BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS/BULLETIN
Bull. Int. Math. Virtual Inst., **14**(1)(2024), 169–180
DOI: 10.7251/BIMVI2401169A

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

RICCI YAMABE SOLITON ON GENERALIZED SASAKIAN SPACE FORM

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Abstract. The aim of this paper is to study Ricci Yamabe soliton in generalized Sasakian spaceform satisfying the conditions: $R{\cdot}S = L_S Q(g,S)$, $Q{\cdot}R = 0$, $Q \cdot R = 0$, $P \cdot S = fQ(g,S)$ and $R \cdot R = L_R Q(S,R)$. We obtain the conditions for Ricci Yamabe solitons to be shrinking, expanding or steady.

1. Introduction

Guler and Crasmareanu [**9**] introduced a generalization of Ricci flow and Yamabe flow known as Ricci-Yamabe flow given by

$$(1.1) \qquad \frac{\partial}{\partial t} g(X,Y) = -2pS(X,Y) + qrg(X,Y), \qquad g(0) = g_0.$$

A solution to the Ricci-Yamabe flow is called the Ricci-Yamabe soliton (RYS) and its (g, V, p, λ) on a Riemannaian manifold (M, g) such that

$$(1.2) \qquad L_V\ g(X,Y) + 2pS(X,Y) + (2\lambda - qr)g(X,Y) = 0.$$

Following two cases arise from Ricci-Yamabe soliton:
Case 1: Yamabe soliton, if $p = 0$, then

$$(1.3) \qquad L_V\ g(X,Y) + (2\lambda - qr)g(X,Y) = 0.$$

2020 *Mathematics Subject Classification.* Primary 53D10.

Key words and phrases. Sasakian space form, Ricci-Yamabe soliton, Ricci pseudo-symmetric manifold, Ricci generalized pseudo-symmetric manifold.

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Communicated by Dusko Bogdanic.

Case 2: Ricci soliton, if $q = 0$, then

$$L_V \ g(X,Y) + 2pS(X,Y) + 2\lambda g(X,Y) = 0. \tag{1.4}$$

We strongly encourage authors to read the related works in [**1, 10, 15, 17, 18, 22**–**26**]. Many authors have investigated the geometrical and physical aspects of different spaces and these developments can be found in [**2**–**8, 11**–**14, 16, 19**–**21, 27**–**30**].

2. Preliminaries

An n-dimensional smooth manifold (M, g) is almost contact metric structure (ϕ, ξ, η, g) if it satisfies the following relations:

$$\phi^2(X) = -X + \eta(X)\xi, \quad \phi(\xi) = 0, \quad \eta(\xi) = 1, \quad g(X,\xi) = \eta(X), \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

for all vector X, Y and M. In the view of the above relation, we have

$$g(\phi X, Y) = -g(X, \phi Y), \quad g(\phi X, X) = 0. \tag{2.3}$$

An n-dimensional generalized Sasakian spaceform (GSSF) is given by

$$\begin{aligned} R(X,Y)Z =& f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi X \\ &+ 2g(X,\phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(z)X \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{2.4}$$

for all vector X, Y, Z on M, where $f_1, \ f_2, \ f_3$ are function on M, R denotes curvature tensor, and $f_1 = \dfrac{c+3}{4}$, $f_2 = \dfrac{c-1}{4}$ and $f_3 = \dfrac{c-1}{4}$.
In a GSSF, the following relations hold:

$$S(X,Y) = [(n-1)f_1 + 3f_2 - f_3]g(X,Y) - [3f_2 + (n-2)f_3]\eta(X)\eta(Y), \tag{2.5}$$

$$QX = [(n-1)f_1 + 3f_2 - f_3]X - [3f_2 + (n-2f_3)]\eta(X)\xi \tag{2.6}$$

$$R(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y] \tag{2.7}$$

$$R(\xi,X)Y = -R(X,\xi)Y = (f_1 - f_3)[g(X,Y)\xi - \eta(Y)X]. \tag{2.8}$$

The Projective curvature tensor of an n-dimensional GSSF is defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y]. \tag{2.9}$$

We define tensor $R \cdot T$ and $Q(g,T)$ by

$$\begin{aligned} (R(X,Y)\cdot T)(X_1, X_2, X_3, \ldots, X_K) &= -T(R(X,Y)X_1, X_2, X_3, \ldots, X_K) \\ - T(X_1, R(X,Y)X_2, X_3, \ldots, X_K) &- \cdots T(X_1, X_2, X_3, \ldots, R(X,Y)X_K) \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} Q(g,T)(X_1, X_2, X_3 \ldots X_K; X, Y) &= -T((X \wedge_g Y)X_1, X_2, X_3 \ldots X_K) - \\ T(X_1, (X \wedge_g Y)X_2, X_3, \ldots, X_K) &- \cdots - T(X_1, X_2, X_3, \ldots, (X \wedge_g Y)X_K) \end{aligned} \tag{2.11}$$

with

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y. \tag{2.12}$$

We define the tensor $R \cdot R$ and $R \cdot S$ on (M^n, g) by

$$\begin{aligned}(R(X,Y) \cdot R)(U,V)W = &R(X,Y)R(U,V)W - R(R(X,Y)U,V)W \\ &- R(U,R(X,Y)V)W - R(U,V)R(X,Y)W\end{aligned} \tag{2.13}$$

and

$$(R(X,Y) \cdot S)(U,V) = -S(R(X,Y)U,V) - S(U,R(X,Y)V), \tag{2.14}$$

where S is Ricci tensor and R is curvature tensor.

$$S(X,\xi) = (n-1)(f_1 - f_3)\eta(X), \tag{2.15}$$

$$Q(\xi) = (n-1)(f_1 - f_2)\xi, \tag{2.16}$$

$$g(QX,Y) = S(X,Y), \tag{2.17}$$

where S is Ricci tensor and Q is Ricci opeartor.

Definition 2.1. *A GSSF (M^n, g) is said to be Ricci -pseudo-symmetric if the tensor $R \cdot S$ and $Q(g,S)$ are linearly dependent. This is equivalent to*

$$R \cdot S = L_S Q(g,S), \tag{2.18}$$

where L_S is some function.

Definition 2.2. *A GSSF (M^n, g) is said to be Ricci generalized pseudo-symmetric if the tensor $R \cdot R$ and $Q(S,R)$ are linearly dependent. This is equivalent to*

$$R \cdot R = L_R Q(S,R), \tag{2.19}$$

where L_R is some function.

Definition 2.3. *A GSSF is said to be projective Ricci pseudo symmetric if the tensors $P \cdot S$ and $Q(g,S)$ are linearly dependent. This is equivalent to*

$$P \cdot S = FQ(g,S), \tag{2.20}$$

where F is a some function.

3. RYS in GSSF

Let (g,ξ,λ) be a RYS in an n-dimensional GSSF M, from (1.2) we have

$$L_V \ g(X,Y) + 2PS(X,Y) + (2\lambda - qr)g(X,Y) = 0. \tag{3.1}$$

Putting $V = \xi$ in (3.1), we obtain

$$L_\xi \ g(X,Y) + 2pS(X,Y) + (2\lambda - qr)g(X,Y) = 0, \tag{3.2}$$

for any $X, Y \in TM^n$, where L_ξ is the Lie derivative operator along the vector field ξ, S is the Ricci tensor field of the metric g and λ is real constant and we have

$$L_\xi \ g(X,Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 0. \tag{3.3}$$

By virtue of (3.3) and (3.2), we get

$$S(X,Y) = -\left(\frac{\lambda}{p} - \frac{qr}{2p}\right) g(X,Y). \tag{3.4}$$

Ricci-Yamabe soliton is called shrinking, steady or expanding according as λ is negative, zero or positive, respectively.

4. RYS in Ricci pseudo-symmetric GSSF

We consider a Ricci pseudo-symmetric GSSF. Then from (2.18), we have

$$(R(X,Y)\cdot S)(U,V) = L_S Q(g,S)(X,Y;U,V). \tag{4.1}$$

Using (2.11) in (4.1), we have

$$(R(X,Y)\cdot S)(U,V) = L_S((X\wedge_g Y).S)(U,V). \tag{4.2}$$

With the help of (2.11) and (2.14), we get from (4.2)

$$\begin{aligned} -S(R(X,Y)U,V) - S(U,R(X,Y)V) = L_S[&-S((X\wedge_g Y)U,V) \\ &- S(U,(X\wedge_g Y)V)]. \end{aligned} \tag{4.3}$$

Using (2.12)in (4.3), we obtain

$$\begin{aligned} -S(R(X,Y)U,V) - S(U,R(X,Y)V) = L_S[&-g(Y,U)S(X,V) \\ &+ g(X,U)S(Y,V) - g(Y,V)S(U,X) + g(X,V)S(U,Y)]. \end{aligned} \tag{4.4}$$

Putting $X = U = \xi$ in (4.4), we have

$$(L_S + (f_1 - f_3))[S(Y,V) - g(Y,V)(f_1 - f_3)(n-1)] = 0. \tag{4.5}$$

From this, we conclude that either $L_S = (f_3 - f_1)$ or the manifold is an Einstein manifold of the form

$$S(Y,V) = (f_1 - f_3)(n-1)g(Y,V). \tag{4.6}$$

Hence we state the following result:

LEMMA 4.1. *A Ricci-pseudo symmetric GSSF is an Einstein manifold with* $L_S \neq (f_3 - f_1)$.

If a Ricci-pseudo symmetric GSSF admits RYS, then by virtue of (4.6) and (3.4) we obtain

$$\lambda = -(f_1 - f_3)(n-1)p + \frac{qr}{2}. \tag{4.7}$$

Now substituting f_1 and f_3 in (4.7), we get

$$\lambda = -(n-1)p + \frac{qr}{2}. \tag{4.8}$$

Thus, we have the following Theorem:

THEOREM 4.1. *A RYS in Ricci pseudo-symmetric GSSF is shrinking or steady or expanding accordingly as:*

$$-(n-1)p+\frac{qr}{2}>0, \quad -(n-1)p+\frac{qr}{2}=0, \quad or\ -(n-1)p+\frac{qr}{2}<0, \tag{4.9}$$

provided $L_S \neq (f_3-f_1)$.

If $p=0$, then from (4.8) we have $\lambda=\frac{qr}{2}$. Then by Theorem 4.1, the following corollary is immediate:

COROLLARY 4.1. *A q-Yamabe soliton in Ricci pseudo-symmetric GSSF is shrinking or expanding or steady accordingly as:*

$$\frac{qr}{2}>0,\ \frac{qr}{2}<0,\ or\ \frac{qr}{2}=0. \tag{4.10}$$

If $q=0$, then from (4.8) we have $\lambda=-(n-1)p$. Then by Theorem 4.1, the following corollary is immediate:

COROLLARY 4.2. *A p-Ricci soliton in Ricci pseudo-symmetric GSSF is shrinking or expanding or steady as:*

$$-(n-1)p>0,\ -(n-1)p<0,\ or\ -(n-1)p=0. \tag{4.11}$$

5. RYS in Ricci generalized pseudo-symmetric GSSF

Consider a Ricci generalized pseudo-symmetric GSSF. From (2.19), we have

$$R\cdot R=L_R Q(S,R). \tag{5.1}$$

Using(2.11) in (5.1), we get,

$$(R(X,Y)\cdot R)(U,V)W=L_R((X\wedge_S Y)\cdot R)(U,V)W. \tag{5.2}$$

Using (2.11) in (2.14), we get

$$\begin{aligned} &R(X,Y)R(U,V)W-R(R(X,Y)U,V)W-R(U,R(X,Y)V)W\\ &-R(U,V)R(X,Y)W=L_R[(X\wedge_S Y)R(U,V)W\\ &-R((X\wedge_S Y)U,V)W-R(U,(X\wedge_S Y)V)W-R(U,V)(X\wedge_S Y)W]. \end{aligned} \tag{5.3}$$

Using (2.12) in (5.3), we get

$$\begin{aligned} &R(X,Y)R(U,V)W-R(R(X,Y)U,V)W-R(U,R(X,Y)V)W\\ &-R(U,V)R(X,Y)W=L_R[S(Y,R(U,V)W)X-S(X,R(U,V)W)Y\\ &-S(Y,U)R(X,V)W+S(X,U)R(Y,V)W-S(Y,V)R(U,X)W\\ &+S(X,V)R(U,Y)W-S(Y,W)R(U,V)X+S(X,W)R(U,V)Y]. \end{aligned} \tag{5.4}$$

Putting $X = U = \xi$ in (5.4), we have

$$\begin{aligned}(5.5)\quad -(f_1 - f_3)^2 g(V,W)Y + (f_1 - f_3)R(Y,V)W + (f_1 - f_3)^2 g(Y,W)V\\ = L_R[-S(Y,W)(f_1 - f_3)\eta(V)\xi - (f_1 - f_3)^2 + g(V,W)Y\\ + (n-1)(f_1 - f_3)R(Y,V)W + (n-1)(f_1 - f_3)^2\eta(V)g(Y,W)\xi\\ - S(Y,W)(f_1 - f_3)\eta(V)\xi + S(Y,W)V(f_1 - f_3)\\ + (n-1)g(Y,V)\xi\eta(W)].\end{aligned}$$

Taking the inner product with respect to Z in (5.5), we get

$$\begin{aligned}(5.6)\quad -(f_1 – f_3)^2 g(V,W)g(Y,Z) - g(R(Y,V)W,Z)(f_1 - f_3)\\ + g(Y,W)g(V,Z)(f_1 - f_3)^2 = l_R[-S(Y,V)\eta(W)g(\xi,Z)\\ - (f_1 - f_3)^2(n-1)g(V,W)g(Y,Z)\\ + (n-1)(f_1 - f_3)g(Y,Z) + (n-1)(f_1 - f_3)g(R(Y,V)W,Z)\\ + (n-1)(f_1 - f_3)^2\eta(V)g(Y,W)\eta(Z) - S(Y,W)(f_1 - f_3)\eta(V)\eta(Z)\\ - S(Y,W)(f_1 - f_3)\eta(V)\eta(Z) + S(Y,W)(f_1 - f_3)g(V,Z)\\ + (f_1 - f_3)g(Y,V)\eta(Z)\eta(W)].\end{aligned}$$

Putting $V = W = e_i$ in (5.6), we get

$$(5.7)\qquad (f_1 - f_3)(nL_R - 1)[S(Y,Z) - (nL_R - 1)g(Y,Z)(n-1)(f_1 - f_3)] = 0.$$

Hence, we conclude that either $(f_1 = f_3)$ or $L_R = 1/n$ or the manifold is an Einstein manifold

$$(5.8)\qquad S(Y,Z) = (n-1)(f_1 - f_3)g(Y,Z).$$

Hence we state the following Lemma:

LEMMA 5.1. *A Ricci generalized pseudo symmetric GSSF is an Einstein manifold with $(f_1 \neq f_3)$ or $L_R \neq 1/n$.*

If Ricci generalized pseudo symmetric GSSF admits RYS, then by virtue of (3.4) and (5.8) we have

$$(5.9)\qquad \lambda = -(n-1)p + \frac{qr}{2}.$$

Thus we have the following Theorem:

THEOREM 5.1. *A RYS in Ricci generalized pseudo-symmetric GSSF is shrinking or steady or expanding accordingly as:*

$$(5.10)\qquad -(n-1)p + \frac{qr}{2} > 0,\ -(n-1)p + \frac{qr}{2} = 0,\ \textit{or}\ -(n-1)p + \frac{qr}{2} < 0$$

provided $(f_1 \neq f_3)$ or $L_R \neq 1/n$.

If $p = 0$, then from (5.9) we get $\lambda = \frac{qr}{2}$. Then by Theorem 5.1, the following corollary is immediate:

COROLLARY 5.1. *A q-Yamabe soliton in Ricci generalized pseudo-symmetric GSSF is shrinking or expanding or steady accordingly as:*

$$\frac{qr}{2} > 0, \ \frac{qr}{2} < 0, \ or \ \frac{qr}{2} = 0. \tag{5.11}$$

If $q = 0$, then from (5.9) we get $\lambda = -(n-1)p$. Then by Theorem 5.1, the following corollary is immediate:

COROLLARY 5.2. *A p-Ricci soliton in Ricci generalized pseudo-symmetric GSSF is shrinking or expanding or steady as:*

$$-(n-1)p > 0, \ -(n-1)p < 0, \ or \ -(n-1)p = 0. \tag{5.12}$$

6. RYS in GSSF satisfying the curvature condition $Q \cdot R = 0$

Consider a GSSF satisfying the curvature condition

$$Q \cdot R = 0. \tag{6.1}$$

Then

$$(Q \cdot R)(X,Y)Z = 0. \tag{6.2}$$

Using(2.13) in (6.2), we get

$$(Q(R(X,Y)Z) - R(QX,Y)Z - R(X,QY)Z - R(X,Y)QZ = 0. \tag{6.3}$$

Putting $X = Z = \xi$ in (6.3), we get

$$\begin{aligned}(f_1 - f_3)\eta(Y)Q\xi - R(Q\xi,Y)\xi - (f_1 - f_3)\eta(QY)\xi \\ - (f_1 - f_3)S(Y,\xi)\xi + (f_1 - f_3)S(\xi,\xi)Y = 0.\end{aligned} \tag{6.4}$$

Taking inner product with ξ in (6.4), we get

$$\begin{aligned}(f_1 - f_3)\eta(Y)g(Q\xi,\xi) - g(R(Q\xi,Y)\xi,\xi) \\ - 2(f_1 - f_3)S(Y,\xi)\xi + (f_1 - f_3)S(\xi,\xi)\eta(Y) = 0.\end{aligned} \tag{6.5}$$

But

$$g(R(Q\xi,Y)\xi,\xi) = 0. \tag{6.6}$$

In the view of (6.5), (6.6) and (3.2), we have

$$\lambda = -(n-1)(f_1 - f_3)p + \frac{qr}{2}. \tag{6.7}$$

Now substituting f_1 and f_3 in (6.7) ,we obtain

$$\lambda = -(n-1)p + \frac{qr}{2}. \tag{6.8}$$

Hence we state the following Theorem:

THEOREM 6.1. *A RYS in GSSF satisfying the curvature condition* $Q.R = 0$ *is shrinking or steady or expanding accordingly as:*

$$-(n-1)p + \frac{qr}{2} > 0, \ -(n-1)p + \frac{qr}{2} = 0, \ -(n-1)p + \frac{qr}{2} < 0. \tag{6.9}$$

If $p = 0$, then from (6.9) we get $\lambda = \frac{qr}{2}$. Then by Theorem 6.1, the following corollary is immediate:

Corollary 6.1. *A q-Yamabe soliton in GSSF satisfying the curvature condition $Q.R = 0$ is shrinking or expanding or steady accordingly as:*

$$\frac{qr}{2} > 0, \frac{qr}{2} < 0, \frac{qr}{2} = 0. \tag{6.10}$$

If $q = 0$, then from (6.9) we get $\lambda = -(n-1)p$. Then by Theorem 6.1, the following corollary is immediate:

Corollary 6.2. *A p-Ricci soliton in GSSF satisfying the curvature condition $Q.R = 0$ is shrinking or expanding or steady as:*

$$-(n-1)p > 0,\ -(n-1)p < 0,\ -(n-1)p = 0. \tag{6.11}$$

7. RYS in Projective Ricci pseudo-symmetric GSSF

We consider a Projective Ricci pseudo- symmetric GSSF. Then from (2.20), we have

$$(P(X,Y)\cdot S)(U,V) = FQ(g,S)(X,Y;U,V) \tag{7.1}$$

which implies that

$$(P(X,Y)\cdot S)(U,V) = F((X\wedge_g Y)\cdot S)(U,V). \tag{7.2}$$

This equation can be written as

$$\begin{aligned} -S(P(X,Y)U,V)-&S(U,P(X,Y)V) \\ &= F[-S((X\wedge_g Y)U,V) - S(U,(X\wedge_g Y)V)]. \end{aligned} \tag{7.3}$$

Using (2.12) and (2.14) in (7.3), we obtain

$$\begin{aligned} -S(P(X,Y)U,V) - S(U,P(X,Y)V) \\ =F[-g(Y,U)S(X,V) + g(X,U)S(Y,V) \\ - g(Y,V)S(U,X) + g(X,V)S(U,Y)]. \end{aligned} \tag{7.4}$$

Putting $X = U = \xi$ in (7.4) and applying the values of f_1 and f_3, we have

$$(F-2)S(Y,V) = -(n-1)(F+1)g(Y,V). \tag{7.5}$$

Hence, we conclude that $F = 2$ or the manifold is an Einstein manifold of the form

$$S(Y,V) = (1-n)(F+1)g(Y,V). \tag{7.6}$$

Thus, we state the following Lemma:

Lemma 7.1. *A Projective Ricci-pseudo symmetric GSSF is an Einstein manifold with $F \neq 2$, $F \neq 1$.*

If a Projective Ricci-pseudo symmetric GSSF admits RYS, then by virtue of (7.6) and (3.4) we obtain

$$\lambda = (F+1)(n-1)p + \frac{qr}{2}. \tag{7.7}$$

Hence, we state the following Theorem:

THEOREM 7.1. *A RYS in Projective Ricci pseudo-symmetric GSSF is shrinking or steady or expanding accordingly as:*

$$(F+1)(n-1)p+\frac{qr}{2}<0,\ \ (F+1)(n-1)p+\frac{qr}{2}=0,\ \ or\ \ (F+1)(n-1)p+\frac{qr}{2}>0. \tag{7.8}$$

If $p=0$, then from (7.7) we get $\lambda=\frac{qr}{2}$. Then by Theorem 7.1, the following corollary is immediate:

COROLLARY 7.1. *A q-Yamabe soliton in Ricci pseudo-symmetric GSSF is shrinking or expanding or steady accordingly as:*

$$\frac{qr}{2}<0,\ \ \frac{qr}{2}>0,\ \ or\ \ \frac{qr}{2}=0 \tag{7.9}$$

If $q=0$, then from (7.7) we get $\lambda=-(n-1)p$. Then by Theorem 7.1, the following Corollary is immediate:

COROLLARY 7.2. *A p-Ricci soliton in Ricci pseudo-symmetric GSSF is shrinking or expanding or steady as:*

$$(F+1)(n-1)p<0,\ \ (F+1)(n-1)p>0,\ \ or\ \ (F+1)(n-1)p=0. \tag{7.10}$$

8. RYS in GSSF satisfying the Projective curvature condition $Q\cdot P=0$

Let the Ricci Yamabe soliton in GSSF satisfies the Projective curvature condition $Q\cdot P=0$. Then

$$(Q\cdot R)(X,Y)Z=0 \tag{8.1}$$

which implies that

$$(Q(P(X,Y)Z)-P(QX,Y)Z-P(X,QY)Z-P(X,Y)QZ=0. \tag{8.2}$$

Putting $X=Z=\xi$ in (8.2), we obtain

$$Q(P(\xi,Y)\xi)-P(Q\xi,Y)\xi-P(\xi,QY)\xi-P(\xi,Y)Q\xi=0. \tag{8.3}$$

Taking the inner product with ξ in (8.3), we have

$$(f_1-f_3)[-2S(Y,\xi)+(f_1-f_3)(n-2)\eta(Y)]=0. \tag{8.4}$$

In the view of (3.4) and (8.4) and applying the values of f_1, f_3, we obtain

$$\lambda=\frac{qr}{2}-p(n-2). \tag{8.5}$$

Therefore, we have the following Theorem:

THEOREM 8.1. *A RYS in GSSF with satisfying the Projective curvature condition $Q\cdot R=0$ is shrinking or steady or expanding accordingly as:*

$$\frac{qr}{2}-p(n-2)>0,\ \ \frac{qr}{2}-p(n-2)=0,\ \ or\ \ \frac{qr}{2}-p(n-2)<0. \tag{8.6}$$

If $p=0$, then from (8.5) we get $\lambda=\frac{qr}{2}$. Then by Theorem 8.1, the following Corollary is immediate:

COROLLARY 8.1. *A q-Yamabe soliton in GSSF with satisfying the Projective curvature condition $Q \cdot R = 0$ is shrinking or steady or expanding accordingly as:*

$$\frac{qr}{2} < 0, \quad \frac{qr}{2} = 0, \quad or \ \frac{qr}{2} > 0. \tag{8.7}$$

If $q = 0$, then from (8.5) we get $\lambda = -(n-2)p$. Then by Theorem 8.1, the following Corollary is immediate:

COROLLARY 8.2. *A p-Ricci soliton in GSSF with satisfying the Projective curvature condition $Q \cdot R = 0$ is shrinking or steady or expanding accordingly as:*

$$-(n-2)p > 0, \quad -(n-2)p = 0, \quad or \ -(n-2)p < 0. \tag{8.8}$$

Acknowledgments

The authors would like to thank the anonymous reviewer for their comments and suggestions.

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Received by editors 31.1.2024; Revised version 16.9.2024; Available online 30.9.2024.

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