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A UNIFIED DIFFERENCE METHOD FOR NUMERICAL SOLUTION OF THE BOUNDARY VALUE PROBLEMS IN ORDINARY DIFFERENTIAL EQUATIONS OF THE ORDER FOUR

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Abstract. In this article we propose a unified difference method for the numerical solution for the fourth order boundary value problems. The boundary problem is transformed into an equivalent system of boundary value problems. We discuss convergence analysis of the proposed method. Numerical experiments are performed to approve the efficiency and accuracy of the proposed method.

1. Introduction

Ordinary differential equations and corresponding boundary value problems are used to describe many physical phenomena. A fourth order differential equation and corresponding BVPshave a very important role in study of the theory of shells in natural sciences. Any literary work on the application of a fourth order differential equation and corresponding BVPs are there in studies of theory and application of elasticity [**1**–**3**], deformation of structures [**4**], deformation of elastic membrane [**5**] and effects of soil settlement [**6**].

In this article we consider following fourth order BVPs,

$$y^{(4)}(x) = \alpha y^{(3)}(x) + f(x, y, y', y'', y^{(3)}), \quad a < x < b, \tag{1.1}$$

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subject to the boundary conditions

$$y(a) = \beta, \quad y(b) = \gamma$$

$$y''(a) = \beta_0, \quad y''(b) = \gamma_0$$

where α, β, γ, β_0 and γ_0 are real constant and $f(x, y, y', y'', y^{(3)})$ is continuous in domain of definition of the problem.

For the approximate numerical solution of fourth order boundary value problems (1.1), a variety of methods have been introduced. However, numerical methods are available for solving problem (1.1) directly without reducing to an equivalent lower order system of differential equations. These existing methods for solving problem (1.1) employ spline methods [**7**], finite difference methods [**8**], finite element methods [**9**] and references therein.

We can find the theorems on uniqueness, the existence and convergence of the solution of the problem (1.1) in [**10, 11**]. We have assumed the existence and uniqueness of the solution of the problem (1.1). So we will not consider any specific assumption on forcing function $f(x, y, y', y'', y^{(3)})$ to ensure the existence and uniqueness of the solution to the problem (1.1).

The emphasis in this article will be on the development of an indirect method for the numerical solution of the fourth order boundary value problem. Thus, we will reduce the problem (1.1) to an equivalent system of second order ODEs. We will develop and employ unified difference method to solve the problem (1.1).

We have presented our work in this article as follows. In the next section we developed a unified difference method. In Section 3, we have discussed convergence of the proposed method under appropriate condition. The application of the proposed method to the test problems and illustrative numerical results so produced to show the efficiency in Section 4. Discussion and conclusion on the performance of the proposed method are presented in Section 5.

2. Unified difference method

In this section we propose unified difference method for the numerical solution of the problem (1.1). Let us introduce an intermediate variable $u(x)$ such that

$$u(x) = y''(x) \tag{2.1}$$

and the boundary conditions are

$$y(a) = \beta, \quad y(b) = \gamma.$$

This intermediate variable enable us to transform problem (1.1) into following an equivalent problem

$$u''(x) - \alpha u'(x) = f(x, y(x), y'(x), u(x), u'(x)). \tag{2.2}$$

and transformed boundary conditions are

$$u(a) = \beta_0 \quad \text{and} \quad u(b) = \gamma_0.$$

Thus, problem (1.1) is reduced into an equivalent coupled system of differential equations (2.1) - (2.2) subject to boundary and transformed boundary conditions.

We substitute domain $[a, b]$ by a discrete set of points and we wish to determine the numerical solution of the problem at these discrete points. Thus we define N finite numbers of $a = x_0 < x_1 < x_2 ...... < x_{N+1} = b$ nodal points in the domain of $[a,b]$ using a uniform step length h such that $x_i = a+ih, \quad i = 0, 1, 2, ..., N+1$. We wish to determine the numerical approximation of the solution $y(x)$ of the problem (1) at the nodal points $x_i, \quad i = 1, 2, ..., N$. We denote the numerical approximation of $y(x)$ at node $x = x_i$ as y_i, $i = 1, 2, .., N$. Let us denote f_i as the approximation of the theoretical value of the source function $f(x, y(x), y'(x), v(x), v'(x))$ at node $x = x_i, \quad i = 0, 1, 2, ..., N + 1$ and similarly we have defined other notations in the present article. Thus, the finite difference method reduces the problems (2.1) - (2.2) to the following discrete problems at node $x = x_i$,

$$\begin{aligned} y''_i &= u_i, \\ u''_i - \alpha u'_i &= f_i, \quad i = 0, 1, \cdots, N + 1. \end{aligned} \tag{2.3}$$

subject to the boundary conditions

$$y_0 = \beta, \quad y_{N+1} = \gamma, \quad u_0 = \beta_0 \quad \text{and} \quad u_{N+1} = \gamma_0$$

Let define following approximations,

$$\begin{aligned} \overline{y}'_i &= \frac{y_{i+1} - y_{i-1}}{2h} \\ \overline{u}'_i &= \frac{u_{i+1} - u_{i-1}}{2h} \\ \text{and} \quad \overline{f}_i &= f(x_i, y_i, \overline{y}'_i, u_i, \overline{u}'_i). \end{aligned} \tag{2.4}$$

Hence, following the ideas in [**12**–**14**], we propose following unified difference method for the numerical solution of the (1.1),

$$\begin{aligned} y_{i+1} - 2y_i + y_{i-1} &= h^2 u_i \\ u_{i+1} - (1 + \exp(\alpha h))u_i + \exp(\alpha h)u_{i-1} &= \frac{h^2}{2}(1 + \exp(\alpha h))\overline{f}_i. \end{aligned} \tag{2.5}$$

Thus we have obtained at each nodal point $x_i, i = 1, 2, ..., N$ the system of equations (2.5). The solution of the system of equations (2.5) is the approximate numerical solution of the problem (1.1).

3. Convergence analysis

We will consider following test equation for convergence analysis of the proposed method (2.5).

$$y^{(4)} = \alpha y^{(3)} + f(x, y, y', y'', y^{(3)}), \qquad a < x < b. \tag{3.1}$$

subject to the boundary conditions $y(a) = \beta, y(b) = \gamma, y''(a) = \beta_0$ and $y''(b) = \gamma_0$. Let us define Y_i, y_i, U_i and u_i are respectively exact and approximate solution of equations in (2.5). Define

$$F_i = f(x_i, Y_i, Y_i', U_i, U_i') \quad \text{and} \quad f_i = f(x_i, y_i, y_i', u_i, u_i').$$

Hence we linearize F_i and

$$F_i - f_i = (Y_i - y_i)G_i + (Y_i' - y_i')\dot{G}_i + (U_i - u_i)H_i + (U_i' - u_i')\dot{H}_i$$

where

$$G_i = (\frac{\partial f}{\partial y})_i, \quad \dot{G}_i = (\frac{\partial f}{\partial y'})_i, \quad H_i = (\frac{\partial f}{\partial u})_i \quad \text{and} \quad \dot{H}_i = (\frac{\partial f}{\partial u'})_i.$$

Define error term in approximate solution of system of equations (2.5),

$$\epsilon_i = Y_i - y_i \quad \text{and} \qquad \delta_i = U_i - u_i, \quad i = 1, 2, \cdots, N.$$

Hence,

$$\overline{F}_i - \overline{f}_i = \epsilon_i G_i + \frac{1}{2h}(\epsilon_{i+1} - \epsilon_{i-1})\dot{G}_i + \delta_i H_i + \frac{1}{2h}(\delta_{i+1} - \delta_{i-1})\dot{H}_i \tag{3.2}$$

and

$$\epsilon_{i+1} - 2\epsilon_i + \epsilon_{i-1} = h^2\delta_i + T_i \tag{3.3}$$

$$\delta_{i+1} - (1 + \exp(\alpha h))\delta_i + \exp(\alpha h)\delta_{i-1} = \frac{h^2}{2}(1 + \exp(\alpha h))(\overline{F}_i - \overline{f}_i) + \overline{T}_i.$$

where T_i and $\dot{T}_i$ are

$$T_i = \frac{h^4}{12}y_i^{(4)}$$

$$\overline{T}_i = \frac{h^4}{12}(y_i^{(4)} + 2\alpha y_i^{(3)} + \alpha^3 y_i' - (1 + \exp(h\alpha))(y_i^{(3)}\dot{G}_i + u_i^{(3)}\dot{H}_i)), \quad i = 1, 2, \cdots, N.$$

Using (3.2) in (3.3) and boundary conditions, we have following error equation,

$$\mathbf{JE} = \mathbf{T} \tag{3.4}$$

where

$$\mathbf{J} = \begin{pmatrix} \mathbf{A}_{1,1} & \vdots & \mathbf{A}_{1,2} \\ \cdots & \cdots & \cdots \\ \mathbf{A}_{2,1} & \vdots & \mathbf{A}_{2,2} \end{pmatrix}_{2N \times 2N},$$

$$\mathbf{E} = (\epsilon_1, \cdots, \epsilon_N, \delta_1, \cdots, \delta_N)^T,$$
$$\mathbf{T} = (T_1, \cdots, T_N, \overline{T}_1, \cdots, \overline{T}_N)^T.$$

Further,

$$\mathbf{A}_{1,1} = \begin{pmatrix} -2 & 1 & & 0 \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \\ 0 & & 1 & -2 \end{pmatrix}_{N \times N}, \mathbf{A}_{1,2} = \begin{pmatrix} -h^2 & & & 0 \\ & -h^2 & & \\ & & \ddots & \\ 0 & & & -h^2 \end{pmatrix}_{N \times N},$$

$$\mathbf{A}_{2,1} = -\frac{h}{4}(1+\exp(\alpha h))\begin{pmatrix} 2hG_1 & \dot{G}_1 & & 0 \\ -\dot{G}_2 & 2hG_2 & \dot{G}_2 & \\ & \ddots & \ddots & \\ 0 & & -\dot{G}_{N-1} & 2hG_N \end{pmatrix}_{N\times N},$$

and $\mathbf{A}_{2,2} = \mathbf{A}+\mathbf{B}$,

$$\mathbf{A} = \begin{pmatrix} -(1+\exp(\alpha h)) & 1 & & 0 \\ \exp(\alpha h) & -(1+\exp(\alpha h)) & 1 & \\ & \ddots & \ddots & \\ 0 & & \exp(\alpha h) & -(1+\exp(\alpha h)) \end{pmatrix}_{N\times N},$$

$$\mathbf{B} = -\frac{h}{4}(1+\exp(\alpha h))\begin{pmatrix} 2hH_1 & \dot{H}_1 & & 0 \\ -\dot{H}_2 & 2hH_2 & \dot{H}_2 & \\ & \ddots & \ddots & \\ 0 & & -\dot{H}_{N-1} & 2hH_N \end{pmatrix}_{N\times N}.$$

Let

$$G = \max_{1\leqslant i\leqslant N}|G_i|, \quad \dot{G} = \max_{1\leqslant i\leqslant N}|\dot{G}_i|, H = \max_{1\leqslant i\leqslant N}|H_i|, \quad \dot{H} = \max_{1\leqslant i\leqslant N}|\dot{H}_i|$$

So it is easy to calculate $||\mathbf{A}_{2,1}||$ and $||\mathbf{B}||$. Matrix $\mathbf{A}_{1,1}$ is invertible [**15**]. We determined $\mathbf{A}^{-1} = (a_{l,m})$ explicitly where ,

$$a_{l,m} = \begin{cases} \frac{(1-\exp(lh\alpha))(\exp(-Nh\alpha)-\exp((1-m)h\alpha))}{(\exp(h\alpha)-1)(\exp(-Nh\alpha)-\exp(h\alpha))}, & l\leqslant m \\ \frac{(1-\exp(mh\alpha))(\exp(-(N-l)h\alpha)-\exp(h\alpha))}{(\exp(h\alpha)-1)(\exp(-Nh\alpha)-\exp(h\alpha))}, & l\geqslant m \end{cases}$$

It is easy to see that that $\mathbf{A}_{2,2}$ is invertible [**16**]. Let us assume $||\mathbf{A}^{-1}\mathbf{B}|| < 1$ [**17**] then $||\mathbf{A}_{2,2}^{-1}|| \leqslant \frac{||\mathbf{A}^{-1}||}{1-||\mathbf{A}^{-1}\mathbf{B}||}$. Let us define

$$\overline{V} = ||\mathbf{A}_{1,2}\mathbf{A}_{2,2}^{-1}|| + 1.0 \quad \text{and} \quad \underline{V} = ||\mathbf{A}_{2,1}\mathbf{A}_{1,1}^{-1}|| + 1.0$$

and assume $\overline{V}\underline{V} < \overline{V}+\underline{V}$ then $\mathbf{J}$ is invertible [**18**]. Moreover,

$$||\mathbf{J}^{-1}|| \leqslant \frac{||\mathbf{A}_{1,1}^{-1}||\overline{V}\underline{V}}{1-||\mathbf{A}^{-1}\mathbf{B}||-(\underline{V}-1)||\mathbf{A}_{1,2}||||\mathbf{A}^{-1}||} \tag{3.5}$$

But

$$||\mathbf{A}_{1,1}^{-1}|| \leqslant \frac{(b-a)^2}{8h^2} \tag{3.6}$$

Therefore, from (3.5) and (3.6) we have,

$$||\mathbf{J}^{-1}|| \leqslant \frac{(b-a)^2\overline{V}\underline{V}}{8(1-||\mathbf{A}^{-1}\mathbf{B}||-(\underline{V}-1)||\mathbf{A}_{1,2}||||\mathbf{A}^{-1}||)h^2} \tag{3.7}$$

Let

$M = \max\{y_i^{(4)}, (y_i^{(4)}+2\alpha y_i^{(3)}+\alpha^3 y_i'-(1+\exp(h\alpha))(y_i^{(3)}\dot{G}_i+u_i^{(3)}\dot{H}_i))\}$, for all x in [a,b].

Thus

$$||\mathbf{T}|| \leqslant \frac{h^4}{12}M \tag{3.8}$$

From (3.4), (3.7) and (3.8), we obtained

$$||\mathbf{E}|| \leqslant \frac{h^2(b-a)^2\overline{V}\underline{V}M}{96(1-||\mathbf{A}^{-1}\mathbf{B}||-(\underline{V}-1)||\mathbf{A}_{1,2}||||\mathbf{A}^{-1}||)} \tag{3.9}$$

Thus from (3.9), we find $||\mathbf{E}||$ is bounded and $||\mathbf{E}||$ tends to zero as $h \longrightarrow 0$. So we conclude that proposed unified difference method (2.5) converge and the order of the convergence is at least $O(h^2)$.

4. Numerical results

To test the computational efficiency of our proposed method, we have considered two model problems. In each model problem, we took a uniform step size h. In Table 1 - Table 2, we have shown EMY the maximum absolute error in the solution $y(x)$ of the problem (1.1) and EMU the maximum absolute error in the second derivative of solution, i.e. $y''(x) = u(x)$ of the problems (1.1) for different values of N. In computation following formulas were used,

$$EMY = \max_{1\leqslant i\leqslant N} |y(x_i) - y_i|.$$

$$EMU = \max_{1\leqslant i\leqslant N} |u(x_i) - u_i|.$$

We have used Gausss-Seidel and Newton-Raphson iteration method to solve respectively the system of linear and nonlinear equations arised from equation (6). All computations were performed on a Windows 7 Home Basic operating system in the GNU FORTRAN environment version 99 compiler (2.95 of gcc) on Intel Core i3-2330M, 2.20 Ghz PC. The solutions are computed on N nodes and iteration is continued until either the maximum difference between two successive iterates is less than 10^{-10} or the number of iteration reached 10^3.

Problem 1. The model non-linear problem given by

$$y^{(4)}(x) = \alpha y^{(3)}(x) + y(x)(y'(x) + y''(x) + y^{(3)}(x)) + f(x), \quad 0 < x < 1$$

subject to boundary conditions

$$y(0) = 0 \quad , \quad y(1) = \exp(\alpha)$$

$$y''(0) = \alpha^2 \quad \text{and} \quad y''(1) = \alpha^2 \exp(\alpha).$$

where $f(x)$ is calculated so that the analytical solution of the problem is $y(x) = \exp(\alpha x)$. The EMY and EMU computed by method (2.5) for different values of N and α are presented in Table 1.

Problem 2. The model non-linear problem given by

$$y^{(4)}(x) = \alpha y^{(3)}(x) + y(x)(y^{(3)}(x) + 1.0) + x^2(y''(x))^2 + f(x), \quad 0 < x < 1$$

subject to boundary conditions

$$y(0) = \exp(A) \quad , \quad y(1) = \exp(\alpha + A)$$
$$y''(0) = \alpha^2 \exp(A) \quad \text{and} \quad y''(1) = \alpha^2 \exp(\alpha + A).$$

where $f(x)$ is calculated so that the analytical solution of the problem is $y(x) = \exp(\alpha x + A)$. The EMY and EMU computed by method (2.5) for different values of N, α and A are presented in Table 2.

Table 1. Maximum absolute error (Problem 1).

α	$Error$	Maximum absolute error			
		$N = 16$	$N = 32$	$N = 64$	$N = 128$
1.25	EMY	.33450127(-3)	.67234039(-4)	.23841858(-6)	.23841858(-6)
	EMU	.15315562(-2)	.34160912(-3)	.43302774(-4)	.41425228(-5)
-1.25	EMY	.42498112(-4)	.84638596(-5)	.59604645(-7)	.59604645(-7)
	EMU	.12620911(-3)	.25954098(-4)	.22724271(-6)	.61839819(-6)
1.75	EMY	.22068024(-2)	.52833557(-3)	.30755997(-4)	.47683716(-6)
	EMU	.13934404(-1)	.33753216(-2)	.46448410(-3)	.82701445(-5)
-1.75	EMY	.13494492(-3)	.31858683(-4)	.65565109(-6)	.59604645(-7)
	EMU	.39522350(-3)	.89570880(-4)	.23879111(-5)	.78603625(-6)

Table 2. Maximum absolute error (Problem 2).

α, A	$Error$	Maximum absolute error			
		$N = 16$	$N = 32$	$N = 64$	$N = 128$
1.75,	EMY	.15602112(-1)	.37746429(-2)	.52261353(-3)	.38146973(-5)
2.0	EMU	.96046448(-1)	.22895813(-1)	.49972534(-2)	.57220459(-3)
1.75	EMY	.15324354(-3)	.36269426(-4)	.17881393(-6)	.59604645(-7)
-2.0	EMU	.20349026(-3)	.47326088(-4)	.47683716(-6)	.23841858(-6)
-1.75	EMY	.23412704(-3)	.73432922(-4)	.47683716(-5)	.47683716(-6)
2.0	EMU	.11478424(-1)	.28619766(-2)	.59127808(-3)	.19073486(-5)
-1.75	EMY	.22955239(-4)	.54612756(-5)	.12665987(-6)	.74505806(-8)
-2.0	EMU	.41127205(-5)	.50663948(-6)	.29802322(-7)	.29802322(-7)

Numerical results, for example 1 for different values of N and α are presented in table 1. The maximum absolute errors in solution decreases with decrease in step size h and the order of accuracy in the result is at least quadratic. It observed from the result, for example 2, the accuracy depends on coefficient of the solution of the problem. Clearly the computational accuracy of the method depends on the constructed solution of the problem.

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