

TRIMMING AND BUILDING FREEZING SETS

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ABSTRACT. We develop new tools for the construction of fixed point sets in digital topology. We define *excludable points* and show that these may be excluded from all freezing sets. We show that articulation points are excludable. We also present results concerning points that must belong to a freezing set and often are easily recognized. These include points of degree 1 and some local extrema.

1. Introduction

Freezing sets are part of the fixed point theory of digital topology. They were introduced in [4] and studied subsequently in [5–11]. Given a digital image (X, κ) , it is desirable to know of a κ -freezing set for X that is as small as possible. A reason for this: Suppose we wish to construct a continuous self-map f on (X, κ) such that all members of a subset A of X are fixed by f . If A is known to be a freezing set for (X, κ) , then it can be concluded that $f = \text{id}_X$, typically in time depending on $\#A$ rather than on $\#X$. This could be a useful savings of time, since often $\#A \ll \#X$.

In this paper, we study conditions that can be used to exclude or force membership in a freezing set. Specifically, let (X, κ) be a digital image.

- In section 2 we give background material.
- In section 3 we show all points of degree 1 in (X, κ) must be included in all freezing sets of (X, κ) .
- In section 4, we introduce excludable sets. Such a subset of X consists of points that may be excluded from any minimal freezing set for (X, κ) .

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- In section 5 we show all articulation points of (X, κ) can be excluded from freezing sets, in the sense that if A is a freezing set and y is an articulation point for (X, κ) , then $A \setminus \{y\}$ is also a freezing set for (X, κ) .
- In section 6 we develop the notion of a *1-coordinate local extreme point* and show that such a point that satisfies an additional condition must be a member of every freezing set of (X, κ) .
- In section 7 we develop bounds on the cardinality of a minimal freezing set for (X, κ) .

2. Preliminaries

We use \mathbb{N} for the set of natural numbers, \mathbb{Z} for the set of integers, and $\#X$ for the number of distinct members of X .

We typically denote a (binary) digital image as (X, κ) , where $X \subset \mathbb{Z}^n$ for some $n \in \mathbb{N}$ and κ represents an adjacency relation of pairs of points in X . Thus, (X, κ) is a graph, in which members of X may be thought of as black points, and members of $\mathbb{Z}^n \setminus X$ as white points, of a picture of some “real world” object or scene.

2.1. Adjacencies. This section is largely quoted or paraphrased from [7].

We use the notations $y \leftrightarrow_{\kappa} x$, or, when the adjacency κ can be assumed, $y \leftrightarrow x$, to mean x and y are κ -adjacent. The notations $y \Leftrightarrow_{\kappa} x$, or, when κ can be assumed, $y \Leftrightarrow x$, mean either $y = x$ or $y \leftrightarrow_{\kappa} x$.

For $x \in X$, let

$$N(X, x, \kappa) = \{y \in X \mid x \leftrightarrow_{\kappa} y\}.$$

The *degree* of x in (X, κ) is $\#N(X, x, \kappa)$. We are especially interested in points that have degree 1.

Let $u, n \in \mathbb{N}$, $1 \leq u \leq n$. Let $X \subset \mathbb{Z}^n$. Points $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in X$ are c_u -adjacent if and only if

- $x \neq y$, and
- for at most u indices i , $|x_i - y_i| = 1$, and
- for all indices j such that $|x_j - y_j| \neq 1$, we have $x_j = y_j$.

The c_u adjacencies are the adjacencies most used in digital topology, especially c_1 and c_n .

In low dimensions, it is also common to denote a c_u adjacency by the number of points that can have this adjacency with a given point in \mathbb{Z}^n . E.g.,

- For subsets of \mathbb{Z}^1 , c_1 -adjacency is 2-adjacency.
- For subsets of \mathbb{Z}^2 , c_1 -adjacency is 4-adjacency and c_2 -adjacency is 8-adjacency.
- For subsets of \mathbb{Z}^3 , c_1 -adjacency is 6-adjacency, c_2 -adjacency is 18-adjacency, and c_3 -adjacency is 26-adjacency.

A sequence $P = \{y_i\}_{i=0}^m$ in a digital image (X, κ) is a κ -path from $a \in X$ to $b \in X$ if $a = y_0$, $b = y_m$, and $y_i \Leftrightarrow_{\kappa} y_{i+1}$ for $0 \leq i < m$.

X is κ -connected [17], or *connected* when κ is understood, if for every pair of points $a, b \in X$ there exists a κ -path in X from a to b .

A (*digital*) κ -closed curve is a path $S = \{s_i\}_{i=0}^{m-1}$ such that $s_0 \leftrightarrow_{\kappa} s_{m-1}$, and $i \neq j$ implies $s_i \neq s_j$. If also $0 \leq i < m$ implies

$$N(S, x_i, \kappa) = \{x_{(i-1) \bmod m}, x_{(i+1) \bmod m}\}$$

then S is a (*digital*) κ -simple closed curve. We say the members of S are *circularly labeled* if they are indexed as described above.

Let $X \subset \mathbb{Z}^n$. The *boundary* of X [16] is

$$Bd(X) = \{x \in X \mid \text{there exists } y \in \mathbb{Z}^n \setminus X \text{ such that } y \leftrightarrow_{c_1} x\}.$$

2.2. Digitally continuous functions. This section is largely quoted or paraphrased from [7].

Digital continuity is defined to preserve connectedness, as at Definition 2.1 below. By using adjacency as our standard of “closeness,” we get Theorem 2.1 below.

DEFINITION 2.1. [2] (generalizing a definition of [17]) *Let (X, κ) and (Y, λ) be digital images. A function $f : X \rightarrow Y$ is (κ, λ) -continuous if for every κ -connected $A \subset X$ we have that $f(A)$ is a λ -connected subset of Y .*

If either of X or Y is a subset of the other, we use the abbreviation κ -continuous for (κ, κ) -continuous.

When the adjacency relations are understood, we will simply say that f is *continuous*. Continuity can be expressed in terms of adjacency of points:

THEOREM 2.1. [2, 17] *A function $f : X \rightarrow Y$ is continuous if and only if $x \leftrightarrow x'$ in X implies $f(x) \Leftrightarrow f(x')$.*

See also [14, 15], where similar notions are referred to as *immersions*, *gradually varied operators*, and *gradually varied mappings*.

A digital *isomorphism* (called *homeomorphism* in [1]) is a (κ, λ) -continuous surjection $f : X \rightarrow Y$ such that $f^{-1} : Y \rightarrow X$ is (λ, κ) -continuous.

The literature uses *path* polymorphically: a (c_1, κ) -continuous function $f : [0, m]_{\mathbb{Z}} \rightarrow X$ is a κ -path if $f([0, m]_{\mathbb{Z}})$ is a κ -path from $f(0)$ to $f(m)$ as described above.

We use id_X to denote the *identity function*, $\text{id}_X(x) = x$ for all $x \in X$.

Given a digital image (X, κ) , we denote by $C(X, \kappa)$ the set of κ -continuous functions $f : X \rightarrow X$.

Given $f \in C(X, \kappa)$, a *fixed point* of f is a point $x \in X$ such that $f(x) = x$. $\text{Fix}(f)$ will denote the set of fixed points of f . We say f is a *retraction*, and the set $Y = f(X)$ is a *retract* of X , if $f|_Y = \text{id}_Y$; thus, $Y = \text{Fix}(f)$.

DEFINITION 2.2. [4] *Let (X, κ) be a digital image. We say $A \subset X$ is a freezing set for X if given $g \in C(X, \kappa)$, $A \subset \text{Fix}(g)$ implies $g = \text{id}_X$. A freezing set A is minimal if no proper subset of A is a freezing set for (X, κ) .*

The following elementary assertion was noted in [4].

LEMMA 2.1. *Let (X, κ) be a connected digital image for which A is a freezing set. If $A \subset A' \subset X$, then A' is a freezing set for (X, κ) .*

Let $X \subset \mathbb{Z}^n$, $x = (x_1, \dots, x_n) \in \mathbb{Z}^n$, where each $x_i \in \mathbb{Z}$. For each index i , the projection map (onto the i^{th} coordinate) $p_i : X \rightarrow \mathbb{Z}$ is given by $p_i(x) = x_i$.

2.3. Tools for determining fixed point sets. In this section, we give some results, mostly from earlier papers, that help us determine fixed point sets for digitally continuous self maps.

THEOREM 2.2. [4] *Let A be a freezing set for the digital image (X, κ) and let $F : (X, \kappa) \rightarrow (Y, \lambda)$ be an isomorphism. Then $F(A)$ is a freezing set for (Y, λ) .*

PROPOSITION 2.1. [13] *Let (X, κ) be a digital image and $f \in C(X, \kappa)$. Suppose $x, x' \in \text{Fix}(f)$ are such that there is a unique shortest κ -path P in X from x to x' . Then $P \subset \text{Fix}(f)$.*

The following lemma may be understood as saying that if q and q' are adjacent with q in a given direction from q' , and if f pulls q further in that direction, then f also pulls q' in that direction.

LEMMA 2.2. [4] *Let $(X, c_u) \subset \mathbb{Z}^n$ be a digital image, $1 \leq u \leq n$. Let $q, q' \in X$ be such that $q \leftrightarrow_{c_u} q'$. Let $f \in C(X, c_u)$.*

- (1) *If $p_i(f(q)) < p_i(q) < p_i(q')$ then $p_i(f(q')) < p_i(q')$.*
- (2) *If $p_i(f(q)) > p_i(q) > p_i(q')$ then $p_i(f(q')) > p_i(q')$.*

The following has been relied on implicitly in several previous papers. It is an extension of Lemma 2.2 showing that a continuous function can pull a digital arc that is monotone with respect to a given coordinate i in the direction of monotonicity.

PROPOSITION 2.2. *Let $(X, c_u) \subset \mathbb{Z}^n$ be a digital image, $1 \leq u \leq n$. Let $q, q' \in X$. Let $f \in C(X, c_u)$.*

- (1) *Suppose $\{q_j\}_{j=0}^m$ is a c_u -path in X such that $q_0 = q$, $q_m = q'$, $p_i(f(q)) < p_i(q)$, and for $0 \leq j < m$ we have $p_i(q_j) < p_i(q_{j+1})$. Then $p_i(f(q_j)) < p_i(q_j)$ for $0 \leq j \leq m$.*
- (2) *Suppose $\{q_j\}_{j=0}^m$ is a c_u -path in X such that $q_0 = q$, $q_m = q'$, $p_i(f(q)) > p_i(q)$, and for $0 \leq j < m$ we have $p_i(q_j) > p_i(q_{j-1})$. Then $p_i(f(q_j)) > p_i(q_j)$ for $0 \leq j \leq m$.*

PROOF. We prove the first assertion; the second is proven similarly. We argue by induction. We know

$$p_i(f(q_0)) = p_i(f(q)) < p_i(q) = p_i(q_0).$$

Suppose we have $p_i(f(q_k)) < p_i(q_k)$ for some $k < m$. Then by Lemma 2.2, $p_i(f(q_{k+1})) < p_i(q_{k+1})$. This completes our induction. \square

DEFINITION 2.3. [5] *Let $\kappa \in \{c_1, c_2\}$. We say a κ -connected set $S = \{x_i\}_{i=1}^n \subset \mathbb{Z}^2$ for $n > 1$ is a (digital) line segment if the members of S are collinear.*

REMARK 2.1. [5] A digital line segment must be vertical, horizontal, or have slope of ± 1 . We say a segment with slope of ± 1 is *slanted*.

LEMMA 2.3. *Let $(X, c_1) \subset \mathbb{Z}^2$ be a connected digital image. Let L be a horizontal or vertical digital line segment of at least 2 points contained in X . Let $f \in C(X, c_1)$. Suppose*

$$(2.1) \quad \text{the endpoints of } L \text{ are in } \text{Fix}(f).$$

Then $L \subset \text{Fix}(f)$.

PROOF. This follows from Proposition 2.1, since L is the unique shortest c_1 -path in X between the endpoints of L . \square

We do not have an analog of Lemma 2.3 for slanted segments, as shown by the following.

EXAMPLE 2.1. *Let*

$$X = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (2, 0)\}.$$

Let $S = \{(0, 2), (1, 1), (2, 0)\}$. Let $f : X \rightarrow X$ be given by

$$f(x) = \begin{cases} x & \text{if } x \neq (1, 1); \\ (0, 0) & \text{if } x = (1, 1). \end{cases}$$

It is easily seen that $f \in C(X, c_1)$, that S is a slanted segment with endpoints $(0, 2)$ and $(2, 0)$ in $\text{Fix}(f)$, but $(1, 1) \in S \setminus \text{Fix}(f)$.

We have the following analog of Lemma 2.3 for the c_2 adjacency.

LEMMA 2.4. *Let $(X, c_2) \subset \mathbb{Z}^2$ be a connected digital image. Let L be a slanted digital line segment contained in X . Let $f \in C(X, c_2)$. Suppose (2.1). Then $L \subset \text{Fix}(f)$.*

PROOF. This follows from Proposition 2.1, since L is the unique shortest c_2 -path in X between the endpoints of L . \square

We do not have an analog of Lemma 2.4 for horizontal or vertical digital line segments, as shown in the following.

EXAMPLE 2.2. *Let $X = [0, 2]_{\mathbb{Z}}^2 \subset \mathbb{Z}^2$. The function $f : X \rightarrow X$ given by*

$$f(x) = \begin{cases} (1, 1) & \text{if } x \in \{(0, 1), (1, 0)\}; \\ x & \text{otherwise,} \end{cases}$$

belongs to $C(X, c_2)$. The digital line segments

$$S_1 = \{(0, 0), (1, 0), (2, 0)\}, \quad S_2 = \{(0, 0), (0, 1), (0, 2)\}$$

have endpoints in $\text{Fix}(f)$ and are respectively horizontal and vertical, but neither of S_1, S_2 is a subset of $\text{Fix}(f)$.

THEOREM 2.3. [4] *Let $X \subset \mathbb{Z}^n$ be finite. Then for $1 \leq u \leq n$, $\text{Bd}(X)$ is a freezing set for (X, c_u) .*

3. Include degree-1 points

Sometimes, there are points of a digital image that are easily recognized as belonging to freezing sets, as we see in the following.

THEOREM 3.1. *Let (X, κ) be a connected digital image. Let A be a freezing set for (X, κ) . Let $x_0 \in X$ be a point that has degree 1. Then $x_0 \in A$.*

PROOF. Let x_1 be the unique member of X such that $x_0 \Leftrightarrow_{\kappa} x_1$. If $x_0 \notin A$ then the function $f : X \rightarrow X$ given by

$$f(x) = \begin{cases} x & \text{if } x \neq x_0; \\ x_1 & \text{if } x = x_0, \end{cases}$$

is easily seen to belong to $C(X, \kappa)$. Also, $f|_A = \text{id}_A$, and $f \neq \text{id}_X$. The assertion follows. \square

4. Excludable sets

We develop the notion of an excludable set and show how this notion helps us determine minimal or small freezing sets.

DEFINITION 4.1. *Let (X, κ) be a connected digital image. Let $W \subset X$. We say W is excludable from freezing sets for (X, κ) (excludable for short) if for every freezing set A of (X, κ) , if $A \setminus W \neq \emptyset$ then $A \setminus W$ is also a freezing set. If $p \in W$ then p is an excludable point.*

REMARK 4.1. By Definition 4.1, if A is a minimal freezing set and W is excludable for (X, κ) , then $A \cap W = \emptyset$.

PROPOSITION 4.1. *Let (X, κ) be a finite connected digital image. Let W be an excludable set for X . Then every subset of W is excludable.*

PROOF. This follows from Lemma 2.1. \square

PROPOSITION 4.2. *Let $F : (X, \kappa) \rightarrow (Y, \lambda)$ be an isomorphism of digital images. If W is an excludable set for X then $F(W)$ is an excludable set for Y .*

PROOF. Let A be a freezing set for X . Let $f \in C(Y, \lambda)$ such that

$$(4.1) \quad f|_{F(A) \setminus F(W)} = \text{id}_{F(A) \setminus F(W)}$$

Then $g = F^{-1} \circ f \circ F \in C(X, \kappa)$.

Let $b \in F(A) \setminus F(W)$. Then $a = F^{-1}(b) \in A \setminus W$. We have

$$g(a) = F^{-1}(f(b)) = F^{-1}(b) = a.$$

Since every $a \in A \setminus W$ satisfies $a = F^{-1}(b)$ for some $b \in F(A) \setminus F(W)$, we have $g|_{A \setminus W} = \text{id}_{A \setminus W}$. Since W is excludable, $g = \text{id}_X$. Therefore,

$$f = F \circ g \circ F^{-1} = F \circ \text{id}_X \circ F^{-1} = \text{id}_Y.$$

Thus, $F(W)$ is excludable. \square

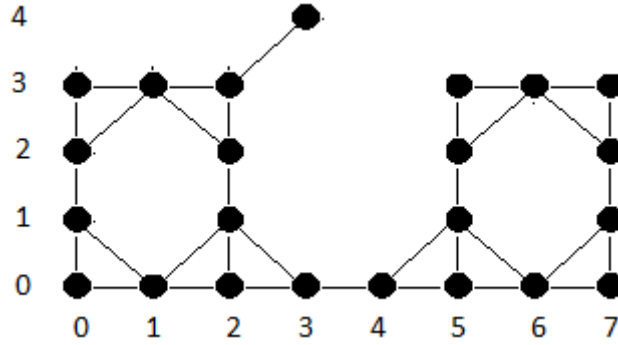


FIGURE 1. A digital image shown with the c_2 adjacency. The set of articulation points is $\{(2, 3), (3, 0), (4, 0)\}$.

5. Articulation points and freezing sets

An *articulation point* or *cut point* of a connected graph (X, κ) is a point $x \in X$ such that $(X \setminus \{x\}, \kappa)$ is not connected (see Figure 1). In this section, we show that articulation points are often excludable, by showing that if the set of articulation points is removed from a freezing set, what is left is often still a freezing set.

LEMMA 5.1. *Let M be the set of articulation points for the connected digital image (X, κ) . Let K be a κ -component of $X \setminus M$. Then there is a κ -retraction of X to $X \setminus K$.*

PROOF. Without loss of generality, $M \neq \emptyset$.

Since X is connected, there exists $x_0 \in X \setminus (K \cup M)$ such that x_0 is κ -adjacent to a point of M . By choice of M , no point of K is adjacent to x_0 . Let $r : X \rightarrow X$ be the function

$$r(x) = \begin{cases} x & \text{if } x \in X \setminus K; \\ x_0 & \text{if } x \in K. \end{cases}$$

It is easily seen that r is a κ -retraction of X to $X \setminus K$. \square

LEMMA 5.2. *Let x_0 be an articulation point for the connected digital image (X, κ) . Let K_1 and K_2 be distinct κ -components of $X \setminus \{x_0\}$. Let $f \in C(X, \kappa)$ such that for some $x_1 \in K_1$ and $x_2 \in K_2$, $\{x_1, x_2\} \subset \text{Fix}(f)$. Then $x_0 \in \text{Fix}(f)$.*

PROOF. Let P_i be a shortest κ -path in X from x_i to x_0 , $i \in \{1, 2\}$. Then $P_1 \cup P_2$ is a path from x_1 to x_2 .

Suppose $f(x_0) \neq x_0$. Then without loss of generality, $f(x_0) \in K_2$. Then $f(P_1)$ is a path from $x_1 = f(x_1)$ to x_0 to $f(x_0)$ that has length greater than that of P_1 , which is impossible. The assertion follows. \square

THEOREM 5.1. *Let W be the set of articulation points for the finite connected digital image (X, κ) , with $W \neq \emptyset$. If W is a proper subset of X , then W is excludable.*

PROOF. Let A be a freezing set for X . Let $f \in C(X, \kappa)$ such that $f|_{A \setminus W} = \text{id}_{A \setminus W}$. Let $x_0 \in W$. Then there exist distinct components K_1, K_2 of $X \setminus \{x_0\}$. By Lemma 5.1, there exists a retraction r of X to $X \setminus K_1$. It follows that $A \cap K_1 \neq \emptyset$, for otherwise $r|_A = \text{id}_A$ yet $r \neq \text{id}_X$, contrary to A being a freezing set. Similarly, $A \cap K_2 \neq \emptyset$.

By Lemma 5.2, $f(x_0) = x_0$. Since x_0 was taken as an arbitrary member of W , we have

$$f|_{A \setminus W} = \text{id}_{A \setminus W} \Rightarrow f|_{A \cup W} = \text{id}|_{A \cup W} \Rightarrow f|_A = \text{id}_A \Rightarrow f = \text{id}_X.$$

Thus W is excludable. \square

REMARK 5.1. Theorem 5.1 implies that if (X, κ) is a wedge of two finite digital images, $(X, \kappa) = (X_1, \kappa) \vee (X_2, \kappa)$, then the ‘‘wedge point’’ of X is excludable, hence does not belong to any minimal freezing set for (X, κ) .

6. 1-D local extrema

We introduce a kind of local extreme point and study the relationship of such a point to a freezing set.

6.1. Definition and relation to freezing sets. Let (X, κ) be a digital image, where $X \subset \mathbb{Z}^n$. Let $x = (x_1, \dots, x_n) \in X$, where each $x_i \in \mathbb{Z}$. We say x is a *1-coordinate local maximum at index i* for (X, κ) if for some index i and all $y = (y_1, \dots, y_n) \in N(X, x, \kappa)$, $x_i > y_i$; and $x' = (x_1, \dots, x_{i-1}, x_i - 1, x_{i+1}, \dots, x_n)$ is a *justifying neighbor at index i* of x . We say x is a *1-coordinate local minimum at index i* for (X, κ) if for some index i and all $y = (y_1, \dots, y_n) \in N(X, x, \kappa)$, $x_i < y_i$; and $x' = (x_1, \dots, x_{i-1}, x_i + 1, x_{i+1}, \dots, x_n)$ is a *justifying neighbor* of x . We say x is a *1-coordinate local extremum for (X, κ)* if x is either a 1-coordinate local maximum or a 1-coordinate local minimum for (X, κ) . Such points are often easily recognized, and, we will show, often are members of minimal freezing sets.

Note if x and x' are, respectively, a 1-coordinate local extremum and its justifying neighbor, then x and x' differ in exactly one index. Thus, $x \leftrightarrow_{c_1} x'$.

Neither being nor not being a 1-coordinate local extremum is necessarily preserved by isomorphism, as the following shows.

EXAMPLE 6.1. *Let*

$$X = \{(x, y) \in \mathbb{Z}^2 \mid |x| + |y| = 2\} \text{ (see Figure 6.1).}$$

Assume the points of X are indexed circularly with $X = \{x_i\}_{i=0}^7$, with $x_0 = (2, 0)$, $x_1 = (1, 1)$, etc. Then (X, c_2) is a digital simple closed curve, for which the ‘‘corner points’’ x_0, x_2, x_4, x_6 are the 1-coordinate local extrema. Let $f : X \rightarrow X$ be the function

$$f(x_i) = x_{i+1 \pmod 8}.$$

Then f is a (c_2, c_2) -isomorphism. For each index i ,

- *if x_i is a 1-coordinate local extremum, then $f(x_i)$ is not a 1-coordinate local extremum; and*

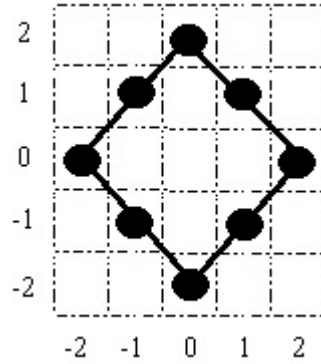


FIGURE 2. The digital image (X, c_2) of Example 6.1. The “corner” points $(2, 0)$, $(0, 2)$, $(-2, 0)$, $(0, -2)$ are 1-coordinate local extrema. None of them has a justifying neighbor in X .

- if x_i is not a 1-coordinate local extremum, then $f(x_i)$ is a 1-coordinate local extremum.

We have the following.

THEOREM 6.1. *Let (X, c_u) be a connected digital image, where $X \subset \mathbb{Z}^n$ and $1 \leq u \leq n$. Let $x_0 = (x_1, \dots, x_n) \in X$, where each $x_i \in \mathbb{Z}$. Suppose, for some index i , x_0 is a 1-coordinate local extremum for (X, c_u) with a justifying neighbor $x' \in X$ at index i . Let A be a freezing set for (X, c_u) . Then $x_0 \in A$.*

PROOF. There are at most u indices j , one of which is $j = i$, at which $|x_j - p_j(x')| = 1$ and for all other indices k , $x_k = p_k(x')$.

Therefore, if $A \subset X$ and $x_0 \notin A$, consider the function $f : X \rightarrow X$ given by

$$f(x) = \begin{cases} x & \text{if } x \neq x_0; \\ x' & \text{if } x = x_0, \end{cases}$$

It follows easily that $f \in C(X, c_u)$. Further, $f|_A = \text{id}_A$ and $f \neq \text{id}_X$. Hence $x_0 \notin A$ implies A is not a freezing set. The assertion follows. \square

6.2. Example. We demonstrate how articulation points and 1-coordinate local extrema can help us determine small freezing sets in the following.

EXAMPLE 6.2. *Let*

$$X = \{(0, 1), (1, 2), (2, 1), (3, 0), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3), (5, 2)\}.$$

See Figure 3.

Let

$$A = \{(0, 1), (3, 0), (4, 3), (5, 2)\}.$$

We claim A is a minimal freezing set for (X, c_2) . We show this as follows.

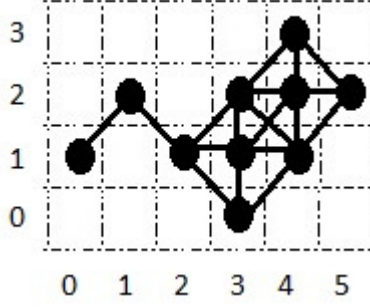


FIGURE 3. The image (X, c_2) of Example 6.2. Articulation points: $(1, 2)$, $(2, 1)$. 1-coordinate local extrema with justifying points: $(3, 0)$, $(4, 3)$, $(5, 2)$.

Theorem 2.3 tells us there is a minimal freezing set B for (X, c_2) that is a subset of $Bd(X)$. Therefore $(3, 1)$ and $(4, 2)$ can be excluded from B . We proceed to show $B = A$.

We must have $(0, 1) \in B$, by Theorem 3.1.

Since each of $(1, 2)$ and $(2, 1)$ is an articulation point of (X, c_2) , by Theorem 5.1, they are excluded from B . Note also that $(1, 2)$ is a 1-coordinate local maximum in X , but lacks a justifying neighbor in X , so Theorem 6.1 does not apply to $(1, 2)$.

Each of $(3, 0)$, $(4, 3)$, and $(5, 2)$ is a 1-coordinate local extreme point of X with justifying neighbors $(3, 1) \in X$, $(4, 2) \in X$, and $(4, 2) \in X$, respectively, so by Theorem 6.1, $\{(3, 0), (4, 3), (5, 2)\} \subset B$.

Thus $A \subset B$.

Let $f \in C(X, c_2)$ such that $f|_B = \text{id}_B$.

Since $(2, 1)$ belongs to the unique shortest-length c_2 -path between the fixed points $(0, 1)$ and $(3, 0)$, Proposition 2.1 lets us conclude $(2, 1) \in \text{Fix}(f)$. Since $(4, 1)$ belongs to the unique shortest-length c_2 -path between the fixed points $(3, 0)$ and $(5, 2)$, Proposition 2.1 or Lemma 2.4 lets us conclude $(4, 1) \in \text{Fix}(f)$. Similarly, $(3, 2)$ belongs to the unique shortest-length c_2 -path between the fixed points $(2, 1)$ and $(4, 3)$, so Theorem 2.1 or Lemma 2.4 lets us conclude $(3, 2) \in \text{Fix}(f)$. Since f is an arbitrary member of $C(X, c_2)$, we conclude that

$$\{(2, 1), (4, 1), (3, 2)\} \subset X \setminus B.$$

Thus $A = B$. Hence by choice of B , A is minimal.

7. Bounds on size of freezing set

How small, and how big, can a freezing set be? We provide bounds on the size of a freezing set in the following.

THEOREM 7.1. *Let (X, c_u) be a connected finite digital image, where $X \subset \mathbb{Z}^n$, $n > 1$, and $1 \leq u \leq n$. Let D_1 be the set of points that have degree 1 in (X, c_u) . Let T be the set of 1-coordinate local extrema of (X, c_u) that have justifying points*

in X . Let W be the set of articulation points of (X, c_u) . Then there is a minimal freezing set $A \subset Bd(X)$ for (X, c_u) such that

$$\#(D_1 \cup T) \leq \#A.$$

If $\emptyset \neq W$ and W is a proper subset of X , then

$$\#(D_1 \cup T) \leq \#A \leq \#Bd(X) - \#W.$$

PROOF. By Theorem 2.3, there is a minimal freezing set $A \subset Bd(X)$ for (X, c_u) . We have $T \subset Bd(X)$, $D_1 \subset Bd(X)$, and, since $n > 1$, $W \subset Bd(X)$. Thus the conclusion follows from Theorems 5.1, 3.1, and 6.1. \square

REMARK 7.1. We need $n > 1$ in Theorem 7.1, since if $X = [0, 2]_{\mathbb{Z}} \subset \mathbb{Z}$, we have that 1 is an articulation point for (X, c_1) but is not a member of $Bd(X)$.

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