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TWO VARIABLE HIGHER-ORDER GENERALIZED FUBINI POLYNOMIALS

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Abstract. This article attempts to present two variable higher-order generalized Fubini polynomials $N_{n,\lambda}^{(r)}(x,y)$. The results obtained here include various families of multilinear and multilateral generating functions, various properties, as well as some special cases for two variable higher-order generalized Fubini polynomials $N_{n,\lambda}^{(r)}(x,y)$. Finally, we get several interesting results of this two variable higher-order generalized Fubini polynomials and obtain an integral representation.

1. Introduction

Generating functions branch a prominent part in the exploration of colorful useful parcels of the rows which they induce. They are used in chancing certain parcels and formulas for numbers and polynomials in a very diverse disquisition subjects, really, in modern combinatorics. For a regular preface to, and several interesting operations of the various styles of carrying direct, bilinear, bilateral or mixed multilateral generating functions for a quite wide kind of rows of special functions (and polynomials) in one, two and more variables, among important ample literature, we relate to the expansive study by Srivastava and Manocha [**17**]. There are numerous studies deal with polynomials and their generating functions. A many of references to special polynomials and their generating functions are pass in the monographs [**8**]- [**16**]. Looking at it the other way, generating functions have some applications in many fields similar to applied mathematics, algebra, statistics, combinatorics, and physics. The Fubini-type polynomials happen in combinatorial

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mathematics and play a significant part in the proposition and operation of mathematics, therefore numerous number proposition and combinatorics experts have remarkably studied their parcels and attained series of fascinating results [**1**]- [**9**].

Special functions and figures have important places in colorful classification of mathematics, theoretical drugs, and engineering. The problems occurring in fine drugs and engineering are framed in terms of dicriminational equations . Utmost of these equations can only be treated by using colorful families of special polynomials which give new means of fine analysis. They are extensively used in computational models of scientific and engineering problems. Additionally, these special polynomials enable the derivate of different useful individualities in a fairly simple way and help in introducing new families of special polynomials. Throughout this composition, we use the coming after memos and delineations.

Let $\mathbb{N} = \{1, 2, 3, \cdots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ denotes the set of integer numbers, $\mathbb{R}$ express the set of real numbers and $\mathbb{C}$ express the set of complex numbers. We originally flash back the classical two variable Fubini polynomials by the following generating function [**1**], [**2**], [**6**]- [**9**]:

$$\sum_{n=0}^{\infty} N_n(x, y)\frac{t^n}{n!} = \frac{e^{xt}}{1 - y\left(e^t - 1\right)}. \tag{1.1}$$

While $x = 0$ in (1.1), the two-variable Fubini polynomials $N_n(x, y)$ reduce to the normal Fubini polynomials given by [**1**], [**2**], [**6**]- [**9**]:

$$\sum_{n=0}^{\infty} N_n(y)\frac{t^n}{n!} = \frac{1}{1 - y\left(e^t - 1\right)}. \tag{1.2}$$

Substituting y by in (1.2), we have the known Fubini numbers $N_n(1) := N_n$ as follows [**1**], [**2**], [**6**]- [**9**]:

$$\sum_{n=0}^{\infty} N_n\frac{t^n}{n!} = \frac{1}{2 - e^t}.$$

For more information about the applications of the normal Fubini polynomials and numbers, [**1**], [**2**], [**6**]- [**9**], and see also the references cited therein. Let us give a short list of these polynomials and numbers as follows [**10**]:

$$N_0(y) = 1,\ N_1(y) = y,\ N_2(y) = y + 2y^2,\ N_3(y) = y + 6y^2 + 6y^3,$$

$$N_4(y) = y + 14y^2 + 36y^3 + 24y^4,$$

$$N_0 = 1,\ N_1 = 1,\ N_2 = 3,\ N_3(y) = 13,\ N_4(y) = 75.$$

The two variable Fubini polynomials $N_n^{(r)}(x, y)$ of order r are defined by

$$\sum_{n=0}^{\infty} N_n^{(r)}(x, y)\frac{t^n}{n!} = \frac{e^{xt}}{\left(1 - y\left(e^t - 1\right)\right)^r}$$

where r is a positive integer.

Here, in this paper y will be an discretionary but fixed real number in order that $N_n^{(r)}(x,y)$ are polynomials in x for each fixed y. Note here that

$$N_n^{(r)}(x,y) \sim \left(\left(1-y\left(e^t-1\right)\right)^r, t\right).$$

Especially, if $r=1$, then $N_n(x,y) = N_n^{(1)}(x,y)$ are named two variable Fubini polynomials and they were introduced by Kargin in [**1**].

This paper enterprises with the following main objectives:

∘ Carrying theorems giving multilinear and multilateral generating function relations for the two variable advanced order generalized Fubini polynomials and agitating their special cases.

∘ Inferring colorful rush relations for the two variable advanced-order generalized Fubini polynomials.

We now define the the two variable higher-order generalized Fubini polynomials as follows.

2. Two variable higher-order generalized Fubini polynomials

Now we attain new generating function for the two variable higher-order generalized Fubini polynomials $N_{n,\lambda}^{(r)}(x,y)$.

Definition 2.1. *The two variable higher-order generalized Fubini polynomials $N_{n,\lambda}^{(r)}(x,y)$ are defined via the following exponential generating function:*

$$\sum_{n=0}^{\infty} N_{n,\lambda}^{(r)}(x,y)\frac{t^n}{n!} = \frac{e^{\lambda xt}}{\left(1-y\left(e^{\lambda t}-1\right)\right)^r}. \tag{2.1}$$

For some special cases of (2.1), we have

$$N_{n,1}^{(r)}(x,y) = N_n^{(r)}(x,y) \text{ and } N_{n,1}^{(r)}(0,y) = N_n^{(r)}(y).$$

We can rewrite (2.1) as

$$\begin{aligned}
\sum_{n=0}^{\infty} N_{n,\lambda}^{(r)}(x,y)\frac{t^n}{n!} &= \left(1-y\left(e^{\lambda t}-1\right)\right)^{-r} e^{\lambda xt} \\
&= \sum_{n=0}^{\infty} N_n^{(r)}(y)\frac{\lambda^n t^n}{n!}\sum_{h=0}^{\infty}\frac{(\lambda xt)^h}{h!} \\
&= \sum_{n=0}^{\infty}\sum_{h=0}^{\infty} N_n^{(r)}(y)\lambda^{n+h}\frac{x^h t^{n+h}}{n!h!} \\
&= \sum_{n=0}^{\infty}\sum_{h=0}^{n} N_{n-h}^{(r)}(y)\lambda^n x^h\frac{t^n}{(n-h)!h!} \\
&= \sum_{n=0}^{\infty}\left[\sum_{h=0}^{n}\binom{n}{h}N_{n-h}^{(r)}(y)\lambda^n x^h\right]\frac{t^n}{n!}.
\end{aligned}$$

Comparison the coefficient of $t^n/n!$ yields

$$N_{n,\lambda}^{(r)}(x,y)=\sum_{h=0}^{n}\binom{n}{h}N_{n-h}^{(r)}(y)\lambda^n x^h.$$

THEOREM 2.1. *Formula holds for the two-variable higher-order generalized Fubini polynomials* $N_{n,\lambda}^{(r)}(x,y)$:

$$N_{n,\lambda}^{(r_1+r_2)}(x_1+x_2,y)=\sum_{h=0}^{n}\binom{n}{h}N_{n-h,\lambda}^{(r_1)}(x_1,y)N_{h,\lambda}^{(r_2)}(x_2,y). \tag{2.2}$$

PROOF. Replacing r by r_1+r_2 and x by x_1+x_2 in (2.1), we get

$$\begin{aligned}
\sum_{n=0}^{\infty}N_{n,\lambda}^{(r_1+r_2)}(x_1+x_2,y)\frac{t^n}{n!} &= \frac{e^{\lambda(x_1+x_2)t}}{\left(1-y\left(e^{\lambda t}-1\right)\right)^{(r_1+r_2)}}\\
&= \frac{e^{\lambda x_1 t}}{\left(1-y\left(e^{\lambda t}-1\right)\right)^{r_1}}\frac{e^{\lambda x_2 t}}{\left(1-y\left(e^{\lambda t}-1\right)\right)^{r_2}}\\
&= \sum_{n=0}^{\infty}N_{n,\lambda}^{(r_1)}(x_1,y)\frac{t^n}{n!}\sum_{h=0}^{\infty}N_{h,\lambda}^{(r_2)}(x_2,y)\frac{t^h}{h!}\\
&= \sum_{n=0}^{\infty}\sum_{h=0}^{\infty}N_{n,\lambda}^{(r_1)}(x_1,y)N_{h,\lambda}^{(r_2)}(x_2,y)\frac{t^{n+h}}{n!h!}\\
&= \sum_{n=0}^{\infty}\sum_{h=0}^{n}\binom{n}{h}N_{n-h,\lambda}^{(r_1)}(x_1,y)N_{h,\lambda}^{(r_2)}(x_2,y)\frac{t^n}{n!},
\end{aligned}$$

the last equality which is the required proof. □

3. Generating function for two variable higher-order generalized Fubini polynomials

In this part of the study, first of all we acquire a few families of bilinear and bilateral generating functions for the two variable higher-order generalized Fubini polynomials $N_{n,\lambda}^{(r)}(x,y)$ which are generated by (2.1) and given clearly by (2.2) by using the similar way technique in [**11**]- [**16**].

THEOREM 3.1. *Let*

$$\Theta_{n,p}^{\mu,\psi}(x,y;y_1,...,y_s;\xi):=\sum_{h=0}^{[n/p]}a_h N_{n-ph,\lambda}^{(r)}(x,y)\Omega_{\mu+\psi h}(y_1,...,y_s)\frac{\xi^h}{(n-ph)!}, \tag{3.1}$$

$$(a_h\neq 0).$$

If

$$\Lambda_{\mu,\psi}[y_1,...,y_s;\xi]:=\sum_{h=0}^{\infty}a_h\Omega_{\mu+\psi h}(y_1,...,y_s)\xi^h,\ \ (a_h\neq 0)$$

then, for every non negative integer p we have

$$\sum_{n=0}^{\infty}\Theta_{n,p}^{\mu,\psi}\left(x,y;y_1,...,y_s;\frac{\eta}{t^p}\right)t^n=\frac{e^{\lambda xt}}{\left(1-y\left(e^{\lambda t}-1\right)\right)^r}\Lambda_{\mu,\psi}\left[y_1,...,y_s;\eta\right]. \tag{3.2}$$

PROOF. If we express the left-hand side of (3.2) by T and use (3.1),

$$T=\sum_{n=0}^{\infty}\sum_{h=0}^{[n/p]}a_hN_{n-ph,\lambda}^{(r)}(x,y)\Omega_{\mu+\psi h}(y_1,...,y_s)\eta^h\frac{t^{n-ph}}{(n-ph)!}.$$

Replacing n by $n+ph$

$$\begin{aligned}T &= \sum_{n=0}^{\infty}\sum_{h=0}^{\infty}a_hN_{n,\lambda}^{(r)}(x,y)\Omega_{\mu+\psi h}(y_1,...,y_s)\eta^h\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}N_{n,\lambda}^{(r)}(x,y)\frac{t^n}{n!}\sum_{h=0}^{\infty}a_h\Omega_{\mu+\psi h}(y_1,...,y_s)\eta^h\\ &= \frac{e^{\lambda xt}}{\left(1-y\left(e^{\lambda t}-1\right)\right)^r}\Lambda_{\mu,\psi}\left[y_1,...,y_s;\eta\right]\end{aligned}$$

which finishes proof. □

By using alike thought, we as well as attain the next result instantaneously.

THEOREM 3.2. *Corresponding to an identically non-vanishing function $\Omega_\mu\left(y_1,\ldots,y_s\right)$ of r complex variables $y_1,\ldots,y_s$ $(s\in\mathbb{N})$ and of complex order μ,ψ let*

$$\begin{aligned}&\Lambda_{\mu,\psi}^{n,p}\left(x_1+x_2,y;y_1,\ldots,y_s;t\right)\\ &:=\sum_{h=0}^{[n/p]}a_hN_{n-ph,\lambda}^{(r_1+r_2)}(x_1+x_2,y)\Omega_{\mu+\psi h}\left(y_1,\ldots,y_s\right)t^h\quad(a_h\neq0)\end{aligned}$$

and the notation $[n/p]$ means the greatest integer less than or equal n/p. Then, for $p\in\mathbb{N}$, we have

$$\begin{aligned}&\sum_{h=0}^{n}\sum_{r=0}^{[h/p]}\binom{n-pr}{h-pr}a_rN_{n-h,\lambda}^{(r_1)}\left(x_1,y\right)N_{h-pr,\lambda}^{(r_2)}\left(x_2,y\right)\Omega_{\mu+\psi r}\left(y_1,\ldots,y_s\right)t^r\\ &=\Lambda_{\mu,\psi}^{n,p}\left(x_1+x_2,y;y_1,\ldots,y_s;t\right)\end{aligned} \tag{3.3}$$

on condition that each member of (3.3) exists.

PROOF. Let T express the first member of the claim (3.3). Then, upon substituting for the polynomials $N_{n,\lambda}^{(r_1+r_2)}\left(x_1+x_2,y\right)$ from the (2.2) into the left hand side of (3.3), we attain

$$\begin{aligned}
T &= \sum_{r=0}^{[k/p]}\sum_{h=0}^{n-pr}\binom{n-pr}{h}a_r N_{n-h-pr,\lambda}^{(r_1)}(x_1,y)\,N_{h,\lambda}^{(r_2)}(x_2,y)\,\Omega_{\mu+\psi r}(y_1,\ldots,y_s)\,t^r \\
&= \sum_{r=0}^{[n/p]}a_r\left(\sum_{h=0}^{n-pr}\binom{n-pr}{h}N_{n-h-pr,\lambda}^{(r_1)}(x_1,y)\,N_{h,\lambda}^{(r_2)}(x_2,y)\right)\Omega_{\mu+\psi r}(y_1,\ldots,y_s)\,t^r \\
&= \sum_{r=0}^{[n/p]}a_r N_{n-pr,\lambda}^{(r_1+r_2)}(x_1+x_2,y)\,\Omega_{\mu+\psi r}(y_1,\ldots,y_s)\,t^r \\
&= \Lambda_{\mu,\psi}^{n,p}(x_1+x_2,y;y_1,\ldots,y_s;t)\,.
\end{aligned}$$

□

4. Further properties of $N_{n,\lambda}^{(r)}(x,y)$ and some applications

Now we discuss some applications of Theorem 3.1 and Theorem 3.2. If the multivarible function $\Omega_{\mu+\psi k}(y_1,\ldots y_s)$ $s\in\mathbb{N}$ is expressed in terms of simplicity functions of one and more variables, then we can give further applications of the above theorems.
For example, taking

$$\Omega_{\mu+\psi h}(y_1,\ldots,y_s)=\Phi_{\mu+\psi h}^{(\alpha)}(y_1,\ldots,y_s)$$

in Theorem 3.1, where the multivarible polynomials $\Phi_{\mu+\psi h}^{(\alpha)}(y_1,\ldots,y_s)$ are generated by (see [**11**])

$$\sum_{n=0}^{\infty}\Phi_n^{(\alpha)}(x_1,\ldots,x_s)\,t^n=(1-x_1t)^{-\alpha}\,e^{(x_2+\cdots+x_s)t}, \tag{4.1}$$

$$\left(\alpha\in\mathbb{C},\ \ |t|<\left\{|x_1|^{-1}\right\}\right),$$

we attain the following result, which provides a bilateral generating function for multivariable polynomials $\Phi_n^{(\alpha)}(x_1,\ldots,x_s)$ and two variable higher-order generalized Fubini polynomials $N_{n,\lambda}^{(r)}(x,y)$.

Corollary 4.1. *If*

$$\Lambda_{\mu,\psi}[y_1,...,y_s;\xi]\ \ :\ =\sum_{h=0}^{\infty}a_h\Phi_{\mu+\psi h}^{(\alpha)}(y_1,\ldots,y_s)\,\xi^h, \qquad (a_h\neq 0,\ \mu,\psi\in\mathbb{C})$$

and

$$\Theta_{n,p}^{\mu,\psi}\left(x,y;y_{1,\cdots},y_s;\xi\right)\ \ :\ =\sum_{h=0}^{[n/p]}a_hN_{n-ph,\lambda}^{(r)}(x,y)\Omega_{\mu+\psi h}(y_1,\ldots,y_s)\,\frac{\xi^h}{(n-ph)!}, \qquad (n\in\mathbb{N}_0,\ p\in\mathbb{N})$$

then we have

$$\sum_{n=0}^{\infty} \Theta_{n,p}^{\mu,\psi}\left(x,y;y_{1,\dots,}y_s;\frac{\eta}{t^p}\right)t^n = \frac{e^{\lambda xt}}{\left(1-y\left(e^{\lambda t}-1\right)\right)^r}\Lambda_{\mu,\psi}\left[y_1,\dots,y_s;\eta\right] \tag{4.2}$$

on condition that each member of (4.2) exists.

REMARK 4.1. Using (4.1) and taking

$$a_h = 1\ \ (h\in\mathbb{N}_0),\ \mu=0 \textit{ and } \ \psi=1$$

in Corollary 4.1, we find that

$$\sum_{n=0}^{\infty}\sum_{h=0}^{[n/p]} N_{n-ph,\lambda}^{(r)}(x,y)\Phi_h^{(\alpha)}\left(y_1,\dots,y_s\right)\eta^k t^{n-ph}$$

$$= \frac{e^{\lambda xt}}{\left(1-y\left(e^{\lambda t}-1\right)\right)^r}\left(1-y_1\eta\right)^{-\alpha}e^{(y_2+\dots+y_s)\eta},\ \left(|\eta|<|y_1|^{-1}\right).$$

If we choose

$$r=2 \ \textit{ and } \Omega_{\mu+\psi h}\left(y_1,y_2\right) = N_{\mu+\psi h,\lambda}^{(r)}(y_1,y_2)$$

in Theorem 3.1, then we have the following bilinear generating functions $N_{\mu+\psi h,\lambda}^{(r)}(y_1,y_2)$.

COROLLARY 4.2. *If*

$$\Lambda_{\mu,\psi}\left[y_1,y_2;\xi\right] := \sum_{h=0}^{\infty} a_h N_{\mu+\psi h,\lambda}^{(r)}(y_1,y_2)\xi^h$$

$$(a_h\neq 0,\ \mu,\ \psi\in\mathbb{C})$$

and

$$\Theta_{n,p}^{\mu,\psi}\left(x,y;y_{1,}y_2;\xi\right) := \sum_{h=0}^{[n/p]} a_h N_{n-ph,\lambda}^{(r)}(x,y)N_{\mu+\psi h,\lambda}^{(r)}(y_1,y_2)\frac{\xi^h}{(n-hp)!}$$

$$(n\in\mathbb{N}_0,\ p\in\mathbb{N})$$

then we have

$$\sum_{n=0}^{\infty} \Theta_{n,p}^{\mu,\psi}\left(x,y;y_{1,}y_2;\frac{\eta}{t^p}\right)t^n = \frac{e^{\lambda xt}}{\left(1-y\left(e^{\lambda t}-1\right)\right)^r}\Lambda_{\mu,\psi}\left[y_1,y_2;\eta\right] \tag{4.3}$$

on condition that each member of (4.3) exist.

REMARK 4.2. Using (2.1) and taking

$$a_h = 1\ \ (k\in\mathbb{N}_0),\ \mu=0 \textit{ and } \ \psi=1$$

in Corollary 4.2, we find that

$$\sum_{n=0}^{\infty}\sum_{h=0}^{[n/p]} N_{n-ph,\lambda}^{(r)}\left(x,y\right)N_{h,\lambda}^{(r)}\left(y_1,y_2\right)\frac{\eta^h t^{n-ph}}{(n-ph)!}$$

$$= \frac{e^{\lambda xt}}{\left(1-y\left(e^{\lambda t}-1\right)\right)^{r}} \frac{e^{\lambda y_1 \eta}}{\left(1-y_2\left(e^{\lambda \eta}-1\right)\right)^{r}}.$$

If we choose

$$r = 2 \textit{ and } \Omega_{\mu+\psi h}\left(y_1, y_2\right) = N^{(r_3)}_{\mu+\psi h,\lambda}(y_1, y_2)$$

in Theorem 3.2, then we have the following bilinear generating functions $N^{(r_3)}_{\mu+\psi h,\lambda}(y_1, y_2)$.

COROLLARY 4.3. *If*

$$\begin{aligned}
&\Lambda^{n,p}_{\mu,\psi}\left(x_1+x_2; y; y_1, y_2; t\right) \\
&:= \sum_{h=0}^{[n/p]} a_h N^{(r_1+r_2)}_{n-ph,\lambda}(x_1+x_2, y) N^{(r_3)}_{\mu+\psi h,\lambda}\left(y_1, y_2\right) t^h \\
&\left(a_h \neq 0,\ \mu, \psi \in \mathbb{C}\right)
\end{aligned}$$

then, we have

$$\sum_{h=0}^{n} \sum_{r=0}^{[h/p]} \binom{n-pr}{h-pr} a_r N^{(r_1)}_{n-h,\lambda}\left(x_1, y\right) N^{(r_2)}_{h-pr,\lambda}\left(x_2, y\right) N^{(r_3)}_{\mu+\psi r,\lambda}(y_1, y_2) t^r$$
$$= \Lambda^{n,p}_{\mu,\psi}\left(x_1+x_2, y; y_1, y_2; t\right).$$

Moreover for every convenient choice of the coefficients a_h $(h \in \mathbb{N}_0)$ if the multivarible functions $\Omega_{\mu+\psi h}\left(y_1, \ldots, y_s\right)$, $s \in \mathbb{N}$ are expressed as an suitable product of several simpler functions, the claim of Theorem 3.1 and Theorem 3.2 can be applied in order to derive various families of multilinear and multilateral generating functions for the two variable higher-order generalized Fubini polynomials $N^{(r)}_{n,\lambda}(x, y)$.

THEOREM 4.1. *For $r \geqslant 1$ and $n \geqslant 1$, we have*

$$\frac{\partial}{\partial x} N^{(r)}_{n,\lambda}(x, y) = \lambda n N^{(r)}_{n-1,\lambda}(x, y). \tag{4.4}$$

PROOF. By (2.1), we have

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty} N^{(r)}_{n,\lambda}(x, y)\frac{t^n}{n!}\right) &= \frac{\partial}{\partial x}\left(\frac{e^{\lambda xt}}{\left(1-y\left(e^{\lambda t}-1\right)\right)^{r}}\right) \\
\sum_{n=0}^{\infty} \frac{\partial}{\partial x} N^{(r)}_{n,\lambda}(x, y)\frac{t^n}{n!} &= \lambda t \frac{e^{\lambda xt}}{\left(1-y\left(e^{\lambda t}-1\right)\right)^{r}} \\
&= \lambda \sum_{n=0}^{\infty} N^{(r)}_{n,\lambda}(x, y)\frac{t^{n+1}}{n!} \\
&= \lambda \sum_{n=1}^{\infty} N^{(r)}_{n-1,\lambda}(x, y)\frac{t^{n}}{(n-1)!}.
\end{aligned}$$

Especially, for $n=0$, we get

$$N_{0,\lambda}^{(r)}(x,y)=1,\quad \frac{\partial}{\partial x}N_{0,\lambda}^{(r)}(x,y)=0$$

and

$$\sum_{n=1}^{\infty}\frac{\partial}{\partial x}N_{n,\lambda}^{(r)}(x,y)\frac{t^n}{n!}=\lambda\sum_{n=1}^{\infty}N_{n-1,\lambda}^{(r)}(x,y)\frac{t^n}{(n-1)!}. \tag{4.5}$$

□

Comparison the coefficients on both sides of (4.5), we attain the following theorem.

THEOREM 4.2. *For $r\geqslant 1$ and $n\geqslant 0$, we have*

$$(1+y)\frac{\partial}{\partial y}N_{n,\lambda}^{(r)}(x,y)+rN_{n,\lambda}^{(r)}(x,y)$$

$$=\sum_{h=0}^{n}\binom{n}{h}\lambda^h\left[y\frac{\partial}{\partial y}N_{n-h,\lambda}^{(r)}(x,y)+rN_{n-h,\lambda}^{(r)}(x,y)\right].$$

PROOF. By (2.1), we have

$$\begin{aligned}
\frac{\partial}{\partial y}\left(\sum_{n=0}^{\infty}N_{n,\lambda}^{(r)}(x,y)\frac{t^n}{n!}\right) &= \frac{\partial}{\partial y}\left(\frac{e^{\lambda xt}}{\left(1-y\left(e^{\lambda t}-1\right)\right)^r}\right),\\
\sum_{n=0}^{\infty}\frac{\partial}{\partial y}N_{n,\lambda}^{(r)}(x,y)\frac{t^n}{n!} &= -r\frac{e^{\lambda xt}\left(1-e^{\lambda t}\right)}{\left(1-y\left(e^{\lambda t}-1\right)\right)^{r+1}},
\end{aligned}$$

$$\left(1-y\left(e^{\lambda t}-1\right)\right)\sum_{n=0}^{\infty}\frac{\partial}{\partial y}N_{n,\lambda}^{(r)}(x,y)\frac{t^n}{n!}$$

$$= -r\sum_{n=0}^{\infty}N_{n,\lambda}^{(r)}(x,y)\frac{t^n}{n!}+r\sum_{n=0}^{\infty}\sum_{h=0}^{\infty}N_{n,\lambda}^{(r)}(x,y)\frac{t^n}{n!}\frac{(\lambda t)^h}{h!},$$

$$(1+y)\sum_{n=0}^{\infty}\frac{\partial}{\partial y}N_{n,\lambda}^{(r)}(x,y)\frac{t^n}{n!}-y\sum_{n=0}^{\infty}\sum_{h=0}^{\infty}\frac{\partial}{\partial y}N_{n,\lambda}^{(r)}(x,y)\frac{t^n}{n!}\frac{(\lambda t)^h}{h!}$$

$$= r\sum_{n=0}^{\infty}\sum_{h=0}^{\infty}N_{n,\lambda}^{(r)}(x,y)\frac{t^n}{n!}\frac{(\lambda t)^h}{h!}-r\sum_{n=0}^{\infty}N_{n,\lambda}^{(r)}(x,y)\frac{t^n}{n!},$$

$$(1+y)\sum_{n=0}^{\infty}\frac{\partial}{\partial y}N_{n,\lambda}^{(r)}(x,y)\frac{t^n}{n!}-y\sum_{n=0}^{\infty}\sum_{h=0}^{n}\lambda^h\frac{\partial}{\partial y}N_{n-h,\lambda}^{(r)}(x,y)\frac{t^n}{(n-h)!h!}$$

$$= r\sum_{n=0}^{\infty}\sum_{h=0}^{n}\lambda^hN_{n-h,\lambda}^{(r)}(x,y)\frac{t^n}{(n-h)!h!}-r\sum_{n=0}^{\infty}N_{n,\lambda}^{(r)}(x,y)\frac{t^n}{n!}. \tag{4.6}$$

Comparison the coefficients on both sides of (4.6),

$$(1+y)\frac{\partial}{\partial y}N_{n,\lambda}^{(r)}(x,y)-y\sum_{h=0}^{n}\binom{n}{h}\lambda^{h}\frac{\partial}{\partial y}N_{n-h,\lambda}^{(r)}(x,y)$$

$$= r\sum_{h=0}^{n}\binom{n}{h}\lambda^{h}N_{n-h,\lambda}^{(r)}(x,y)-rN_{n,\lambda}^{(r)}(x,y),$$

$$(1+y)\frac{\partial}{\partial y}N_{n,\lambda}^{(r)}(x,y)+rN_{n,\lambda}^{(r)}(x,y)$$

$$= r\sum_{h=0}^{n}\binom{n}{h}\lambda^{h}N_{n-h,\lambda}^{(r)}(x,y)+y\sum_{h=0}^{n}\binom{n}{h}\lambda^{h}\frac{\partial}{\partial y}N_{n-h,\lambda}^{(r)}(x,y)$$

so the proof is complete. □

THEOREM 4.3. *For $n \geqslant 0$, we have*

$$N_{n+1,\lambda}^{(r)}(x,y)=\lambda\left[xN_{n,\lambda}^{(r)}(x,y)+yr\sum_{h=0}^{n}\binom{n}{h}N_{n-h,\lambda}^{(r)}(x,y)\,N_{h,\lambda}(1,y)\right].$$

PROOF. If we take the derivative of (2.1), with respect to t either sides of the expression, we have

$$\frac{d}{dt}\left(\sum_{n=0}^{\infty}N_{n,\lambda}^{(r)}(x,y)\frac{t^{n}}{n!}\right) = \frac{d}{dt}\left(\frac{e^{\lambda xt}}{\left(1-y\left(e^{\lambda t}-1\right)\right)^{r}}\right),$$

$$\sum_{n=1}^{\infty}N_{n,\lambda}^{(r)}(x,y)\frac{t^{n-1}}{(n-1)!}$$

$$= \lambda x\frac{e^{\lambda xt}}{\left(1-y\left(e^{\lambda t}-1\right)\right)^{r}}+y\lambda r\frac{e^{\lambda xt}}{\left(1-y\left(e^{\lambda t}-1\right)\right)^{r}}\frac{e^{\lambda t}}{\left(1-y\left(e^{\lambda t}-1\right)\right)},$$

$$\sum_{n=0}^{\infty}N_{n+1,\lambda}^{(r)}(x,y)\frac{t^{n}}{n!}$$

$$= \lambda x\sum_{n=0}^{\infty}N_{n,\lambda}^{(r)}(x,y)\frac{t^{n}}{n!}+y\lambda r\sum_{n=0}^{\infty}\sum_{h=0}^{\infty}N_{n,\lambda}^{(r)}(x,y)N_{h,\lambda}(1,y)\frac{t^{n}}{n!}\frac{t^{h}}{h!},$$

$$\sum_{n=0}^{\infty}N_{n+1,\lambda}^{(r)}(x,y)\frac{t^{n}}{n!}$$

$$= \lambda x\sum_{n=0}^{\infty}N_{n,\lambda}^{(r)}(x,y)\frac{t^{n}}{n!}+y\lambda r\sum_{n=0}^{\infty}\sum_{h=0}^{n}N_{n-h,\lambda}^{(r)}(x,y)N_{h,\lambda}(1,y)\frac{t^{n}}{(n-h)!h!},$$

$$\sum_{n=0}^{\infty} N_{n+1,\lambda}^{(r)}(x,y)\frac{t^n}{n!}$$

$$= \lambda x \sum_{n=0}^{\infty} N_{n,\lambda}^{(r)}(x,y)\frac{t^n}{n!} + y\lambda r \sum_{n=0}^{\infty}\sum_{h=0}^{n}\binom{n}{h} N_{n-h,\lambda}^{(r)}(x,y) N_{h,\lambda}(1,y)\frac{t^n}{n!}.$$

On equating like powers of $t^n/n!$ on both sides in the above expression and alter some simplification we arrive at our desired result. □

THEOREM 4.4. *The following integral representation*

$$\int_{\alpha}^{\beta} N_{n,\lambda}^{(r)}(x,y)dx = \frac{N_{n+1,\lambda}^{(r)}(\beta,y) - N_{n+1,\lambda}^{(r)}(\alpha,y)}{\lambda(n+1)}. \tag{4.7}$$

PROOF. From (4.4), we derive that

$$\begin{aligned}\int_{\alpha}^{\beta} N_{n,\lambda}^{(r)}(x,y)dx &= \int_{\alpha}^{\beta} \frac{1}{\lambda(n+1)}\frac{\partial}{\partial x} N_{n+1,\lambda}^{(r)}(x,y)dx \\ &= \frac{1}{\lambda(n+1)}\int_{\alpha}^{\beta} \frac{\partial}{\partial x} N_{n+1,\lambda}^{(r)}(x,y)dx \\ &= \frac{1}{\lambda(n+1)}\left[N_{n+1,\lambda}^{(r)}(\beta,y) - N_{n+1,\lambda}^{(r)}(\alpha,y)\right]\end{aligned}$$

which means the asserted result (4.7). □

References

1. L. Kargın, Some formulae for products of Fubini polynomials with applications. *arXiv 2016, arXiv:1701.01023v1*.
2. N. Kilar and Y. Şimşek, A new family of Fubini type numbers and polynomials associated with Apostol-Bernoulli numbers and polynomials. *J. Korean Math. Soc.*, **54**(2017), 1605-1621.
3. T. Kim, D. S. Kim, and G.-W. Jang, A note on degenerate Fubini polynomials. *Proc. Jangjeon Math. Soc.*, **20**(4)(2017), 521-531.
4. M. Masjed-Jamei, M. R. Beyki, and W. Koepf, A new type of Euler polynomials and numbers. *Mediterr. J. Math.*, **15**(3)(2018), 17 pages.
5. M. Masjed-Jamei and W. Koepf, Symbolic computation of some power-trigonometric series. *J. Symb. Comput.*, **80**(2)(2017), 273-284.
6. L. Kargın, Exponential polynomials and its applications to the related polynomials and numbers. *arXiv 2016, arXiv:1503.05444v2*.
7. D. S. Kim, T. Kim, H.-I. Kwon, and J.-W Park, Two variable higher-order Fubini polynomials. *J. Korean Math. Soc.*, **55**(2018), 975-986.
8. D. D. Su and Y. He, Some identities for the two variable Fubini polynomials. *Mathematics*, **7**(2)(2019), 115.
9. T. Kim, D. S. Kim, H. Lee, and J. Kwon, On degenerate generalized Fubini Polynomials. *AIMS Mathematics*, **7**(7)(2022), 12227–12240.
10. W. Khan and N. K S, A New Class of Hermite-Fubini Polynomials and Its Properties. *Preprints.org 2019, 2019040333*.

11. N. Özmen and E. Erkus-Duman, Some results for a family of multivariable polynomials. *AIP Conf. Proc.*, **1558**(2013), 1224-1127.
12. N. Özmen and E. Erkus-Duman, Some generating functions for a class of hypergeometric polynomials. *Gazi University Journal of Science*, **31**(4)(2018), 1179-1190.
13. N. Özmen, Bilateral and bilinear generating functions for the modified generalized Sylvester polynomials. *Facta Universitatis (NIS)*, **33**(2)(2018), 179-293.
14. N. Özmen, Some new generating functions for the modified Laguerre polynomials. *Advances in Applied Mathematcis and Mechanics*, **11**(6)(2019), 1398-1414.
15. D. Korkmaz-Düzgun and E. Erkus-Duman, Generating functions for k-hypergeometric functions. *J. Appl. Phys. Math.*, **9**(3)(2019), 119-126.
16. M. Ağca and N. Özmen, Miscellaneous Properties of Generalized Fubini Polynomials. *Commun. Adv. Math. Sci.*, **6**(1)(2023), 19-30.
17. H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*. Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York,Chichester, Brisbane, and Toronto, 1984.

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