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SECURE DOUBLE DOMINATION IN GRAPHS

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ABSTRACT. A double dominating (DD) set D_d of a graph $G = (V, E)$ is called a secure double dominating (SDD)-set in G, if for every vertex $u \in V - D_d$, there exists $v \in D_d$ is adjacent to u such that $(D_d - \{v\}) \cup \{u\}$ is a DD-set of G. The secure double domination number $\gamma_{ds}(G)$ is the minimum cardinality of an SDD - set of G. In this paper, many bounds on $\gamma_{ds}(G)$ are obtained and its exact values for some standard graphs are found. Also, its relationship with other parameters is investigated.

1. Introduction

The graphs $G = (V, E)$ considered here are simple, finite and undirected without isolated vertices, loops and multiple edges. In general, $\langle X \rangle$ denotes the sub graph induced by the set of vertices X. For any undefined term in this paper, we refer to Harary [**10**].

A set $D \subseteq V$ of vertices in a graph G is called a dominating set if every vertex $u \in V - D$ is adjacent to at least one vertex $v \in D$. The domination number $\gamma(G)$ equals the minimum cardinality of a dominating set of G. Books on domination and its related parameters [**5, 7, 9, 11–15, 19, 20**] have given sufficient impetus to the expansive growth of this area.

Let D be a minimum dominating set of G. If $V - D$ contains a dominating set D_i of G, then D_i is called an inverse dominating (ID) set of G with respect to D. The inverse domination number $\gamma_i(G)$ is the minimum cardinality of an ID-set of G. This concept was introduced by Kulli and Sigarkanti [**18**] and studied by Domke et al., [**8**].

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A dominating set $D \subseteq V$ is a secure dominating (SD) set D_s of G, if for each $u \in V - D$, there exists a vertex $v \in D$ such that $uv \in E(G)$ and $(D - \{v\}) \bigcup \{u\}$ is a dominating set of G. The secure domination number of $\gamma_s(G)$ is the minimum cardinality of an SD-set of G. This concept of protection was initiated by Cockayne et al., [**4**] and studied by [**2**, **6**, **17**, **21**].

A set $D_d \subseteq V$ is a double dominating (DD)-set for G, if each vertex in V is dominated by at least two vertices in D_d. The double domination number $\gamma_d(G)$ is the minimum cardinality of a DD-set of G. This concept was introduced by Harary and Haynes [**12**] and studied by [**3**, **22**, **23**].

Many applications of domination in graphs can be extended to secure double domination. For example, if we think of each vertex in a dominating set as a fileserver for a computer network, then each computer in the network has direct access to a fileserver. It is sometimes reasonable to assume that this access is available even when one of the fileserver goes down. A secure double dominating set provides the desired fault tolerance for such cases because each computer has access to at least two fileservers and each of the fileservers has direct access to at least one backup fileserver.

In this paper, we introduce the concept of secure double domination as follows.

A DD-set $D_d \subseteq V$ is a secure double dominating (SDD)-set of G, if for every vertex $u \in V - D_d$, there exists $v \in D_d$ is adjacent to u such that $(D_d - \{v\}) \cup \{u\}$ is a DD-set of G. The secure double domination number $\gamma_{ds}(G)$ is the minimum cardinality of an SDD - set of a graph G. A γ_{ds} - set is a minimum SDD-set of G. Similarly, other dominating related parameters can be expected. Note that $\gamma_{ds}(G)$ is defined only if G has no isolated vertices. First, we start with a couple of observations.

OBSERVATION 1.1. *If $D_d = \{u, v\}$ is an SDD-set of G, then u, v are called a secure double dominating vertices of G. A vertex u of G is said to be a γ_{ds}-required vertex of G if u lies in every γ_{ds}-set of G.*

OBSERVATION 1.2. *Let G be a connected graph with $p \geqslant 4$. Then DD - set D_d is an SDD - set if and only if for any two vertices $\{v, w\} \in D_d$, there exists a vertex $u \in V - D_d$ such that $N(u) \cap D_d = \{v, w\}$.*

OBSERVATION 1.3. *Let T be a tree with $p \geqslant 7$. Then every end-vertices and its supporting vertices must contained in SDD - set in T.*

2. Specific familes of graphs

The following computed values of $\gamma_{ds}(G)$ are stated without proof.

PROPOSITION 2.1.

(i) *For any complete graph K_p with $p \geqslant 4$,*

$$\gamma_{ds}(K_p) = 2.$$

(ii) *For any path* P_p,

$$\gamma_{ds}(P_p) = \begin{cases} \textit{does not exist}, & \textit{if } p \leqslant 6 \\ p-n, & \textit{if } 4n+3 \leqslant p \leqslant 4n+6;\ n \geqslant 1. \end{cases}$$

(iii) *For any cycle* C_p *with* $p \geqslant 3$,

$$\gamma_{ds}(C_p) = \begin{cases} p-1, & \textit{if } 3 \leqslant p \leqslant 7 \\ p-n, & \textit{if } 4n+4 \leqslant p \leqslant 4n+7;\ n \geqslant 1. \end{cases}$$

(iv) *For any complete bipartite graph* $K_{r,t}$ *with* $1 \leqslant r \leqslant t$,

$$\gamma_{ds}(K_{r,t}) = \begin{cases} \textit{does not exist}, & \textit{if } r=1;\ t \geqslant 1 \\ t+1, & \textit{if } r=2;\ t \geqslant 2 \\ 4, & \textit{if } r=3;\ t=3 \\ 5, & \textit{if } r=3,4;\ t \geqslant 4 \\ 6, & \textit{if } r \geqslant 5;\ t \geqslant 5. \end{cases}$$

To prove our next results, we make use of the following definitions:

A star is a graph $K_{1,p-1}$; $p \geqslant 3$ composed of a central vertex x_1 and $(p-1)$-other vertices only connected to x_1. A tree T_1 is a 2-subdivided star of $K_{1,p-1}$; $p \geqslant 3$, whose edges are subdivided twice, that is each edge is replaced by a path of length 3 by adding a vertex of degree 2. A tree T^* consists of a path P_n with $n \geqslant 1$ (This path is referred to as the spine of the tree T^*) and a collection of T_i, where each vertex on P_n is joined to central vertex x_i for $i = 1, 2, \ldots, n$ from each T_i (Here, $T_1 \cong T^*$, if $n = 1$ in T^*). A tree T^{**} is consisting of a path P_n with $n \geqslant 2$ and collection of P_4, where each vertex on P_n is joined to one end-vertex from each P_4, see Figure-1.

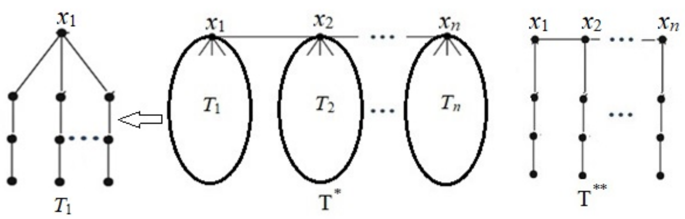

FIGURE 1. Tress of T_1, T^* and T^{**}.

PROPOSITION 2.2. *Let* T^* *and* T^{**} *be a special type of tree. Then*

(i) $\gamma_{ds}(T^*) = p - n$, *if* $n \geqslant 1$.

(ii) $$\gamma_{ds}(T^{**}) = \begin{cases} p-1, & \textit{if } 2 \leqslant n \leqslant 4 \\ 4n - \left\lceil \frac{n}{2} \right\rceil + 1, & \textit{if } n \geqslant 5. \end{cases}$$

3. Bounds and characterizations

THEOREM 3.1. *For any connected graph G with $p \geqslant 4$,*

$$\gamma_t(G) \leqslant \gamma_d(G) \leqslant \gamma_{ds}(G).$$

Furthermore, the bounds are achieved on the graph $G \cong K_p$; $p \geqslant 4$.

PROOF. Since every SDD-set is a DD-set and every DD-set is a TD-set of a connected graph G with $p \geqslant 4$. Hence the result follows.
Furthermore, we have $\gamma_t(K_p) = \gamma_d(K_p) = \gamma_{ds}(K_p) = 2$ for $p \geqslant 4$. □

Naturally, every SDD-set is an SD-set and union of any two disjoint SD-set is an SDD-set of a connected graph G. Hence, we have $\gamma_s(G) \leqslant \gamma_{ds}(G) \leqslant 2\gamma_s(G)$. These bounds can be improved slightly as follows.

THEOREM 3.2. *For any connected graph G with $p \geqslant 4$,*

$$\gamma_s(G) + 1 \leqslant \gamma_{ds}(G) \leqslant p - \gamma_s(G) + 2.$$

Furthermore, the lower bound is achieved on the graph $G \cong K_p$; $p \geqslant 4$, and the upper bound is achieved when $G \cong P_7$.

PROOF. Let G be a connected graph with $p \geqslant 4$. If D_d is a γ_{ds}-set of G, then for any $u \in V - D_d$, and $v \in D_d$, u must be adjacent to v, this implies that $D_d - \{v\} \cup \{u\}$ is a γ_s-set of a graph G, and $\gamma_s(G) \leqslant \gamma_{ds}(G) - 1$. Thus the lower bound follows.

If D_d is a SDD-set of G. If $F = \{u_1, u_2, ..., u_k\}$ be the set of vertices in G. Then $F \cup H = D_d$, where $H \subseteq V(G) - F$ forms an SD-set of G such that $|N[u]| \cap D_d \geqslant 2$ for all $u \in V(G) - D_d$ and it follows that $|D_d| \cup |D_s| \leqslant k + 2 \leqslant p + 2$. Hence $\gamma_{ds}(G) + \gamma_s(G) \leqslant p + 2$. Thus the upper bound follows.
Furthermore, if $\gamma_s(K_p) = 1$ and $\gamma_{ds}(K_p) = 2$ for $p \geqslant 4$, then the lower bound is achieved and if $\gamma_s(P_7) = 3$ and $\gamma_{ds}(P_7) = 6$, then the upper bound is achieved. □

THEOREM 3.3. *For any connected spanning subgraph H of a connected graph G with $p \geqslant 7$,*

$$\gamma_{ds}(G) \leqslant \gamma_{ds}(H).$$

Furthermore, the bound is achieved on the graph $G \cong C_{4n+3}; n \geqslant 1$.

PROOF. Since every SDD-set of a connected spanning subgraph H is an SDD-set of a connected graph G with $p \geqslant 7$. Hence the result follows.
By Proposition 2.1 (ii) and (iii), the bound is achieved on the connected spanning subgraph $H \cong P_{4n+3}; n \geqslant 1$ of a connected graph $G \cong C_{4n+3}; n \geqslant 1$. □

In the next few results, we obtain the bounds on $\gamma_{ds}(G)$ in terms of order, minimum/maximum degree and edge independence number of a graph. Further, to prove the next result, we make use of the following graph G^*.

THEOREM 3.4. *For any connected graph G with $p \geqslant 4$,*

$$2 \leqslant \gamma_{ds}(G) \leqslant p - 1.$$

Furthermore, the lower bound is attained if and only if $G \cong G^*$ or K_p; $p \geqslant 3$ and an upper bound attained if and only if $G \cong P_p$; $7 \leqslant p \leqslant 10$ or C_p; $3 \leqslant p \leqslant 7$ or T_1.

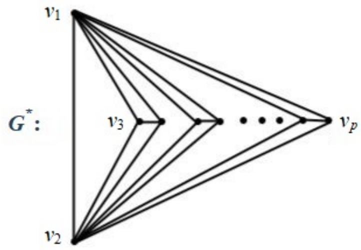

FIGURE 2. The graph G^*.

PROOF. Let G be a connected graph with $p \geqslant 4$. If every SDD-set is a DD-set D_d of G, then we have $2 \leqslant \gamma_d(G) \leqslant \gamma_{ds}(G)$. Therefore, for every SDD-set exist atleast two vertices and atmost $V - \{v\}$ vertices, where $v \in V - D_d$. Hence the lower and upper bounds are follows.
Now, we prove $\gamma_{ds}(G) = 2$ if and only if $G \cong K_p$; $p \geqslant 3$.
On the contrary, suppose the graph $G \cong G^*$ or K_p; $p \geqslant 3$, then there exists at least three vertices u, v and w such that u and v are adjacent and w is adjacent to u and v which form a cycle of length two, this implies that the set $\{u, v\}$ is an SDD-set of G, which is a contradiction. This proves the necessity and the sufficiency is obvious.
Next, we prove $\gamma_{ds}(G) = p - 1$ if and only if $G \cong P_p$; $7 \leqslant p \leqslant 10$ or C_p; $3 \leqslant p \leqslant 7$ or T_1.
On the contrary, suppose $\gamma_{ds}(G) = p - 1$ and the given condition is not satisfied, then there exists a Path P_p with $p \geqslant 11$ vertices and Cycle C_p with $p \geqslant 8$ vertices. By Proposition 2.1 (ii) and (iii), we have $\gamma_{ds}(G) < p - 1$, which is a contradiction. This proves the necessity and the sufficiency is obvious. □

THEOREM 3.5. *For any connected graph G with $p \geqslant 4$,*

$$\frac{2p}{\Delta(G) + 1} \leqslant \gamma_{ds}(G) \leqslant p - \delta(G) + 1.$$

Furthermore, the bounds are achieved when $G \cong K_p$; $p \geqslant 4$.

PROOF. Let D_d be a γ_{ds}-set of a connected graph G with $p \geqslant 4$. If t is the number of edges with one end in D_d and the other end in $V - D_d$. Since every vertex in D_d has at least one neighbor in D_d,

$$t \leqslant (\Delta(G) - 1)|D_d| = (\Delta(G) - 1)\gamma_{ds}(G).$$

Also every vertex in $V - D_d$ is adjacent to at least two vertices in D_d and so $t \geqslant 2|V - D_d| = 2(p - \gamma_{ds}(G))$. Since $2p - 2\gamma_{ds}(G) \leqslant (\Delta(G) - 1)\gamma_{ds}(G)$. Hence the lower bound follows.

Let D_d be a γ_{ds}-set of a connected graph G with $p \geqslant 4$. If $\delta(G) = 1$, then equality holds. Assume that $\delta(G) \geqslant 2$, Let v be a vertex of D_d. Thus $V - D_d$ contains all neighbors of v except one and so $deg(v) - 1 \leqslant |V - D_d| = p - (p - \delta(G) + 1) = \delta(G) - 1$. Thus all the vertices of D_d have the same degree $\delta(G)$, and $|V - D_d| = \delta(G) - 1$. Let u be a vertex of $N(v) \cap D_d$. Then u is adjacent to all the vertices of $V - D_d$ and hence at this point every vertex of $V - D_d$ is securely double dominated by u and v. Thus $D_d = \{u, v\}$ and all the vertices of $V - D_d$ are mutually adjacent. Hence the upper bound follows.

By Proposition 2.1, both lower and upper bounds are achieved when $G \cong K_p$; $p \geqslant 4$. □

THEOREM 3.6. *For any connected graph G with $\delta(G) \geqslant 2$ and $p \geqslant 4$,*

$$\beta_1(G) + 1 \leqslant \gamma_{ds}(G) \leqslant 2\beta_1(G),$$

where $\beta_1(G)$ is an edge independence number of G.
Furthermore, the lower bound is achieved on the graph $G \cong C_p$; $p = 2n; n \geqslant 2$ and an upper bound is achieved on the graph $G \cong K_p$; $p = 2n + 1; n \geqslant 2$.

PROOF. Let G be a connected graph with $\delta(G) \geqslant 2$ and and $p \geqslant 4$. If M is a maximum independent set of edges in G and D_d is the vertices in the set of edges of M. Since $V - D_d$ is an independent set, each $v \in V - D_d$ must have at least two neighbors in D_d. Further, the vertices of D_d secure double dominate themselves. This implies that SDD - set D_d for at least $\mid M \mid + 1$ and atmost twice the $\mid M \mid$ of G. Thus the result follows. □

THEOREM 3.7. *Let D_d be an SDD-set of a connected graph G with $p \geqslant 4$. Then one of the following conditions holds:*

(i) *D_d exists if and only if every vertex in $V - D_d$ contains at least two strong neighbors.*
(ii) $|V - D_d| \leqslant |D_d|$.

PROOF. Let D_d be an SDD-set of a connected graph G with $p \geqslant 4$. Suppose a vertex $u \in V - D_d$ has only one strong neighbors, other vertices in $V - D_d$ have at least two strong neighbors. Then for every $u \in V - D_d$, there exists a vertex $v \in D_d$ such that D_d is an SDD - set of G, which is a contradiction. Hence every vertex in $V - D_d$ should contain at least two strong neighbors. □

4. Inverse secure double domination

Let D_d be a γ_{ds}-set of a connected graph G with $p \geqslant 4$. If $V - D_d$ contains a SDD - set D_d^1 of G, then D_d^1 is called an inverse secure double dominating (ISDD) set with respect to D_d. The inverse secure double domination number $\gamma_{ds}^{-1}(G)$ is the minimum cardinality of an ISDD - set of G. A γ_{ds}^{-1} - set is a minimum ISDD - set of G.

OBSERVATION 4.1. *Not all graphs have an ISDD - set. For example, the graph $G \cong P_p$; $7 \leqslant p \leqslant 10$ or C_p; $3 \leqslant p \leqslant 7$ or T_1.*

An application of ISDD-set is found in a Computer Network. We consider a computer network in which a core group of file servers has the ability to communicate directly with every computer outside the core group. In addition, each file server is directly linked with at least one other backup file server where duplicate information is stored. A minimum core group with this property is the smallest SDD-set for the graph representing the network. If a second important core group is needed then a separate disjoint SDD-set provides duplication in case the first is corrupted in some way. We have $\gamma_{ds}(G) \leqslant \gamma_{ds}^{-1}(G)$. From the point of networks, one may demand $\gamma_{ds}(G) = \gamma_{ds}^{-1}(G)$, whereas many graphs do not enjoy such a property. For example, we consider graph G^* (see, Figure 2). Then $\gamma_{ds}(G^*) = 2$ and $\gamma_{ds}^{-1}(G^*) = p - 2$.

In this case, if p is large in graph G^*, then $\gamma_{ds}^{-1}(G^*)$ is sufficiently large compare to $\gamma_{ds}(G^*)$.

Now, we characterize the $\gamma_{ds}(G) = \gamma_{ds}^{-1}(G)$.

Proposition 4.1. *If K_p is a complete graph with $p \geqslant 4$, then*

$$\gamma_{ds}(K_p) = \gamma_{ds}^{-1}(K_p).$$

Proposition 4.2. *Let G be a simple graph with $p \geqslant 4$. If $\gamma_{ds}(G) = \gamma_{ds}^{-1}(G)$, then G has no γ_{ds}-required vertex.*

Proof. Let G be a graph with $\gamma_{ds}(G) = \gamma_{ds}^{-1}(G)$. Let D_d be a γ_{ds}-set of G. Suppose G contains a γ_{ds}-required vertex u. Then u lies in every γ_{ds}-set of G. Thus $u \in D_d$ and $u \in D_d^{-1}$, which is a contradiction to $D_d^{-1} \subseteq V - D_d$. □

Proposition 4.3. *If u, v are secure double dominating vertices of a simple graph G, then $\gamma_{ds}^{-1}(G) = \gamma_{ds}(G - u - v)$.*

Proof. Since u, v are secure double dominating vertices of G, $\{u, v\}$ is a γ_{ds}-set of G. Thus any γ_{ds}^{-1}-set of G lies in $G - \{u, v\}$ and is a minimum SDD-set of $G - \{u, v\}$. Hence $\gamma_{ds}^{-1}(G) = \gamma_{ds}(G - u - v)$. □

Proposition 4.4. *Let G and H be two nontrivial complete graphs. Then*

$$\gamma_{ds}(G + H) = \gamma_{ds}^{-1}(G + H) = 2.$$

Proof. If G and H are two nontrivial complete graphs, then $G + H$ is a complete graph with at least 4 vertices. By Proposition 4.1, we have $\gamma_{ds}(G + H) = \gamma_{ds}^{-1}(G + H) = 2$. □

Proposition 4.5. *Let G be a connected graph with $p \geqslant 4$ vertices. If a γ_{ds}^{-1}-set exists in G, then*

(i) $\gamma_{ds}(G) + \gamma_{ds}^{-1}(G) \leqslant p$.
(ii) $2 \leqslant \gamma_{ds}^{-1}(G) \leqslant p - 2$.

Furthermore, the complete graph K_4 realizes the sharp bounds on (i) and (ii).

Proof.

(i) This follows from the definition of $\gamma_{ds}^{-1}(G)$.
(ii) By (i) and $2 \leqslant \{\gamma_{ds}(G), \gamma_{ds}^{-1}(G)\}$. We have $\gamma_{ds}^{-1}(G) \leqslant p - \gamma_{ds}(G)$. Thus $2 \leqslant \gamma_{ds}^{-1}(G) \leqslant p - 2$ follows.

By Proposition 4.1, the complete graph K_4 realizes the sharp bounds on (i) and (ii). □

Construct the graph G^{**} as follows: Let $e_i = u_i v_i, 1 \leqslant i \leqslant m$ and $e_{i+1} = v_i u_{i+1}$ be the edges of a cycle C_{2m}. For each $e_i = u_i v_i$, join the vertices $u_i and v_i$ to two new vertices x_i, y_i and also join the vertices x_i to the vertices y_i to form the graph G^{**}, see Figure-3.

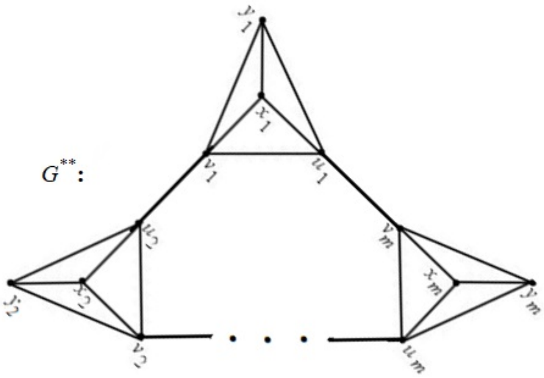

Figure 3. The graph G^{**}.

Proposition 4.6. *Let G^{**} be a graph with $4m$ - vertices. Then*

$$\gamma_{ds}(G^{**}) = \gamma_{ds}^{-1}(G^{**}) = 2m.$$

Proof. In the graph G^{**} of Figure-3, the vertex set $V(G^{**}) = \{u_1, \ldots, u_m, v_1, \ldots, v_m, x_1, \ldots, x_m, y_1, \ldots, y_m\}$. Then $D_d = \{u_1, \ldots, u_m, v_1, \ldots, v_m\}$ is a γ_{ds}-set with $2m$ - vertices and $D_d^{-1} = \{x_1, \ldots, x_m, y_1, ..., y_m\}$ is a γ_{ds}^{-1}-set with $2m$ - vertices. Thus $\gamma_{ds}(G^{**}) = \gamma_{ds}^{-1}(G^{**}) = 2m$. □

Remark 4.1. Let $G_1, G_2, \ldots, G_m$ be the m - connected components of a graph G. Let D_d^i be a γ_{ds}-set of G_i, and $(D_d^i)^{-1}$ be a γ_{ds}^{-1}-set of G_i, for $i = 1, 2, \ldots, m$. Then $D_d^1 \cup D_d^2 \cup \ldots \cup D_d^m$ is a γ_{ds}-set of G and $(D_d^1)^{-1} \cup (D_d^2)^{-1} \cup \ldots \cup (D_d^m)^{-1}$ is a γ_{ds}^{-1}-set of G. Thus $\gamma_{ds}(G) = \sum_{i=1}^{m} \gamma_{ds}(G_i)$ and $\gamma_{ds}^{-1}(G) = \sum_{i=1}^{m} \gamma_{ds}^{-1}(G_i)$.

Theorem 4.1. *Let $G_1, G_2, \ldots, G_m$ be the m - connected components of a graph G. Then $\gamma_{ds}(G) = \gamma_{ds}^{-1}(G)$ if and only if $\gamma_{ds}(G_i) = \gamma_{ds}^{-1}(G_i)$, for $i = 1, 2, \ldots, m$.*

Proof. Let $G_1, G_2, \ldots, G_m$ be the m - connected components of G. By above Remark, $\gamma_{ds}(G) = \sum_{i=1}^{m} \gamma_{ds}(G_i)$ and $\gamma_{ds}^{-1}(G) = \sum_{i=1}^{m} \gamma_{ds}^{-1}(G_i)$.

Therefore, $\gamma_{ds}(G) = \gamma_{ds}^{-1}(G)$ if $\gamma_{ds}(G_i) = \gamma_{ds}^{-1}(G_i)$, for $i = 1, 2, \ldots, m$.

Conversely suppose $\gamma_{ds}(G) = \gamma_{ds}^{-1}(G)$. We have $\gamma_{ds}(G_i) = \gamma_{ds}^{-1}(G_i)$ for $i = 1, 2, \ldots, m$. We now prove that $\gamma_{ds}(G_i) = \gamma_{ds}^{-1}(G_i)$ for $i = 1, 2, \ldots, m$. On the contrary, assume $\gamma_{ds}(G_i) < \gamma_{ds}^{-1}(G_i)$ for some i. Then $\gamma_{ds}(G_j) > \gamma_{ds}^{-1}(G_j)$ for some j; $i \neq j$, which is a contradiction. Thus $\gamma_{ds}(G_i) = \gamma_{ds}^{-1}(G_i)$ for $i = 1, 2, \ldots, m$. □

5. Conclusion and open problems

In this article, we combine two domination-related parameters such as secure domination and double domination to form the secure double domination in graphs. For the comparative advantages, applications, and mathematical properties point of view, many concepts and questions are suggested by this research, among them are the following.

5.1. Secure double domination vs secure 2-domination. A set $D \subseteq V$ is a 2-dominating set if every vertex in $V - D$ has at least two neighbors in D. A 2-dominating set D is secure 2-dominating set if for every vertex u in $V - D$ such that $v \in (D \cap N(u))$ such that $D - \{v\} \cup \{u\}$ is a 2-dominating set. The secure 2-domination number $\gamma_{2s}(G)$ is the minimum cardinality of a secure 2-dominating set of G. For more details, one can refer to [**1**, **16**].

1. Generally, for any graph G with $p \geqslant 4$, we have $\gamma_{2s}(G) \leqslant \gamma_{ds}(G)$. Therefore, characterize when $\gamma_{2s}(G) = \gamma_{ds}(G)$?
2. Analogously, define an inverse secure 2-domination number $\gamma_{2s}^{-1}(G)$ and find some bounds and characterization. Also, characterize when $\gamma_{2s}^{-1}(G) = \gamma_{ds}^{-1}(G)$?

5.2. Unsecure double domination (or, Self double domination). A DD-set D_d is an unsecure double dominating (UDD) set in a connected graph G. If for every vertex $u \in V - D_d$, there exists $v \in D_d$ is adjacent to u such that $(D_d - \{v\}) \cup \{u\}$ is not a DD-set of G. An unsecure double domination number $\gamma_{uds}(G)$ is the minimum cardinality of a UDD-set of G.

1. Find $\gamma_{uds}(G)$ for specific families of graphs. Also, obtain some bounds and characterization of $\gamma_{uds}(G)$
2. Compare $\gamma_{uds}(G)$ with other domination related parameters.

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