold $\overline{M}$. Then the tensör f is parallel if and only if shape operator A_V of satisfies the condition*

$$A_V pU = A_U pV$$

for $V,\ U\ \in T^{\perp}M$.

PROOF. For any $V,\ U\ \in\Gamma(T^{\perp}M)$. from (3.15) , we have

$$\begin{aligned}0&=g\left(\sigma\left(pV,X\right)+FA_VX,U\right)\\&=g(A_UpV,X)-g(A_VX,pU)\\&=g(A_UpV-A_VpU,X)\end{aligned}$$

for any X $\in\ \Gamma(TM)$.

□

THEOREM 4.4. *Let M be a contact pseudo slant submanidold of a para β-Kenmotsu manifold $\overline{M}$. Then $D^{\perp}$ is integrable at all times.*

PROOF. For all $W,\ U\ \in\ \Gamma(D^{\perp})$, from (2.4), we have

$$\left(\overline{\nabla}_W\varphi\right)U=\beta\{g\left(\varphi W,U\right)\xi-\eta\left(U\right)\varphi W\}=0.$$

By using (3.1), (3.2) , (3.5) and (3.6) we have

$$-A_{FU}W+\nabla^{\perp}{}_WFU-P\nabla_WU-F\nabla_WU-p\sigma\left(W,U\right)-f\sigma\left(W,U\right)=0.$$

Comparing the tangent companents, we have

$$A_{FU}W+P\nabla_WU+p\sigma\left(W,U\right)=0 \tag{4.1}$$

interchangin W and U , we get

$$A_{FW}U+P\nabla_UW+p\sigma\left(U,W\right)=0\ . \tag{4.2}$$

Subtracting equation (4.1) from (4.2) and using the fact that is symmetric , we get

$$A_{FU}W-A_{FW}U+P\left[W,U\right]=0,$$

$$P\left[U,W\right]=A_{FU}W-A_{FW}U. \tag{4.3}$$

On the other hand, for all $Z \in \Gamma(TM)$. By using (2.4), (3.1), (3.2), (3.3) and (3.11), we have

$$\begin{aligned} g\left(A_{FU}W - A_{FW}U, Z\right) &= g(\sigma\left(Z, W\right), FU) - g(\sigma\left(U, Z\right), FW) \\ &= g(\sigma\left(Z, W\right), FU) - g(\overline{\nabla}_Z U, FW) \\ &= g\left(\sigma\left(Z, W\right), FU\right) + g(\varphi\overline{\nabla}_Z U, W) \\ &= g\left(\sigma\left(Z, W\right), FU\right) + g\left(-A_{FU}Z + \nabla^{\perp}{}_Z FU, W\right) \\ &= g\left(\sigma\left(Z, W\right), FU\right) - g(A_{FU}Z, W) \\ &= g\left(\sigma\left(Z, W\right), FU\right) - g\left(\sigma\left(Z, W\right), FU\right) = 0 \end{aligned}$$

here

$$A_{FU}W = A_{FW}U.$$

So, from (4.3), for all $W,\ U \in \Gamma(D^{\perp})$, $[U, W] \in \Gamma(\mathrm{D}^{\perp})$, That is, $\mathrm{D}^{\perp}$ is every time integrable.

□

Theorem 4.5. *Let M be a contact pseudo slant submanidold of a para β-Kenmotsu manifold $\overline{M}$. Then the* D_θ *is integrable if and only f*

$$\omega_1\{\nabla_X PY - A_{FY}X - P\nabla_Y X - \ p\sigma\left(X, Y\right) + \beta\eta(Y)PX\} = 0$$

for all $X,\ Y \in \Gamma(D_\theta)$.

Proof. Let $\omega_{\ 1}$ and $\omega_{\ 2}$ the projections on $\mathrm{D}^{\perp}$ and D_θ , respectively. For all $X,\ Y \in \Gamma(D_\theta)$ from (2.4), we have

$$\left(\overline{\nabla}_X\right) Y \ = \beta\{g\left(\varphi X, Y\right) - (Y)\,\varphi X\}.$$

On applying (3.1), (3.2), (3.5) and (3.6), we get

$$\begin{aligned} \nabla_X PY + \sigma\left(X, PY\right) - A_{FY}X + \nabla^{\perp}{}_X FY - P\nabla_X Y - \ F\nabla_X Y - p\sigma\left(X, Y\right) \\ - f\sigma\left(X, Y\right) - \beta\{g\left(\varphi X, Y\right) - \eta\left(Y\right)\varphi X\} = 0. \end{aligned}$$

Comparig the tangential components

$$\nabla_X PY - A_{FY}X - P\nabla_X Y - \ p\sigma\left(X, Y\right) - \beta\{g\left(PX, Y\right) - \eta\left(Y\right)PX\} = 0,$$

$$\nabla_X PY - A_{FY}X - P\nabla_Y X + P\nabla_Y X - P\nabla_X Y - p\sigma\left(X, Y\right) - \beta\{g\left(PX, Y\right)\xi - \eta\left(Y\right)PX\} = 0$$

(4.4)

$$P\left[X, Y\right] = \nabla_X PY - A_{FY}X - P\nabla_Y X - \ p\sigma\left(X, Y\right) - \beta\{g\left(PX, Y\right)\xi - \eta\left(Y\right)PX\}.$$

For $X,\ Y \in \Gamma\left(D_\theta\right),\ [X,\ Y] \in \Gamma\left(D_\theta\right),\ \ so\ \omega_{\ 1}P[X,\ Y] = 0.$

As a result, we conclude our theorem by applying $\omega_{\ 1}$ to both sides of (4.4) equation.

□

THEOREM 4.6. *Let M be a totally umbilical contact pseudo slant submanidold of a para β-Kenmotsu manifold $\overline{M}$. Then at least one of the following satements is true.*

(1) *M is proper contact pseudo slant submanidold,*
(2) $H \in \Gamma(\nu)$,
(3) $Dim\left(D^{\perp}\right) = 1$.

PROOF. Let $X \in \Gamma(D^{\perp})$ and using (2.4), we obtain

$$\left(\overline{\nabla}_X \varphi\right) \mathrm{X} = \beta\{g\left(\varphi \mathrm{X}, \mathrm{X}\right)\xi - \eta\left(\mathrm{X}\right)\varphi \mathrm{X}\} = 0.$$

On applying (3.1), (3.2), (3.5) and (3.6), we get

$$-\mathrm{A_{FX}X} + \nabla^{\perp}{}_X \mathrm{FX} - \mathrm{F}\nabla_\mathrm{X}\mathrm{X} - \mathrm{p}\sigma\left(X, X\right) - f\ \sigma\left(X, X\right) = 0.$$

Comparig the tangential components

$$\mathrm{A_{FX}X} + \mathrm{p}\sigma\left(X, X\right) = 0.$$

Taking the product by $\mathrm{Z} \in \Gamma(\mathrm{D}^{\perp})$,we obtain

$$g(\mathrm{A_{FX}X}, \mathrm{Z}) + g\left(\mathrm{p}\sigma\left(X, X\right), Z\right) = 0.$$

Because M is a totally umbilical and (3.3), we get

$$\begin{aligned} 0 &= g(\mathrm{A_{FX}Z}, \mathrm{X}) + g\left(\mathrm{p}\sigma\left(X, X\right), Z\right) \\ &= g\left(\sigma\left(Z, X\right), FX\right) - g\left(\sigma\left(X, X\right), FZ\right) \\ &= g\left(Z, X\right) g\left(H, FX\right) - g\left(X, X\right) g\left(H, FZ\right) \\ &= g\left(X, X\right) g\left(BH, Z\right) - g\left(Z, X\right) g\left(BH, X\right) \end{aligned}$$

that is

$$g\left(BH, Z\right)\mathrm{X} - g\left(BH, X\right) Z = 0.$$

Here BH is either zero or X and Z are linearly dependent vector fields. If $BH \neq 0$, than dim $\left(\mathrm{D}^{\perp}\right) = 1$. Othervise $H \in \Gamma(\mu)$. Since $\mathrm{D}_\theta \neq 0$ M is contact pseudo slant submanidold. Since $\theta \neq 0$ and $\mathrm{d}_1\mathrm{d}_2 \neq 0$ proper contact pseudo slant submanifold. □

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Received by editors 2.4.2024; Revised version 14.9.2024; Available online 30.9.2024.

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