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TRI-QUASI IDEALS OF SEMIRINGS

Murali Krishna Rao Marapureddy

Abstract. In this paper, as a further generalization of ideals, we introduce the notion of tri-quasi ideals as a generalization of ideals, left ideals, right ideals, tri-ideals, bi-ideals, quasi ideals, interior ideals, bi-interior ideals, weak interior ideals, bi-quasi ideals, quasi-interior ideals and bi-quasi-interior ideals of a semiring. Some charecterizations of a regular semiring and a simple semiring using tri-quasi ideals are given and study the properties of tri-quasi ideals of semirings.

1. Introduction

The algebraic structures play a prominent role in mathematics with wide range of applications. Generalization of ideals of algebraic structures and ordered algebraic structure plays a very remarkable role and also necessary for further advance studies and applications of various algebraic structures. Many mathematicians proved important results and charecterization of algebraic structures by using the concept and the properties of generalization of ideals in algebraic structures.

During 1950-1980, the cocepts of bi-ideals, quasi ideals and interior ideals were studied by many mathematicians and during 1950-2019, the applications of these ideals only studied by mathematicians. Between 1980 and 2016 there have been no new generalization of these ideals of algebraic structures. Then the author [**12–14**] introduced and studied weak interior ideals , bi-interior ideals, bi quasi ideals, quasi interior ideals and bi quasi interior ideals of $\Gamma-$semirings, semirings, $\Gamma-$semigroups, semigroups as a generalizationof bi-ideal, quasi ideal and interior ideal of algebraic structures and charecterized regular algebraic structures as well as simple algebraic structures using these ideals.

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The notion of a semiring was introduced by Vandiver [**23**] in 1934, but semirings had appeared in earlier studies on the theory of ideals of rings. A universal algebra $(S, +, \cdot)$ is called a semiring if and only if $(S, +), (S, \cdot)$ are semigroups which are connected by distributive laws, *i.e.*, $a(b+c) = ab+ac$, $(a+b)c = ac+bc$, for all $a, b, c \in S$. The theory of rings and theory of semigroups have considerable impact on the development of theory of semirings. Semirings play an important role in studying matrices and determinants. Semirings are useful in the areas of theoretical computer science as well as in the solution of graph theory, optimization theory, in particular for studying automata,coding theory and formal languages. Semiring theory has many applications in other branches of mathematics.

We know that the notion of a one sided ideal of any algebraic structure is a generalization of an ideal. The quasi ideals are generalization of left ideal and right ideal whereas the bi-ideals are generalization of quasi ideals. In 1952, the concept of bi-ideals was introduced by Good and Hughes [**2**] for semigroups. The notion of bi-ideals in rings and semigroups were introduced by Lajos and Szasz. Bi-ideal is a special case of (m-n) ideal. In 1976, the concept of interior ideals was introduced by Lajos [**8**] for semigroups. Steinfeld [**22**] first introduced the notion of quasi ideals for semigroups and then for rings. Iseki and Izuka [**4**–**6**] introduced the concept of quasi ideal for a semiring. Quasi ideals bi-ideals in $\Gamma-$semirings studied by Jagtap and Pawar. Henriksen [**3**] and Shabir et al. [**21**] studied ideals in semirings. Murali Krishna Rao et al. [**15, 18, 19**] studied ideals in $\Gamma-$semirings. In this paper,as a further generalization of ideals, we introduce the notion of tri -quasi ideal as a generalization of bi-ideal, quasi ideal, interior ideal, bi-interior ideal, tri-ideal and bi-quasi ideal of a semiring and study some of the properties of tri-quasi ideals of semirings.

2. Preliminaries

In this section we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

Definition 2.1. *[**10**] A set S together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively will be called semiring provided*

(i) *addition is a commutative operation.*
(ii) *multiplication distributes over addition both from the left and from the right.*
(iii) *there exists $0 \in S$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$.*

Definition 2.2. *[**10**] A semiring M is said to be commutative semiring if $xy = yx$, for all $x, y \in M$.*

Definition 2.3. *[**10**] Let M be a semiring. An element $1 \in M$ is said to be unity if for each $x \in M$ such that $x1 = 1x = x$.*

Definition 2.4. *[**10**] In a semiring M with unity 1, an element $a \in M$ is said to be left invertible (right invertible) if there exist $b \in M$, such that $ba = 1(ab = 1)$.*

Definition 2.5. *[**10**] In a semiring M with unity 1, an element $a \in M$ is said to be invertible if there exist $b \in M$, such that $ab = ba = 1$.*

Definition 2.6. *[**10**] A semiring M is said to have zero element if there exists an element $0 \in M$ such that $0 + x = x$ and $0x = x0 = 0$, for all $x \in M$.*

Definition 2.7. *[**10**] An element a in a semiring M is said to be idempotent if $a = aa$.*

Definition 2.8. *[**10**] Every element of M, is an idempotent of M then M is said to be idempotent semiring M.*

Definition 2.9. *[**19**] A semiring M is called a division semiring if for each non-zero element of M has multiplication inverse.*

Definition 2.10. *[**19**] A non-empty subset A of a semiring M is called*

(i) *a -subsemiring of M if $(A, +)$ is a subsemigroup of $(M, +)$ and $AA \subseteq A$.*
(ii) *a quasi ideal of M if A is a subsemiring of M and $AM \cap MA \subseteq A$.*
(iii) *a bi-ideal of M if A is a subsemiring of M and $AMA \subseteq A$.*
(iv) *an interior ideal of M if A is a subsemiring of M and $MAM \subseteq A$.*
(v) *a left (right) ideal of M if A is a subsemiring of M and $MA \subseteq A(AM \subseteq A)$.*
(vi) *an ideal if A is a subsemiring of $M, AM \subseteq A$ and $MA \subseteq A$.*
(vii) *a $k-$ideal if A is a subsemiring of $M, AM \subseteq A, MA \subseteq A$ and $x \in M$, $x + y \in A, y \in A$ then $x \in A$.*
(viii) *a bi-interior ideal of M if A is a subsemiring of M and $MAM \cap AMA \subseteq A$.*
(ix) *a left bi-quasi ideal (right bi-quasi ideal) of M if A is a subsemigroup of $(M, +)$ and $MA \cap AMA \subseteq A$ $(AM \cap AMA \subseteq A)$.*
(x) *a bi-quasi ideal of M if A is a subsemiring of M and Ais a left bi-quasi ideal and a right bi-quasi ideal of M.*
(xi) *a left quasi-interior ideal (right quasi-interior ideal) of M if A is a subsemiring of M and $MAMA \subseteq A$ $(AMAM \subseteq A)$.*
(xii) *a quasi-interior of M if A is a subsemiring of M and Ais a left quasi-interior ideal and a right quasi-interior ideal of M.*
(xiii) *a bi-quasi-interior ideal of M if A is a subsemiring of M and $AMAMA \subseteq A$.*
(xiv) *a left tri-ideal (right tri-ideal) of M if A is a subsemiring of M and $AMAA \subseteq A$ $(AAMA \subseteq A)$.*
(xv) *a tri-ideal of M if A is a subsemiring of M and $AMAA \subseteq A$ and $AAMA \subseteq A$.*
(xvi) *a left(right) weak-interior ideal of M if A is a subsemiring of M and $MAA \subseteq A($ $AAM \subseteq A)$.*
(xvii) *a weak-interior ideal of M if A is a subsemiring of M and Ais a left weak-interior ideal and a right weak-interior ideal)of M.*

3. Tri-quasi ideals of semirings

In this section, we introduce the notion of a tri-quasi ideal as a generalization of bi-ideals, quasi-ideals and interior ideals of semiring and study the properties of tri-quasi ideal of a semiring.

DEFINITION 3.1. *A non-empty subset A of a semiring M is said to be tri-quasi ideal of M if A is a subsemiring of M and $AAMAA \subseteq A$.*

Every tri-quasi ideal of a semiring M need not be bi-ideal, quasi-ideal, interior ideal bi-interior ideal, and bi-quasi ideals of M.

EXAMPLE 3.1. *Let $M = \{0, a, b, c\}$. Define the binary operations " $+$ " and " $\cdot$ " on M, with the following tables*

$+$	0	a	b	c
0	0	a	b	c
a	a	b	c	a
b	b	c	b	b
c	c	a	b	c

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	b	b
c	0	c	c	c

Then $(M, +)$ and $(M.\cdot)$ are semigroups. Therefore M is a semiring.
Let $B = \{0, b\}$. Then B is a tri-quasi ideal of M.

In the following theorem, we mention some important properties and we omit the proofs since proofs are straight forward.

THEOREM 3.1. *Let M be a semring. Then the following hold.*

(1) *Every left ideal is a tri-quasi ideal of M.*
(2) *Every right ideal is a tri-quasi ideal of M.*
(3) *Every quasi ideal is tri-quasi ideal of M.*
(4) *Every ideal is a tri-quasi ideal of M.*
(5) *Intersection of a right ideal and a left ideal of M is a tri-quasi ideal of M.*
(6) *If L is a left ideal and R is a right ideal of a semiring M then $B = RL$ is a tri-quasi ideal of M.*
(7) *Every bi-ideal of a semiring M is a tri-quasi ideal of a semiring M.*
(8) *Every interior ideal of semring M is a tri-quasi ideal of M.*
(9) *Let B be bi-ideal of a semiring M and I be interior ideal of M. Then $B \cap I$ is a tri-quasi ideal of M.*
(10) *Every bi-interior ideal of a semiring M, is a tri-quasi ideal of M.*
(11) *Every left bi-quasi ideal of a semiring M, is a tri-quasi ideal of M.*
(12) *Every right bi-quasi ideal of a semiring M, is a tri-quasi ideal of M.*
(13) *Every bi-quasi ideal of a semiring M, is a tri-quasi ideal of M.*

THEOREM 3.2. *Let M be a semiring. B is a tri-quasi ideal of $M$and $BB = B$ if and only if there exist a left ideal L and a right ideal$R$ such that $RL \subseteq B \subseteq R \cap L$.*

PROOF. Suppose B is a tri-quasi ideal of the semiring$M$.
Then $BBMBB \subseteq B$. Let $R = BM$ and $L = MB$.
Then L and R are left and right ideals of M respectively.

Therefore $RL \subseteq B \subseteq R \cap L$.
Conversely suppose that there exist L and R are left and right ideals of M respectively such that $RL \subseteq B \subseteq R \cap L$ .Then
$BBMBB \subseteq (R \cap L)(R \cap L)M(R \cap L)(R \cap L)$
$\subseteq (R)RML(L)$
$\subseteq RL \subseteq B$.
Hence B is a tri-quasi ideal of the semiring M. □

THEOREM 3.3. *The intersection of a tri-quasi ideal B of semiring M and a right ideal A of M is always tri-quasi ideal of M.*

PROOF. Suppose $C = B \cap A$.

$$CCMCC \subseteq BBMBB \subseteq B$$
$$CCMCC \subseteq AMAMA \subseteq A. \text{ Since } A \text{ is a left ideal of } M$$

Therefore $CACMCM \subseteq B \cap A = C$.

Hence the intersection of a tri-quasi ideal B of the semiring M and a right ideal A of M is always a tri-quasi ideal of M. □

THEOREM 3.4. *Let A and C be subsemirings of M and $B = AC$. If A is the left ideal then B is a tri-quasi-interior ideal of M.*

PROOF. Let A and C be subsemirings of M and $B = AC$ Suppose A is the left ideal of M. Then $B = AC$ is a left ideal.Then by Theorem [3.4], B is a tri-quasi ideal of the semiring M. □

COROLLARY 3.1. *Let A and C be subsemirings of a semiring M and $B = AC$. If C is a right ideal then B is a tri-quasi ideal of M.*

THEOREM 3.5. *Let M be a semiring and T be a non-empty subset of M. Then every subsemiring of T containing $TTMTT$ is a tri-quasi ideal of semiring M.*

PROOF. Let B be a subsemiring of T containing $TTMTT$. Then

$$\begin{aligned} BBMBB &\subseteq TTMTT \\ &\subseteq B. \end{aligned}$$

Therefore $BBMBB \subseteq B$.
Hence B is a tri-quasi ideal of M. □

THEOREM 3.6. *Let M be a semiring.Then B is a tri-quasi ideal of a semiring M if and only if B is a left ideal of some right ideal of a semiring M.*

PROOF. Let B be a tri-quasi ideal of the semiring M. Then $BBMBB \subseteq B$. Therefore BB is a left ideal of right ideal BBM of a semiring M.
Conversely suppose that B is a left ideal of some right ideal R of the semiring M.Then $RB \subseteq B, RM \subseteq R$. Hence $BBMBB \subseteq BMB \subseteq RMB \subseteq RB \subseteq B$. Therefore B is a tri-quasi ideal of the semiring M.

□

Corollary 3.2. *B is a tri-quasi ideal of a semiring M if and only if B is a right ideal of some left ideal of a semiring M.*

Theorem 3.7. *If B is a tri-quasi ideal of a semiring M, T is a subsemiring of M and $T \subseteq B$ then BT is a tri-quasi ideal of M.*

Proof. Obviously, BT is a subsemigroup of $(M, +)$. $BTBT \subseteq BT$.
Hence BT is a subsemiring of M.

$$\Rightarrow BTBTMBTBT \subseteq BBMBBT \subseteq BT.$$

Hence BT is a tri-quasi ideal of the semiring M. □

Theorem 3.8. *Let M be a semiring. If $M = M < a >$,for all $a \in M$ where $< a >$ is the smallest tri-quasi ideal generated by a. Then every tri-quasi ideal of M is a quasi ideal of M.*

Proof. Let B be a tri-quasi ideal of a semiring M and $a \in B$. Then

$$\begin{aligned}
&BBMBB \subseteq B \\
&\Rightarrow M < a > \subseteq MB, \ (BM = M) \\
&\Rightarrow M \subseteq MB \subseteq M \\
&\Rightarrow MB = M \\
&\Rightarrow BM = BMB \subseteq BBMBB \subseteq B \\
&\Rightarrow MB \cap BM \subseteq M \cap BM \subseteq B.
\end{aligned}$$

Therefore B is a quasi ideal of M. □

Theorem 3.9. *The intersection of $\{B_\lambda \mid \lambda \in A\}$ tri-quasi ideals of a semiring M is a tri-quasi-interior ideal of M.*

Proof. Let $B = \bigcap_{\lambda \in A} B_\lambda$. Then B is a subsemiring of M.
Since B_λ is a tri-quasi ideal of M, we have

$$\begin{aligned}
&B_\lambda B_\lambda M B_\lambda B_\lambda \subseteq B_\lambda, \text{ for all } \lambda \in A \\
&\Rightarrow BBMBB \subseteq B.
\end{aligned}$$

Hence B is a tri-quasi ideal of M. □

4. Tri-quasi simple semiring and regular semiring

In this section, we introduce the notion of a tri-quasi simple semiring and characterize the tri-quasi simple semiring using tri-quasi ideals of a semiring and study the properties of minimal tri-quasi ideals of a semiring. We also characterize regular semiring using tri-quasi ideals of a semiring.

Definition 4.1. *A semiring M is a left (right) simple semiring if M has no proper left (right) ideals of M*

Definition 4.2. *A semiring M is said to be simplesemring if M has no proper ideals of M*

Definition 4.3. *A semiring M is said to be tri-quasi simple semiring M if M has no tri-quasi ideals other than M itself.*

Theorem 4.1. *If M is a division semiring then M is a tri-quasi simple semiring.*

Proof. Let B be a proper tri-quasi ideal of the division semring M and $0 \neq a \in B$. Since M is a division semiring, there exist $b \in M$, such that $ab = 1$. Then there exist $x \in M$ such that $abx = x = xab$. Then $x \in BM$. Therefore $M \subseteq BM$. We have $BM \subseteq M$. Hence $M = BM$. Similarly we can prove $MB = M$.

$$\begin{aligned} M &= MB \\ &= BBMBB \subseteq B \\ M &\subseteq B \\ \text{Therefore} M &= B. \end{aligned}$$

Hence a division semiring M has no proper tri-quasi interior ideals. □

Theorem 4.2. *Let M be a left and a right simple semiring.Then M is a tri-quasi simple semiring.*

Proof. Let M be a simplesemring and B be a tri-quasi ideal of M.
Then $BBMBB \subseteq B$ and MB and BM are left and right ideals of M.
Since M is a left and right simple semiring, we have $MB = M$. $BM = M$.Hence

$$\begin{aligned} &BBMBB \subseteq B \\ \Rightarrow &BMB \subseteq B. \Rightarrow M \subseteq B. \end{aligned}$$

□

Theorem 4.3. *Let M be a semiring. Then M is a tri-quasi simple semiring if and only if $(a)_{tqi} = M$, for all $a \in M$, where $(a)_{tqi}$ is the tri-quasi ideal generated by a.*

Proof. Let M be a semiring. Suppose that $(a)_{tqi}$ is a tri-quasi ideal generated by a and M is a bi-quasi -interior simple semiring. Then $(a)_{tqi} = M$, for all $a \in M$.

Conversely suppose that B is a tri-quasi ideal of semiring M and $(a)_{tqi} = M$, for all $a \in M$. Let $b \in B$.
Then $(b)_{tqi} \subseteq B \;\Rightarrow\; M = (b)_{tqi} \subseteq B \subseteq M$.
Therefore M is a tri-quasi simple semiring. □

Theorem 4.4. *Let M be a semiring. M is a tri-quasi simple semiring if and only if $< a >= M$, for all $a \in M$ and where $< a >$ is the smallest tri-quasi ideal generated by a.*

Proof. Let M be a semiring. Suppose M is a tri-quasi simple semiring, $a \in M$ and $B = Ma$.
Then B is a left ideal of M.

Therefore, by Theorem[3.4], B is a tri-quasi ideal of M.
Therefore $B = M$. Hence $Ma = M$, for all $a \in M$.

$$
\begin{aligned}
Ma &\subseteq < a > \subseteq M \\
\Rightarrow M &\subseteq < a > \subseteq M. \\
\text{Therefore } M &= < a > .
\end{aligned}
$$

Suppose $< a >$ is the smallest tri-quasi ideal of M generated by a and $< a >= M$ and A is the tri-quasi ideal and $a \in A$. Then

$$
\begin{aligned}
< a > &\subseteq A \subseteq M \\
\Rightarrow M &\subseteq A \subseteq M.
\end{aligned}
$$

Therefore $A = M$. Hence M is a tri-quasi simple semiring. □

Theorem 4.5. *Let M be a semiring. Then M is a tri-quasi simple semriring if and only if $aaMaa = M$, for all $a \in M$.*

Proof. Suppose M is a tri-quasi simple semiring and $a \in M$.
Therefore $aaMaa$ is a tri-quasi ideal of M.
Hence $aMaMa = M$, for all $a \in M$.
Conversely suppose that $aaMaa = M$, for all $a \in M$.
Let B be a tri-quasi ideal of the semiring M and $a \in B$.

$$
\begin{aligned}
M &= aaMaa \\
M &= aaMaa \\
&\subseteq BBMBB \subseteq B \\
\text{Therefore} M &= B.
\end{aligned}
$$

Hence M is a tri-quasi simple semiring. □

Theorem 4.6. *If B is a minimal tri-quasi ideal of a semiring M then any two non-zero elements of B generated the same right ideal of M.*

Proof. Let B be a minimal tri-quasi ideal of M and $x \in B$. Then $(x)_R \cap B$ is a tri-quasi ideal of M. Therefore $(x)_R \cap B \subseteq B$.
Since B is a minimal tri-quasi ideal of M, we have $(x)_R \cap B = B \Rightarrow B \subseteq (x)_R$.
Suppose $y \in B$. Then $y \in (x)_R$, $(y)_R \subseteq (x)_R$.
Therefore $(x)_R = (y)_R$. Hence the theorem. □

Corollary 4.1. *If B is a minimal tri-quasi ideal of a semiring M then any two non-zero elements of B generates the same left ideal of M.*

Theorem 4.7. *Let M be a semiring and B be a tri-quasi ideal of M. Then B is minimal tri-quasi ideal of M if and only if B is a tri-quasi simple subsemiring.*

Proof. Let B be a minimal tri-quasi ideal of the semiring M and C be a tri-quasi ideal of B. Then $CCBCC \subseteq C$.

Therefore $CCBCC$ is a tri-quasi ideal of M.
Since C is a tri-quasi ideal of B,

$$\begin{aligned}&CCBCC = B\\ \Rightarrow &B = CCBCC \subseteq C\\ \Rightarrow &B = C.\end{aligned}$$

Conversely suppose that B is a tri-quasi simple subsemiring of M. Let C be a tri-quasi ideal of M and $C \subseteq B$.

$$\begin{aligned}&CCBCC = C\\ \Rightarrow &CCBCC \subseteq CCBCC \subseteq BBMBB \subseteq B,\\ \Rightarrow &C \text{ is a tri-quasi ideal of } B\ ,\\ \Rightarrow &B = C.\text{Since } B \text{ is a tri-quasi simple semiring.}\end{aligned}$$

Hence B is a minimal tri-quasi ideal of M. □

THEOREM 4.8. *Let M be a semiring and $B = RL$, where L and R are minimal left and right ideals of M repectively. Then B is a minimal tri-quasi ideal of M.*

PROOF. Obviously $B = RL$ is a tri-quasi ideal of M. Let A be a tri-quasi ideal of M such that $A \subseteq B$. Then MAA is a left ideal of M.

$$\begin{aligned}\Rightarrow MAA &\subseteq MBB\\ &= MRLRL\\ &\subseteq L,\ \text{since } L \text{ is a left ideal of } M.\\ \text{Similarly, we can prove } AAM &\subseteq R\\ \text{Therefore } MAA &= L,\ AAM = R\\ \text{Hence}\ \ B &= AAMMAA\\ &\subseteq AAMAA.\\ &\subseteq A\end{aligned}$$

Therefore $A = B$. Hence B is a minimal tri-quasi ideal of M.

□

THEOREM 4.9. *Let M be a regular idempotent semiring. Then B is a tri-quasi ideal of M if and only if $BBMBB = B$, for all tri-quasi ideals B of M.*

PROOF. Suppose M is a regular semiring, B is a tri-quasi ideal of M and $x \in B$. Then $BBMBB \subseteq B$ and there exist $y \in M$, such that $x = xxyxx \in BBMBB$. Therefore $x \in BBMBB$.
Hence $BBMBB = B$.

Conversely suppose that $BBMBB = B$, for all tri-quasi ideals B of M.
Let $B = R \cap L$, where R is a right ideal and L is a left ideal of M.
Then B is a tri-quasi ideal of M.

Therefore $(R \cap L)M(R \cap L)M(R \cap L) = R \cap L$

$$\begin{aligned} R \cap L &= (R \cap L)(R \cap L)MM(R \cap L)(R \cap L) \\ &\subseteq RMLML \\ &\subseteq RL \\ &\subseteq R \cap L \text{ (since } RL \subseteq L \text{ and } RL \subseteq R). \end{aligned}$$

Therefore $R \cap L = RL$. Hence M is a regular semiring. □

THEOREM 4.10. *Let M be a regular commutative semiring. Then every tri-quasi ideal of M is an ideal of M.*

PROOF. Let B be a tri-quasi ideal of M.and $C = BBMBB$.
Then $C = BBMBB = B$
$\Rightarrow BM = CM \subseteq CMC$, since M is regular
$\Rightarrow BM \subseteq BBMBBMBBMBB \subseteq B$.

□

THEOREM 4.11. *M is regular semiring if and only if $AB = A \cap B$ for any right ideal A and left ideal B of $-$semiring M.*

THEOREM 4.12. *Let B be a subsemiring of a regular idempotent semiring M. B can be represented as $B = RL$, where R is a right ideal and L is a left ideal of M if and only if B is a tri-quasi ideal of M.*

PROOF. Suppose $B = RL$, where R is right ideal of M and L is a left ideal of M.

$$\begin{aligned} BBMBB &= RLRLMRLRL \\ &\subseteq RL = B. \end{aligned}$$

Hence B is a tri-quasi ideal of the semiring M.

Conversely suppose that B is a tri-quasi ideal of the regular idempotent semiring M.Then $BBMBB = B$. Let $R = BM$ and $L = MB$.
Then $R = BM$ is a right ideal of M and $L = MB$ is a left ideal of M.

$$\begin{aligned} BM \cap MB &\subseteq BBMBB = B \\ \Rightarrow & BM \cap MB \subseteq B \\ \Rightarrow & R \cap L \subseteq B. \\ \text{We have } B \subseteq BM = R &\text{ and } B \subseteq MB = L \\ \Rightarrow & B \subseteq R \cap L \\ \Rightarrow & B = R \cap L = RL, \text{ since } M \text{ is a regular semiring.} \end{aligned}$$

Hence B can be represented as RL, where R is the right ideal and L is the left ideal of M. Hence the theorem. □

The following theorem is a necessary and sufficient condition for semiring M to be regular using tri-quasi ideal of M.

Theorem 4.13. *M is a regular semiring if and only if $B \cap I \cap L \subseteq BIL$, for any tri-quasi ideal B, ideal I and left ideal L of M.*

Proof. Suppose M be a regular semiring, B, I and L are tri-quasi ideal, ideal and left ideal of M respectively.
Let $a \in B \cap I \cap L$. Then $a \in aMa$, since M is regular.

$$\begin{aligned} a \in aMa &\subseteq aMaMa \\ &\subseteq BIL \end{aligned}$$

Hence $B \cap I \cap L \subseteq BIL$.
Conversely suppose that $B \cap I \cap L \subseteq BIL$, for any tri-quasi ideal B, ideal I and left ideal L of M. Let R be a right ideal and L be left ideal of M. Then by assumption, $R \cap L = R \cap M \cap L \subseteq RML \subseteq RL$. We have $RL \subseteq R$, $RL \subseteq L$. Therefore $RL \subseteq R \cap L$. Hence $R \cap L = RL$.
Thus M is a regular semiring. □

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Murali Krishna Rao Marapureddy, Department of Mathematics, Sankethika Institute of Tech. and Management, Visakhapatnam, Andhra Pradesh, postal code 530 040, INDIA.

Email address: mmarapureddy@gmail.com