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SOME RESULTS FOR INTERPOLATIVE CARISTI MAPPINGS IN A NEW EXTENSION OF M_B-METRIC SPACES

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Abstract. In this study, we express the concept of M_b-metric space, which is a combination of M $-$metric space and $b-$metric spaces, and this space is the generalization of both spaces. We present some interpolative type contraction mappings and prove fixed point theorems in such spaces. Finally, we give some examples for such mappings and spaces.

1. Introduction and preliminaries

Metric spaces are one of the spaces where fixed point theory has been studied most intensively. In recent years, many authors have defined different generalizations of metric spaces and have worked on them on very different aspects of fixed point theory. Some of the generalizations of the metric space are $b-$metric, extended $b-$metric, $M-$metric and extended M_b-metric spaces.

Now we will give the definitions of the spaces expressed above, respectively, as follows.

The concept of $b-$metric space, which is a generalization of metric spaces, was defined by Bakhtin [**8**] in 1989. Since then, many authors have examined the fixed points of different transformation classes on these spaces and have proven different theorems for different iteration methods using the convexity of this space (see [**9**], [**15**], [**16**], [**20**], [**22**]).

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Definition 1.1. *[**8**] Assume that $\Lambda \neq \emptyset$, $b \in \mathbb{R}$ and $b \geqslant 1$. The $\rho_b : \Lambda \times \Lambda \rightarrow \mathbb{R}^+$ mapping that satisfies the following conditions is called a $b-$metric and the pair (Λ, ρ_b) is called $b-$metric space. If for all $u, v, w \in \Lambda$,*
(i) $\rho_b(u,v) = 0 \Longleftrightarrow u = v$;
(ii) $\rho_b(u,v) = \rho_b(v,u)$;
(iii) $\rho_b(u,w) \leqslant b\left[\rho_b(u,v) + \rho_b(v,w)\right]$.

After, Kamran et al. [**16**] introduced extended $b-$metric spaces which is a generalization of the concept of $b-$metric space. They also proved some fixed point theorems for mappings defined on such spaces.

Definition 1.2. *[**16**] Assume that $\Lambda \neq \emptyset$ and $\varphi : \Lambda \times \Lambda \rightarrow [1, \infty)$. The $\rho_\varphi : \Lambda \times \Lambda \rightarrow [0, \infty)$ mapping that satisfies the following conditions is called an extended $b-$metric and the pair (Λ, ρ_φ) is said to be an extended $b-$metric space. If for all $u, v, w \in \Lambda$,*
(i) $\rho_\varphi(u,v) = 0 \Longleftrightarrow u = v$;
(ii) $\rho_\varphi(u,v) = \rho_\varphi(v,u)$;
(iii) $\rho_\varphi(u,w) \leqslant \varphi(u,w)\left[\rho_\varphi(u,v) + \rho_\varphi(v,w)\right]$.

In 2019, Aydi et al. [**7**] replaced the modified triangle inequality with a functional triangle inequality. And, they defined the notion of extended $b-$metric spaces as follows. They also established some fixed point results for nonlinear contractive mappings in such spaces.

Definition 1.3. *[**7**] Assume that $\Lambda \neq \emptyset$ and $\varphi : \Lambda^3 \rightarrow [1, \infty)$. The $\rho_\varphi : \Lambda \times \Lambda \rightarrow [0, \infty)$ mapping that satisfies the following conditions is called a new extended $b-$metric and the pair (Λ, ρ_φ) is said to be a new extended $b-$metric space. If for all $u, v, w \in \Lambda$,*
(i) $\rho_\varphi(u,v) = 0 \Longleftrightarrow u = v$;
(ii) $\rho_\varphi(u,v) = \rho_\varphi(v,u)$;
(iii) $\rho_\varphi(u,w) \leqslant \varphi(u,v,w)\left[\rho_\varphi(u,v) + \rho_\varphi(v,w)\right]$.

Very recetly, Asadi [**4**] first introduced the concept of $M-$metric space which includes the partial metric space.

Definition 1.4. *[**4**] Assume that $\Lambda \neq \emptyset$. The $m : \Lambda \times \Lambda \rightarrow \mathbb{R}^+$ mapping that satisfies the following conditions is called an $M-$metric and the pair (Λ, m) is said to be an $M-$metric space. If for all $u, v, w \in \Lambda$,*
(i) $m(u,u) = m(v,v) = m(u,v) \Longleftrightarrow u = v$,
(ii) $m_{uv} \leqslant m(u,v)$,
(iii) $m(u,v) = m(v,u)$,
(iv) $m(u,v) - m_{uv} \leqslant (m(u,w) - m_{uw}) + (m(w,v) - m_{wv})$.

Motivated by above studies Ozgur et al. [**19**] introduce the notion of an extended M_b-metric space by using extended $b-$metric space and $M-$metric space. They also proved some widely known fixed point theorems such as the Banach's fixed-point theorem, the Kannan's fixed-point theorem and the Chatterjea's fixed-point theorem.

DEFINITION 1.5. *[**19**] Assume that $\Lambda \neq \emptyset$ and $\varphi : \Lambda^2 \to [1,\infty)$. The $m_\varphi : \Lambda \times \Lambda \to \mathbb{R}^+$ mapping that satisfies the following conditions is called an extended M_b-metric and the pair (Λ, m_φ) is said to be an extended M_b-metric space. If for all $u, v, w \in \Lambda$,*

(i) $m_\varphi(u,u) = m_\varphi(u,v) = m_\varphi(v,v) \Longleftrightarrow u = v$,

(ii) $m_{\varphi u,v} \leqslant m_\varphi(u,v)$,

(iii) $m_\varphi(u,v) = m_\varphi(v,u)$,

(iv) $m_\varphi(u,v) - m_{\varphi_{u,v}} \leqslant \varphi(u,v) \begin{bmatrix} \left(m_\varphi(u,w) - m_{\varphi_{u,w}}\right) \\ +\left(m_\varphi(w,v) - m_{\varphi_{w,v}}\right) \end{bmatrix}$.

Considering the different spaces defined above, many authors have obtained some widely known fixed point theorems and many fixed point results for different mapping classes (see [**1**], [**2**], [**3**], [**4**], [**5**], [**6**], [**10**], [**11**], [**12**], [**13**], [**14**], [**16**], [**17**], [**18**], [**21**]).

2. Main results

Based on the spaces given in the Introduction and Preliminaries section and the studies carried out, we defined a space that we called a new extended M_b-metric space as follows. After, we give the definition of interpolative Caristi type contractive mapping using concept of interpolative type mappings in such spaces. Finally, we prove some fixed point theorems in this metric space.

DEFINITION 2.1. *Assume that $\Lambda \neq \emptyset$ and $\varphi : \Lambda^3 \to [1,\infty)$. The $m_\varphi : \Lambda^3 \to [1,\infty)$ mapping that satisfies the following conditions is called a new extended M_b-metric and the pair (Λ, m_φ) is said to be a new extended M_b-metric space (shortly "ne$M_b$ms"). If for all $u, v, w \in \Lambda$,*

(1) $m_\varphi(u,u) = m_\varphi(v,v) = m_\varphi(u,v) \Longleftrightarrow u = v$,

(2) $m_{\varphi_{u,v}} \leqslant m_\varphi(u,v)$,

(3) $m_\varphi(u,v) = m_\varphi(v,u)$,

(4) $m_\varphi(u,v) - m_{\varphi_{u,v}} \leqslant \varphi(u,v,w)\left[m_\varphi(u,w) - m_{\varphi_{u,w}} + m_\varphi(v,w) - m_{\varphi_{w,v}}\right]$.

REMARK 2.1. If we take $\varphi(u,v,w) = \varphi(u,v)$, Definition 2.1 coincides with Definition 1.5. Moreover, taking $\varphi(u,v,w) = 1$, it reduces to the definition of an $M-$metric space.

EXAMPLE 2.1. *Let $\Lambda = C([0,1],\mathbb{R})$ be the set of all continuous real valued functions on $[0,1]$. We suppose that the functions $\varphi : \Lambda^3 \to [1,\infty)$ and $m_\varphi : \Lambda^2 \to [0,\infty)$ defined by*

$$\varphi(f(w), g(w), h(w)) = |f(w)| + |g(w)| + |h(w)| + 3$$

and

$$m_\varphi(f(w), g(w)) = \sup\left\{|f(w) - g(w)|^2 : w \in [0,1]\right\}.$$

Then (Λ, m_φ) is a ne$M_b$ms.

Some topological notions on a ne$M_b$ms are given below.

EXAMPLE 2.2. *$\Lambda = \{2,4,6\}$ and the function $\varphi : \Lambda^3 \to [1,\infty)$ be defined by*

$$\varphi(u,v,w) = u+v+w$$

for all $u,v \in \Lambda$.Let us define the function $m_\varphi : \Lambda^2 \to [0,\infty)$ as

$$\begin{aligned} m(2,2) &= 1, m(4,4) = 2, m(6,6) = 3 \\ m(2,4) &= m(4,2) = 4, \\ m(2,6) &= m(6,2) = 5, \\ m(4,6) &= m(6,4) = 10, \end{aligned}$$

for all $u,v \in \Lambda$.Then m_φ is a neM_bms. But it is not an $M-$metric or partial metric. Indeed, for $u=4, v=6$ and $w=2$, we have

$$m_\varphi(4,6) - m_{\varphi_{4,6}} = 8 \leqslant \left[\left(m_\varphi(4,2) - m_{\varphi_{4,2}}\right) + \left(m_\varphi(2,6) - m_{\varphi_{2,6}}\right)\right] = 7$$

and

$$m_\varphi(4,6) = 10 \leqslant m_\varphi(4,2) + m_\varphi(2,6) - m_\varphi(6,6) = 8.$$

This is a contradiction. Thus the condition of partial metric spaces and the condition (4) given in Definition 1.4 are not satisfied.

DEFINITION 2.2. *Let (Λ, m_φ) be a neM_bms. Then*

(a) a sequence $\{u_n\}$ in Λ converges to a point $u \Leftrightarrow$

$$\lim_{n\to\infty} m_\varphi(u_n,u) - m_{\varphi_{u_n,u}} = 0,$$

(b) a sequence $\{u_n\}$ in Λ is said to be $m_\varphi-$Cauchy sequence if

$$\lim_{n,m\to\infty} m_\varphi(u_m,u_n) - m_{\varphi_{u_m,u_n}},$$

and

$$\lim_{n,m\to\infty} M_{\varphi_{u_m,u_n}} - m_{\varphi_{u_m,u_n}}$$

exist and finite;

(c) a neM_bms is said to be $m_\varphi-$complete if every $m_\varphi-$Cauchy sequence $\{u_n\}$ converges to a point u such that

$$\lim_{n\to\infty} m_\varphi(u_n,u) - m_{\varphi_{u_n,u}} = 0$$

and

$$\lim_{n\to\infty} M_{\varphi_{u_n,u}} - m_{\varphi_{u_n,u}} = 0.$$

LEMMA 2.1. *Let (Λ, m_φ) be a neM_bms. If the sequence $\{u_n\}$ in Λ converges two point u and v with $u \neq v$, then we have $m_\varphi(u,v) - m_{\varphi_{u,v}} = 0$.*

PROOF. Let $u_n \to u$ and $u_n \to v$ with $u \neq v$. Then we obtain

$$\lim_{n\to\infty} m_\varphi(u_n,u) - m_{\varphi_{u_n,u}} = 0$$

and

$$\lim_{n\to\infty} m_\varphi(u_n,v) - m_{\varphi_{u_n,v}} = 0.$$

From the conditions (3) and (4) in Definition 2.1, we get

$$m_\varphi(u,v) - m_{\varphi_{u,v}} \leqslant \varphi(u,v,w)\left[m_\varphi(u,u_n) - m_{\varphi_{u,u_n}} + m_\varphi(u_n,v) - m_{\varphi_{u_n,v}}\right]$$

and

$$\lim_{n\to\infty}\left[m_\varphi(u,v)-m_{\varphi_{u,v}}\right]\leqslant\lim_{n\to\infty}\varphi(u,v,w)\left[\begin{array}{c}\lim_{n\to\infty}m_\varphi(u,u_n)-m_{\varphi_{u,u_n}}\\+\lim_{n\to\infty}m_\varphi(u_n,v)-m_{\varphi_{u_n,v}}\end{array}\right].$$

Using the condition (2) given in Definition 2.1, we get $m_\varphi(u,v)-m_{\varphi_{u,v}}=0$. □

From the proof of the above lemma, it is clearly seen that the limit of a sequence is not to unique.

Lemma 2.2. *Let (Λ, m_φ) be a neM$_b$ms such that m_φ is continuous. Then every convergent sequence has a unique limit.*

Lemma 2.3. *Suppose that (Λ, m_φ) is a neM$_b$ms, the mapping m_φ is continuous and $L:\Lambda\to\Lambda$ is a mapping. If there exists $k\in[0,1)$ such that*

$$m_\varphi(Lu,Lv)\leqslant km_\varphi(u,v)\tag{2.1}$$

for all $u,v\in\Lambda$, then the sequence $\{u_n\}$ is defined by $u_{n+1}=Lu_n$. If $u_n\to w$ as $n\to\infty$, then $Lu_n\to Lw$ as $n\to\infty$.

Proof. Taking $m_\varphi(Lu_n,Lw)=0$, then $m_{\varphi_{Lu_n,Lw}}=0$. From $m_{\varphi_{Lu_n,Lw}}\leqslant m_\varphi(Lu_n,Lw)$, we obtain

$$m_\varphi(Lu_n,Lw)-m_{\varphi_{Lu_n,Lw}}\to 0 \text{ as } n\to\infty$$

which implies that $Lu_n\to Lw$ as $n\to\infty$.

So, we may assume that $m_\varphi(Lu_n,Lw)>0$, from (2.1) we have $m_\varphi(Lu_n,Lw)\leqslant m_\varphi(u_n,w)$. So there are the following two steps:

Step 1: If $m_\varphi(w,w)\leqslant m_\varphi(u_n,u_n)$, then it is easy to see that $m_\varphi(u_n,u_n)\to 0$, which implies that $m_\varphi(w,w)=0$ and since $m_\varphi(Lw,Lw)<m_\varphi(w,w)=0$ we deduce that

$$m_\varphi(Lw,Lw)=m_\varphi(w,w)=0 \text{ and } m_\varphi(u_n,w)\to 0,$$

on the other words we have

$$m_\varphi(Lu_n,Lw)\leqslant m_\varphi(u_n,w)\to 0.$$

Hence, $m_\varphi(Lu_n,Lw)-m_{\theta_{Lu_n,Lw}}\to 0$ and thus $Lu_n\to Lw$.

Step 2: If $m_\varphi(w,w)\geqslant m_\varphi(u_n,u_n)$, and once again it is easy to see that $m_\varphi(u_n,u_n)\to 0$, which implies that $m_{\varphi_{u_n,w}}\to 0$ and since $m_\varphi(Lu_n,Lw)<m_\varphi(u_n,w)\to 0$ then we have $m_\varphi(Lu_n,Lw)-m_{\varphi_{Lu_n,Lw}}\to 0$ and thus $Lu_n\to Lw$ as desired. □

Definition 2.3. *Suppose that (Λ, m_φ) is a neM$_b$ms and $L:\Lambda\to\Lambda$ is a mapping. This mapping is called*

a) interpolative Kannan-Caristi contractive mapping (shortly "IKCCM") if there exists a function $g:\Lambda\to[0,\infty)$ and $\eta\in(0,1)$ such that

$$m_\varphi(Lu,Lv)\leqslant[g(u)-g(Lu)]^\eta.m_\varphi(u,Lu)^\eta m_\varphi(v,Lv)^{1-\eta}$$

for all $u,v\in\Lambda-Fix(L)$,

b) interpolative Reich-Caristi contractive mapping (shortly "IRCCM") if there exists a function $g:\Lambda\to[0,\infty)$ and $w,\eta\in(0,1)$ such that

$$m_\varphi\left(Lu,Lv\right)\leqslant\left[g\left(u\right)-g\left(Lu\right)\right]^{w+\eta}.m_\varphi\left(u,v\right)^{w}.m_\varphi\left(u,Lu\right)^{\eta}.m_\varphi\left(v,Lv\right)^{1-w-\eta}$$

for all $u,v\in\Lambda-Fix\left(L\right)$.

c) interpolative Jaggi-Caristi contractive mapping (shortly "IJCCM") if there exists a function $g:\Lambda\to[0,\infty)$ and $w\in(0,1)$ such that

$$m_\varphi\left(Lu,Lv\right)\leqslant\left[g\left(u\right)-g\left(Lu\right)\right]^{w}.m_\varphi\left(u,v\right)^{w}.\left[\frac{m_\varphi\left(u,Lu\right).m_\varphi\left(v,Lv\right)}{m_\varphi\left(u,v\right)}\right]^{1-w}$$

for all $u,v\in\Lambda-Fix\left(L\right)$.

d) interpolative Das–Gupta-Caristi contractive mapping (shortly "IDGCCM") if there exists a function $g:\Lambda\to[0,\infty)$ and $w\in(0,1)$ such that

$$m_\varphi\left(Lu,Lv\right)\leqslant\left[g\left(u\right)-g\left(Lu\right)\right]^{w}.m_\varphi\left(u,v\right)^{w}.\left[\frac{\left(1+m_\varphi\left(u,Lu\right)\right).m_\varphi\left(v,Lv\right)}{1+m_\varphi\left(u,v\right)}\right]^{1-w}$$

for all $u,v\in\Lambda-Fix\left(L\right)$.

EXAMPLE 2.3. *Let $\Lambda=\{2,4,6\}$ and the mapping $m_\varphi:\Lambda^2\to[0,\infty)$*

$$\begin{aligned} m_\varphi(0,0) &= 1, m_\varphi(2,2)=2, m_\varphi(4,4)=3\\ m_\varphi(0,2) &= m_\varphi(2,0)=4,\\ m_\varphi(0,4) &= m_\varphi(4,0)=5,\\ m_\varphi(2,4) &= m_\varphi(4,2)=6, \end{aligned}$$

Assume that the mapping $L:\Lambda\to\Lambda, L(u)=4-u$, $g:\Lambda\to\Lambda, g(u)=3u$ and $\eta=2/3$. Then the mapping L is an IKCCM. For $u=0$, $v=4$ and $\forall u,v\in\Lambda-Fix(L)$,

$$m_\varphi\left(L(0),L(4)\right)\leqslant\left[g\left(0\right)-g\left(L(0)\right)\right]^{\eta}.m_\varphi(0,L(0))^{\eta}m_\varphi\left(4,L(4)\right)^{1-\eta}$$

$$m_\varphi\left(4,0\right)\leqslant\left[0-12\right]^{2/3}.m_\varphi(0,4)^{2/3}m_\varphi\left(4,0\right)^{1-2/3}$$

$$m_\varphi\left(4,0\right)\leqslant\left(-12\right)^{2/3}.m_\varphi(0,4)^{2/3}m_\varphi\left(4,0\right)^{1/3}.$$

LEMMA 2.4. *Let (Λ,m_φ) be a neM_bms and $\{u_n\}$ be a sequence such that $m_\varphi\left(u_n,u_{n+1}\right)\leqslant km_\varphi\left(u_{n-1},u_n\right)$ for all $n\geqslant 2$, where $k\in(0,1)$ and $\lim_{n,m\to\infty}\eta\left(u_n,u_{n+1},u_m\right)<\frac{1}{k}$, then $\{u_n\}$ is $m_\varphi-$Cauchy sequence in Λ.*

PROOF. From the inequality in the expression of the lemma, we write

$$\begin{aligned} m_\varphi\left(u_n,u_{n+1}\right) &\leqslant km_\varphi\left(u_{n-1},u_n\right)\\ &\leqslant k^2m_\varphi\left(u_{n-2},u_{n-1}\right)\\ &\ \ \vdots\\ &\leqslant k^{n-1}m_\varphi\left(u_1,u_2\right). \end{aligned}$$

For all $n \in \mathbb{N}$ and $p = 1, 2, \ldots,$ we get

$$\begin{aligned} m_\varphi(u_n, u_{n+p}) - m_{\varphi_{u_n,u_{n+p}}} &\leqslant \eta(u_n, u_{n+1}, u_{n+p})\,[m_\varphi(u_n, u_{n+1}) - m_{\varphi_{u_n,u_{n+1}}} \\ &\quad + m_\varphi(u_{n+1}, u_{n+p}) - m_{\varphi_{u_{n+1},u_{n+p}}}]. \end{aligned}$$

Then

$$\begin{aligned} m_\varphi(u_{n+1}, u_{n+p}) - m_{\varphi_{u_{n+1},u_{n+p}}} &\leqslant \eta(u_{n+1}, u_{n+2}, u_{n+p})\,[m_\varphi(u_{n+1}, u_{n+2}) - m_{\varphi_{u_{n+1},u_{n+2}}} \\ &\quad + m_\varphi(u_{n+2}, u_{n+p}) - m_{\varphi_{u_{n+2},u_{n+p}}}] \end{aligned}$$

and

$$\begin{aligned} m_\varphi(u_{n+2}, u_{n+p}) - m_{\varphi_{u_{n+2},u_{n+p}}} &\leqslant \eta(u_{n+2}, u_{n+3}, u_{n+p})\,[m_\varphi(u_{n+2}, u_{n+3}) - m_{\varphi_{u_{n+2},u_{n+3}}} \\ &\quad + m_\varphi(u_{n+3}, u_{n+p}) - m_{\varphi_{u_{n+3},u_{n+p}}}] \\ &\ \ \vdots \end{aligned}$$

This implies that

$$\begin{aligned} m_\varphi(u_n, u_{n+p}) - m_{\varphi_{u_n,u_{n+p}}} &\leqslant \eta(u_n, u_{n+1}, u_{n+p})\,.m_\varphi(u_n, u_{n+1}) + \\ &\quad \eta(u_n, u_{n+1}, u_{n+p})\,.\eta(u_{n+1}, u_{n+2}, u_{n+p})\,.m_\varphi(u_{n+1}, u_{n+2}) \\ &\quad + \eta(u_n, u_{n+1}, u_{n+p})\,.\eta(u_{n+1}, u_{n+2}, u_{n+p})\,. \\ &\quad \eta(u_{n+2}, u_{n+3}, u_{n+p})\,.m_\varphi(u_{n+2}, u_{n+3}) + \ldots \\ &\quad + \eta(u_n, u_{n+1}, u_{n+p}) \ldots \eta(u_{n+p-2}, u_{n+p-1}, u_{n+p}) \\ &\quad .\left[m_\varphi(u_{n+p-2}, u_{n+p-1}) + m_\varphi(u_{n+p-1}, u_{n+p})\right]. \end{aligned}$$

From above inequality, we get

$$\begin{aligned} & m_\varphi(u_n, u_{n+p}) - m_{\varphi_{u_n,u_{n+p}}} \\ \leqslant & \left[\sum_{r=n}^{n+p-1} k^{r-1} \prod_{s=1}^{r} \eta(u_s, u_{s+1}, u_{n+p})\right] m_\varphi(u_1, u_2). \end{aligned}$$

We suppose that $\delta_j^{(n+p)} = k^{j-1} \prod_{s=1}^{j} \eta(u_s, u_{s+1}, u_{n+p})$ for all $j \in \mathbb{N}$. Then for any $p = 1, 2, \ldots,$

$$\begin{aligned} \lim_{n\to\infty} \frac{\delta_{n+1}^{(n+p)}}{\delta_n^{(n+p)}} &= \lim_{n\to\infty} \frac{k^n \prod_{s=1}^{n+1} \eta(u_s, u_{s+1}, u_{n+p})}{k^{n-1} \prod_{s=1}^{n} \eta(u_s, u_{s+1}, u_{n+p})} \\ &= \lim_{n\to\infty} kv(u_{n+1}, u_{n+2}, u_{n+p}) < 1. \end{aligned}$$

Using ratio test, we obtain

$$\begin{aligned} & \lim_{n\to\infty} m_\varphi(u_n, u_{n+p}) - m_{\varphi_{u_n,u_{n+p}}} \\ \leqslant & \lim_{n\to\infty} \left[\sum_{r=n}^{n+p-1} k^{r-1} \prod_{s=1}^{r} \eta(u_s, u_{s+1}, u_{n+p})\right] m_\varphi(u_1, u_2) \\ = & \ 0. \end{aligned}$$

Therefore $\{u_n\}$ is $m_\varphi-$Cauchy sequence in Λ. $\square$

THEOREM 2.1. *Let (Λ, m_φ) be a complete neM$_b$ms such that m_φ is continuous and L be an IKCCM. Let also the sequence $\{u_n\}$ be the Picard iteration generated by $u_0 \in \Lambda$. If $\lim_{n,m\to\infty} \eta(u_n, u_{n+1}, u_m)$ is finite for some $u_0 \in \Lambda$, then the sequence $\{u_n\}$ converges to $u \in \Lambda$. Also if $\lim_{n\to\infty} \eta(u, u_n, Lu)$ is finite then u is a fixed point of L in Λ.*

PROOF. If $u_n = u_{n+1}$ for some $n \geqslant 0$ then L has a fixed point in Λ. So we suppose that $m_\varphi(u_{n-1}, Lu_{n-1}) > 0$ for all $n \in \mathbb{N}$. Taking $u = u_{n-1}$ and $v = u_n$, we have

$$m_\varphi(u_n, u_{n+1}) \leqslant [g(u_{n-1}) - g(u_n)]^\eta . m_\varphi(u_{n-1}, u_n)^\eta . m_\varphi(u_n, u_{n+1})^{1-\eta}$$

which implies that

$$m_\varphi(u_n, u_{n+1})^\eta \leqslant [g(u_{n-1}) - g(u_n)]^\eta . m_\varphi(u_{n-1}, u_n)^\eta .$$

Then

$$m_\varphi(u_n, u_{n+1}) \leqslant [g(u_{n-1}) - g(u_n)] . m_\varphi(u_{n-1}, u_n) . \tag{2.2}$$

Let's take $m_\varphi(u_{n-1}, u_n) = \varphi_n$ for all $n \in \mathbb{N}$. Then from (2.2) we get

$$0 < \frac{\varphi_{n+1}}{\varphi_n} \leqslant [g(u_{n-1}) - g(u_n)] \tag{2.3}$$

$$\vdots$$

Thus for all $l \in \mathbb{N}$, we have

$$\sum_{\text{i}=1}^{l} \frac{\varphi_{i+1}}{\varphi_i} \leqslant [g(u_{i-1}) - g(u_i)] = g(u_0) - g(u_l) . \tag{2.4}$$

From (2.3), we know that the sequence $\{g(u_n)\}_{n\in\mathbb{N}}$ is a monotone decreasing and bounded below. Then it is convergent. Let t be a limit of this sequence. Thus using (2.4) we have

$$\sum_{i=1}^{\infty} \frac{\varphi_{i+1}}{\varphi_i} \leqslant g(u_0) - \lim_{l\to\infty} g(u_l) = g(u_0) - t < \infty.$$

That is $\lim_{l\to\infty} \frac{\varphi_{l+1}}{\varphi_l} = 0$. Since $\lim_{n,m\to\infty} \eta(u_n, u_{n+1}, u_m)$ is finite, there exists some $k \in (0,1)$ such that $\lim_{n,m\to\infty} \eta(u_n, u_{n+1}, u_m) < \frac{1}{k}$. For this $k \in (0,1)$ there exists $n_0 \in \mathbb{N}$ such that $\varphi_{i+1} \leqslant k\varphi_i$ for all $i \geqslant n_0$. From Lemma 2.4, we get that $\{u_n\}$ is a m_φ-Cauchy sequence in Λ. From Theorem 2.1, we know that (Λ, m_φ) is complete. Therefore there exists some $u \in \Lambda$ such that $u_n \to u$ as $n \to \infty$. Next we will show that $Lu = u$. By Lemma 2.3, we obtain

$$\begin{aligned}
\lim_{n\to\infty} m_\varphi(u_n, u) - m_{\varphi_{u_n,u}} &= 0 \\
&= \lim_{n\to\infty} m_\varphi(u_{n+1}, u) - m_{\varphi_{u_{n+1},u}} \\
&= \lim_{n\to\infty} m_\varphi(Lu_n, u) - m_{\varphi_{Lu_n,u}} \\
&= m_\varphi(Lu, u) - m_{\varphi_{Lu,u}} .
\end{aligned}$$

Hence we can find

$$m_\varphi(Lu,u) = m_{\varphi_{Lu,u}}.$$

Since $m_\varphi(Lu,Lv) \leqslant km_\varphi(u,v)$ for all $u,v \in \Lambda$, we obtain

$$M_{\varphi_{u_n,Lu_n}} = m_\varphi(u_n,u_n) \leqslant km_\varphi(u_{n-1},u_{n-1}) \leqslant \ldots \leqslant k^n m_\varphi(u_0,u_0).$$

If we take the limit of above inequality, we get that $M_{\theta_{u_n,Lu_n}} = 0$, which implies that

$$m_\varphi(Lu,u) = m_{\varphi_{Lu,u}} \leqslant M_{\varphi_{u,Lu}} = 0.$$

Then $m_\varphi(Lu,u) = m_\varphi(u,u) = m_\varphi(Lu,Lu)$ and $Lu = u$. Now, we will show that the uniqueness of the fixed point. Let u be a fixed point of L. Hence

$$\begin{aligned} m_\varphi(u,u) &= m_\varphi(Lu,Lu) \\ &\leqslant km_\varphi(u,u) \\ &< m_\varphi(u,u) \text{ since } k \in [0,1). \end{aligned}$$

From the above inequality, we have $m_\varphi(u,u) = 0$. Now let's assume the opposite. That is, L has two fixed points $w \neq \eta \in \Lambda$ such that $Lw = w$ and $Lv = \eta$. Thus,

$$m_\varphi(w,\eta) = m_\varphi(Lw,Lv) \leqslant km_\varphi(w,\eta) < m_\varphi(w,\eta),$$

which implies that $m_\varphi(w,\eta) = 0$, and hence $w = \eta$. Therefore, L has a unique fixed point $w \in \Lambda$ such that $m_\varphi(w,w) = 0$ as desired. □

Theorem 2.2. *Let (Λ, m_φ) be a complete neM$_b$ms such that m_φ is continuous and L be an IRCCM. Let also the sequence $\{u_n\}$ be the Picard iteration generated by $u_0 \in \Lambda$. If $\lim_{n,m\to\infty} \eta(u_n,u_{n+1},u_m)$ is finite for some $u_0 \in \Lambda$, then the sequence $\{u_n\}$ converges to $u \in \Lambda$.Also if $\lim_{n\to\infty} \eta(u,u_n,Lu)$ is finite then u is a fixed point of L in Λ.*

Proof. If $u_n = u_{n+1}$ for some $n \geqslant 0$ then L has a fixed point in Λ. We assume that $m_\varphi(u_{n-1},Lu_{n-1}) > 0$ for all $n \in \mathbb{N}$. For $u = u_{n-1}$ and $v = v_n$, we have

$$\begin{aligned} m_\varphi(u_n,u_{n+1}) \quad &\leqslant \quad [g(u_{n-1}) - g(u_n)]^{w+\eta}.m_\varphi(u_{n-1},u_n)^w \\ &\quad\quad .m_\varphi(u_{n-1},u_n)^\eta m_\varphi(u_n,u_{n+1})^{1-w-\eta} \end{aligned}$$

and

$$m_\varphi(u_n,u_{n+1})^{w+\eta} \leqslant [g(u_{n-1}) - g(u_n)]^{w+\eta}.m_\varphi(u_{n-1},u_n)^{w+\eta}$$

which implies that

$$m_\varphi(u_n,u_{n+1}) \leqslant [g(u_{n-1}) - g(u_n)].m_\varphi(u_{n-1},u_n). \tag{2.5}$$

Let us denote $m_\varphi(u_{n-1},u_n) = f_n$ for all $n \in \mathbb{N}$. Then from (2.5) we obtain

$$0 < \frac{f_{n+1}}{f_n} \leqslant g(u_{n-1}) - g(u_n) \text{ for all } n \geqslant 1.$$

Using a similar method as in Theorem 2.1, we can show that the rest of the proof. □

THEOREM 2.3. *Let* (Λ, m_φ) *be a complete neM*$_b$*ms such that* m_φ *is continuous and* L *be an IJCCM. Let also the sequence* $\{u_n\}$ *be the Picard iteration generated by* $u_0 \in \Lambda$. *If* $\lim_{n,m\to\infty} \eta(u_n, u_{n+1}, u_m)$ *is finite for some* $u_0 \in \Lambda$, *then the sequence* $\{u_n\}$ *converges to* $u \in \Lambda$. *Also if* $\lim_{n\to\infty} \eta(u, u_n, Lu)$ *is finite then* u *is a fixed point of* L *in* Λ.

PROOF. If we take $u_n = u_{n+1}$ for some $n \geqslant 0$, we have the mapping L has a fixed point in Λ. Now we assume that $m_\varphi(u_{n-1}, Lu_{n-1}) > 0$ for all $n \in \mathbb{N}$. Then $u = u_{n-1}$ and $v = u_n$,

$$\begin{aligned} m_\varphi(u_n, u_{n+1}) &\leqslant [g(u_{n-1}) - g(u_n)]^w . m_\varphi(u_{n-1}, u_n)^w \\ &\quad . \left[\frac{m_\varphi(u_{n-1}, u_n) . m_\varphi(u_n, u_{n+1})}{m_\varphi(u_{n-1}, u_n)}\right]^{1-w}. \end{aligned}$$

From the above inequality, we get

$$m_\varphi(u_n, u_{n+1})^w \leqslant [g(u_{n-1}) - g(u_n)]^w . m_\varphi(u_{n-1}, u_n)^w$$

and

$$m_\varphi(u_n, u_{n+1}) \leqslant [g(u_{n-1}) - g(u_n)] . m_\varphi(u_{n-1}, u_n). \tag{2.6}$$

Let us denote $m_\varphi(u_{n-1}, u_n) = f_n$ for all $n \in \mathbb{N}$. Using (2.6), we have

$$0 < \frac{f_{n+1}}{f_n} \leqslant g(u_{n-1}) - g(u_n) \text{ for all } n \geqslant 1.$$

Proceeding in a similar way as in Theorem 2.1, we see that $\{u_n\}$ is convergent to u. The rest of the proof is similar to that of Theorem 2.1. □

THEOREM 2.4. *Let* (Λ, m_φ) *be a complete neM*$_b$*ms such that* m_φ *is continuous and* L *be an IDGCCM. Let also the sequence* $\{u_n\}$ *be the Picard iteration generated by* $u_0 \in \Lambda$. *If* $\lim_{n,m\to\infty} \eta(u_n, u_{n+1}, u_m)$ *is finite for some* $u_0 \in \Lambda$, *then the sequence* $\{u_n\}$ *converges to* $u \in \Lambda$. *Also if* $\lim_{n\to\infty} \eta(u, u_n, Lu)$ *is finite then* u *is a fixed point of* L *in* Λ.

PROOF. If we take $u_n = u_{n+1}$ for some $n \geqslant 0$, we have the mapping L has a fixed point in Λ. Now we assume that $m_\varphi(u_{n-1}, Lu_{n-1}) > 0$ for all $n \in \mathbb{N}$. Then $u = u_{n-1}$ and $v = u_n$,

$$\begin{aligned} m_\varphi(u_n, u_{n+1}) &\leqslant [g(u_{n-1}) - g(u_n)]^w . m_\varphi(u_{n-1}, u_n)^w \\ &\quad . \left[\frac{(1 + m_\varphi(u_{n-1}, u_n)) . m_\varphi(u_n, u_{n+1})}{1 + m_\varphi(u_{n-1}, u_n)}\right]^{1-w} \end{aligned}$$

which implies that

$$m_\varphi(u_n, u_{n+1})^w \leqslant [g(u_{n-1}) - g(u_n)]^w . m_\varphi(u_{n-1}, u_n)^w. \tag{2.7}$$

Let us denote $m_\varphi(u_{n-1}, u_n) = f_n$ for all $n \in \mathbb{N}$. Using (2.7), we have

$$0 < \frac{f_{n+1}}{f_n} \leqslant g(u_{n-1}) - g(u_n) \text{ for all } n \geqslant 1.$$

Proceeding in a similar way as in Theorem 2.1 we see that $\{u_n\}$ is convergent to u. The rest of the proof is similar to that of Theorem 2.1. □

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