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A STUDY ON PROPERTIES OF LARGEST REDUCED SUBMODULES OF FINITE DIMENSIONAL POLYNOMIAL MODULES

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Abstract. Let k be a field with characteristic zero, R be the ring $k[x_1, \cdots, x_n]$, and I be a monomial ideal of R. We study properties of the largest reduced submodules, $\mathfrak{R}(M)$ of finite dimensional polynomial modules. We also introduce a method to calculate generators of $\mathfrak{R}(M)$ and we classify $\mathfrak{R}(M)$, into four types, for which Gorenstein and almost Gorenstein rings can characterize some of them.

1. Introduction

Reduced rings play an important role in algebra. A reduced module as a generalization of a reduced ring was first defined by Lee and Zhou in [**10**]. Reduced modules have since been studied by [**9, 11, 12**] alongside others. A module M is reduced over a commutative ring S if $a^2m = 0$ implies $am = 0$ for all $a \in S, m \in M$. Among other applications, reduced modules form a full subcategory of R-Mod on which the I-torsion functor Γ_I is representable, i.e., if M is an I-reduced R-module, then $\Gamma_I(M) \cong \mathrm{Hom}(R/I, M)$, see [**12**]. Let k be a field and $R := k[x_1, \cdots, x_n]$. Consider the full subcategory, $\mathfrak{C}$ of R-Mod consisting of R-modules of the form $M := R/I$, with $\dim_k(R/I) < \infty$, $\mathfrak{C}_{red}$ contains all reduced submodules of $M \in \mathfrak{C}$ and The largest reduced submodule of M is denoted by $\mathfrak{R}(M)$, [**2**]. The connection between reduced submodules, in particular, $\mathfrak{R}(M)$ and Socle of a module has also been studied in [**2**], where $M \in \mathfrak{C}$. The paper is organize as follows: In Section 2 we study some properties of $\mathfrak{R}(M)$: we show that $\mathfrak{C}_{\mathrm{red}}$ is abelian full subcategory of

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R-Mod (Theorem 2.1) we also prove that the class of ideals I of R forms Oka family, where $\mathfrak{R}(M) = J/I$ (Theorem 2.2). Finally, we show that Koszul cohomologies are reduced modules. In Section 3 for $M := k[x,y]/I$ as $k[x,y]$-module we introduce a general formula to calculate generators of $\mathfrak{R}(M)$ (Theorem 3.1) and classify $\mathfrak{R}(M)$ into four types, where type 4 should be subdivided further into two as type $4A$ and $4B$. The classification is mainly based on a combinatorial object called Young diagram, which is defined as a collection of boxes or cells arranged in left-justified rows, with a (weakly) decreasing number of boxes in each row, [**7**]. We also managed to get a general algebraic formula for some of the types (type 1, 2, and 3). However, couldn't find a general algebraic formula for type $4A$ and $4B$. So, the authors suspected that type $4A$ and type $4B$ should be divided further so that we able to determine a general algebraic formula and characterize them. We left this as an open problem. $\mathfrak{R}(M) = J/I$ is Type 1, type 2, type 3 if J is x-tight and y-tight ideal, principal ideal (generated by a single monomial), pure power ideal (complete intersection) respectively, see Theorem 3.2. In Theorem 3.3 it has been shown that $\mathfrak{R}(M) = J/I$ is type $4A$ and type $4B$ if J is either x-tight or y-tight and neither x-tight nor y-tight ideal of R, respectively. In Section 4 we characterize some types of $\mathfrak{R}(S)$ using Gorenstein and almost Gorenstein rings, when S is a ring. Finally, we posed some open questions.

2. Properties of $\mathfrak{R}(M)$

Definition 2.1. [**2**] Let $M \in \mathfrak{C}$. The largest reduced submodule of M is defined as $\mathfrak{R}(M) = (0 :_M \mathbf{m})$, where $\mathbf{m} = \langle x_1, \cdots, x_n \rangle$.

Let R be a ring. A multiplicatively closed subset of R is a set S in R such that $1 \in S$ and for any two elements $s, s' \in S$, their product ss' is also in S. In this Section, we study some properties of the submodule $\mathfrak{R}(M), M \in \mathfrak{C}$.

Proposition 2.1. $S^{-1}(\mathfrak{R}(M)) = \mathfrak{R}(S^{-1}(M))$.

Proof. To show $S^{-1}(\mathfrak{R}(M)) \subseteq \mathfrak{R}(S^{-1}(M))$, let $y = \frac{m}{s} \in S^{-1}(\mathfrak{R}(M)))$ and $ay \neq 0$ which implies $a\frac{m}{s} \neq 0$, it follows that $am \neq 0$. Since $m \in \mathfrak{R}(M)$, we have $a^2m \neq 0 \Rightarrow a^2\frac{m}{s} \neq 0$ which implies that $a^2y \neq 0$, hence $y \in \mathfrak{R}(S^{-1}(M))$. Conversely, suppose $y = \frac{m}{s} \in \mathfrak{R}(S^{-1}(M))$, where $m \in M$. Let $am \neq 0 \Rightarrow a\frac{m}{s} \neq 0 \Rightarrow ay \neq 0$, by hypothesis this implies that $ay^2 \neq 0 \Rightarrow a\frac{m^2}{s^2} \neq 0$, which follows that $am^2 \neq 0$. Hence, $\mathfrak{R}(S^{-1}(M)) \subseteq S^{-1}\mathfrak{R}(M)$.

□

Lemma 2.1. Let $M \in \mathfrak{C}$ and $\bar{m} \in M$ then there exists a positive integer k such that $\langle x_1, \cdots, x_n \rangle^k \bar{m} = \bar{0}$.

Proof. We can choose a positive integer k in such a way that $\langle x_1, \cdots, x_n \rangle^k \subseteq I$ then $\langle x_1, \cdots, x_n \rangle^k \bar{m} = \bar{0}$. □

[**13**] Let R be a commutative noetherian ring and I, J be ideals of R. For an R-module M, $\Gamma_{I,J}(M) = \{m \in M : I^n m \subseteq Jm, n >> 1\}$ is an R-submodule of M. M is said to be (I, J)-*torsion* (resp. (I, J)-*torsion free*) when $\Gamma_{I,J}(M) = M$ (resp.

$\Gamma_{I,J}(M) = 0$). For an integer i, the i^{th} right derived functor of $\Gamma_{I,J}$ is denoted by $H^i_{I,J}$ and will be referred to as the i^{th} *local cohomology functor* with respect to (I, J).

PROPOSITION 2.2. Let $M \in \mathfrak{C}$. I and J be any monomial ideals of $k[x_1, \cdots, x_n]$. Then M is (I, J)-torsion, i.e., $\Gamma_{I,J}(M) = M$.

PROOF. For $J = 0$, any monomial ideal I of $k[x_1, \cdots, x_n]$ is contained in the maximal ideal $\langle x_1, \cdots, x_n \rangle$ of $k[x_1, \cdots, x_n]$. By Lemma 2.1, there exists $k \in \mathbb{Z}^+$ such that $I^k m_i = 0$ for each generators m_i of $M \in \mathfrak{C}$, then for any $m \in M$, $I^k m = I^k \sum_i f_i m_i = \sum_i f_i I^k m_i = 0$, where $f_i \in k[x_1, \cdots, x_n]$, so $M \subseteq \Gamma_I(M)$. Moreover, $\Gamma_I(M)$ is a submodule of M. Thus $\Gamma_I(M) = M$. Let $m \in M$ for any monomial ideal I, there exists a monomial ideal J such that $I \subseteq J$ and for $n >> 1$ we have $I^n \subseteq I \subseteq J$, then $I^n m \subset Jm$ and thus $m \in \Gamma_{I,J}(M)$. Since $\Gamma_{I,J}(M)$ is a submodule of M we have $\Gamma_{I,J}(M) \subseteq M$. □

PROPOSITION 2.3. Let $M \in \mathfrak{C}$. Then,

1. $H^i_{I,J}(M) = 0$ for all $i > 0$.
2. $H^i_{I,J}(M)$ is an (I, J)-torsion R-module for any $i \geqslant 0$.
3. $M/\Gamma_{I,J}(M)$ is an (I, J)-torsion free R-module.
4. $H^i_{I,J}(M) \cong H^i_{I,J}(M/\Gamma_{I,J}(M))$.

PROOF. From Proposition 2.2, $M \in \mathfrak{C}$ is (I, J)-torsion, then the proof for all of them follows from [**13**, Corollary 1.13]. □

PROPOSITION 2.4. For $M \in \mathfrak{C}$, $\mathfrak{R}$ commutes with both the functors $\Gamma_{I,J}$ and Γ_I, where I and J are monomial ideals.

PROOF. $\mathfrak{R}$, Γ_I and $\Gamma_{I,J}$ are functors over the full subcategory $\mathfrak{C}$ of R-Mod. By Proposition 2.2, $\mathfrak{R}(\Gamma_I(M)) = \mathfrak{R}(M)$ and similarly $\Gamma_I(\mathfrak{R}(M)) = \mathfrak{R}(M)$. Therefore, $\mathfrak{R}(\Gamma_I(M)) = \Gamma_I(\mathfrak{R}(M))$ and $\mathfrak{R}(\Gamma_{I,J}(M)) = \Gamma_{I,J}(\mathfrak{R}(M))$. □

PROPOSITION 2.5. The functor $\mathfrak{R}$ commutes with direct limits.

PROOF. By [**3**, Proposition 3.4.4], the I-torsion functor Γ_I commutes with direct limits, then by Proposition 2.4 $\mathfrak{R}$ commutes with direct limits. □

In general category of reduced modules is not abelian.

THEOREM 2.1. Let $\mathfrak{C}_{\text{red}}$ be the full subcategory R-Mod of all reduced submodules of M in $\mathfrak{C}$. Then, $\mathfrak{C}_{\text{red}}$ is abelian full subcategory of R-Mod.

PROOF. It is enough to show that $\mathfrak{C}_{\text{red}}$ contains kernel and cokernel of a morphism. Consider the homomorphism $f : N_1 \longrightarrow N_2$, where N_1 and N_2 are in $\mathfrak{C}_{\text{red}}$, since reduced modules are closed under submodule, we have $\ker(f) \in \mathfrak{C}_{\text{red}}$ and also $\mathbf{m}(N_2/\text{im}(f)) = 0$, hence $N_2/\text{im}(f) \subseteq \mathfrak{R}(M)$, for some $M \in \mathfrak{C}$, which shows that $\mathfrak{C}_{\text{red}}$ contains the cokernel of f, where $\mathbf{m}$ is a maximal ideal of R.

□

2.1. Family of ideals in which $\mathfrak{R}(M)$ being derived is filter, monoidal and strongly Oka. Let $\mathfrak{F}$ be a family of ideals in $R := k[x_1, \cdots, x_n] \in \mathfrak{T}$. We say

1. $\mathfrak{F}$ is a semifilter if, for all $I, J \leqslant R$, $I \supseteq J \in \mathfrak{F} \Rightarrow I \in \mathfrak{F}$;
2. $\mathfrak{F}$ is a filter if it is a semifilter and $A, B \in \mathfrak{F} \Rightarrow A \cap B \in \mathfrak{F}$;
3. $\mathfrak{F}$ is monoidal if $A, B \in \mathfrak{F} \Rightarrow AB \Rightarrow \mathfrak{F}$;

It is well known that a class of monomial ideals is closed under intersection, see in [**14**].

REMARK 2.1. Note that 0 and R are monomial ideals of $R := k[x_1, \cdots, x_n]$ generated by $\emptyset$ and 1_R respectively.

An ideal family $\mathcal{T}$ in a ring R with $R \in \mathcal{T}$ is said to be *strongly Oka* family if, $(I, J), (I : J) \in \mathcal{T}$, then $I \in \mathcal{T}$. For a monomial ideals I and J. Define $\mathfrak{T} := \left\{ I \leqslant R \mid \mathfrak{R}(M) = \frac{J}{I}, M \in \mathfrak{C} \right\}$. $R \in \mathfrak{T}$ since $M = \frac{R}{R} = 0$, hence $\mathfrak{R}(0) = 0$.

THEOREM 2.2.
1. $\mathfrak{T}$ is semifilter,
2. $\mathfrak{T}$ is filter,
3. $\mathfrak{T}$ is monoidal.
4. The family $\mathfrak{T}$ is strongly Oka.

PROOF.
1. Let $I, J \leqslant R$. $I \supseteq J$ and $J \in \mathfrak{T}$, there exists a finite dimensional $M = \frac{k[x_1, \cdots, x_n]}{J}$ such that $\mathfrak{R}(M) = \frac{J_1}{J}$, if $J = I$, its done. Otherwise, since I contains every generator of J, there exists a finite dimensional $M_1 = \frac{k[x_1, \cdots, x_n]}{J}$ such that $\mathfrak{R}(M_1) = \frac{K}{I}$, for some monomial ideal K. Thus $I \in \mathfrak{T}$.
2. Since $A, B \in \mathfrak{T}$, there exists A_1 and A_2 such that $\mathfrak{R}(M_1) = \frac{A_1}{A}$ and $\mathfrak{R}(M_2) = \frac{B_1}{B}$ and we have $A \cap B \subseteq A_1 \cap B_1$. Let $m \in A_1 \cap B_1$, which implies $m \in A_1$ and $m \in B_1$ and thus by [**2**, Theorem 2.1], $x_i m = 0 \ (\mathrm{mod} A)$ for each i and $x_i m = 0 \ (\mathrm{mod}\ B)$ and hence $x_i m = 0 \ (\mathrm{mod}(A \cap B))$ and thus m is element of the reduced submodule $\mathfrak{R}(K) = \frac{A_1 \cap B_1}{A \cap B}$.
3. Let $A = \langle s \rangle, B = \langle t \rangle \in \mathfrak{T}$, then there exists $\mathfrak{R}(M_1) = \frac{J_1}{A}, \mathfrak{R}(M) = \frac{J_2}{B}$ for some monomial ideals J_1 and J_2. However, $AB = \langle st \rangle$ is a monomial ideal whose generating set is st and $M = \frac{k[x_1, \cdots, x_n]}{AB}$ is a finitely dimensional module and so we have the corresponding $\mathfrak{R}(M) = \frac{J}{AB}$, thus $AB \in \mathfrak{T}$.
4. Let $I + J, (I : J) \in \mathfrak{T}$, which implies $\frac{k[x_1, \cdots, x_n]}{I+J}, \frac{k[x_1, \cdots, x_n]}{(I:J)} \in \mathfrak{C}$, i.e., generators of $I + J$ and $(I : J)$ contains every powers of x_i. Now, let $a \in (I : J)$, i.e., $aJ \subseteq I$ hence, we must have powers of each x_i among generators of I and hence $\frac{k[x_1, \cdots, x_n]}{I} \in \mathfrak{C}$. Thus, $I \in \mathfrak{T}$.

□

2.2. Koszul cohomologies are reduced modules. Let $\mathbf{x} := x_1, x_2, \cdots, x_d$ be a sequence of elements of a ring R, and M an R-module. The *Koszul complex* on $\mathbf{x}$ is the complex

$$K^\bullet(\mathbf{x}; R) = K^\bullet(x_1; R) \otimes_R \cdots \otimes_R K^\bullet(x_d; R),$$

where $K^\bullet\big(x_i;R\big)$, for each $i \leqslant d$, is the complex

$$0 \longrightarrow R \xrightarrow{x_i} R \longrightarrow 0$$

with R in degrees -1 and 0. *The Koszul complex* of $\mathbf{x}$ on M is the complex $K^\bullet\big(\mathbf{x};M\big) = K^\bullet\big(\mathbf{x};R\big) \otimes_R M$. *The Koszul cohomology* of $\mathbf{x}$ on M is $H^j(\mathbf{x};M) = H^j(K^\bullet\big(\mathbf{x};M\big))$ for $j \in \mathbb{Z}$.

Proposition 2.6. Let $\mathbf{x} = x_1, \cdots, x_n$ be a sequence of elements of $R = k[x_1, \cdots, x_n]$ and $M \in \mathfrak{C}$. For each j, $H^j(\mathbf{x};M)$ is a reduced R-module.

Proof. The koszul cohomology $H^j(\mathbf{x};M)$ is a submodule of some factor module of the form $M^l/\text{im } d_{j-1}$. By [**6**, Proposition 6.20], $\langle \mathbf{m} \rangle$ annihilates $H^j(\mathbf{x};M)$. By definition of $\mathfrak{R}(M)$, it follows that $H^j(\mathbf{x};M) \subseteq \mathfrak{R}(M^l/\text{im } d_{j-1})$, since $\mathfrak{R}(M^l/\text{im } d_{j-1})$ is reduced so is its submodule. □

Corollary 2.1. Let $\mathbf{x} = x_1, \cdots, x_n$ (resp. $\mathbf{m} = \langle x_1, \cdots, x_n \rangle$) be a sequence of elements of $R = k[x_1, \cdots, x_n]$ (resp. maximal ideal R), and $M \in \mathfrak{C}$, $N \in \mathfrak{C}_{\text{red}}$. Then the following follows:

1. $H^0(\mathbf{x};M) \cong k$,
2. $H^{-n}(\mathbf{x};M) = \mathfrak{R}(M)$,
3. $H^{-n}(\mathbf{x};N) = H^0(\mathbf{x};N) = N$.

Proof. By [**6**, Exercise 6.8] we have $H^0(\mathbf{x};M) = M/\mathbf{m}M$ and $H^{-n}(\mathbf{x};M) = (0 :_M \mathbf{m})$. Its easy to see that the former is isomorphic with k and the latter holds true by [**2**, Theorem 2.1]. The proof of three is a special case of 1 and 2. □

3. Classification of $\mathfrak{R}(M)$ for $M = k[x,y]/I$ and their characterization

In this Section, for $M := R/I \in \mathfrak{C}$ and $R := k[x,y]$ we find a general algebraic formula that determine $\mathfrak{R}(M)$ and we classify $\mathfrak{R}(M)$ into four types. The lexicographic order in $k[x,y]$ is given by $1 > x > x^2 > \cdots y > y^2 > \cdots$.

Theorem 3.1. Let $M \in \mathfrak{C}$ and $I = \langle m_1, m_2, \cdots, m_t \rangle$, assuming that $m_1 > m_2 > \cdots > m_t$ in the lexicographic order, then the monomial k-basis of $\mathfrak{R}(M)$ is given by:

$$g_i = \frac{lcm(m_i, m_{i+1})}{xy}$$

for $i = 1, \cdots, t-1$(with lcm denoting the least common multiple of the two monomials) and $J = \langle g_1, g_2, \cdots, g_k \rangle$ for $i = 1, \cdots, l$ for $l \leqslant t$ such that $\mathfrak{R}(M) = J/I$.

Proof. Let $V = \{m_1, m_2, \cdots, m_t\}$ be set of generators of I. Assume that $lcm(m_i, m_{i+1}) = n_i$ is located at the junction box of the row and column of the Young diagram which contains the monomials m_i and m_{i+1}. Then dividing this n_i by the monomial xy, means we are shifting back along the diagonal one step, i.e., we are reducing the powers of x and y by one of n_i, denote the resulting monomial g_i, repeating this for other consecutive monomials we get $U = \{g_1, g_2, \cdots, g_l\}$ and the ideal generated by this set denoted by J which we get from the set V containing the monomials $m_i, i = 1, \cdots, t$, thus by [**2**, Theorem 2.1], $\mathfrak{R}(M) = J/I$. □

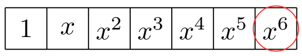
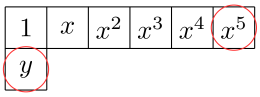
REMARK 3.1. Note that it is also possible to generate an ideal I from the given ideal J of R such that $\mathfrak{R}(M) = J/I$, see the algorithm developed in [**15**].

DEFINITION 3.1. [**4**] An $\mathbf{m}$-primary monomial ideal is called *x-tight* if the power of x in every generator is exactly by one greater than of the preceding generator. That is, I is *x-tight* of order r if and only if $I = \langle x^i y^{b_i} \rangle_{i=0}^{r}$ with $b_0 > \cdots > b_r = 0$. If $J = \langle x^{a_j} y^{s-j} \rangle_{j=0}^{s}$ is an ideal, where $0 = a_0 < \cdots < a_s$, then J is called *y-tight* of order s.

EXAMPLE 3.1. Let $I = \langle x^4, y^3 \rangle$ and $J = \langle x^2, xy, y^4 \rangle$. Then, I is neither x-tight nor y-tight. However, J is x-tight, but not y-tight.

DEFINITION 3.2. Let $R := k[x, y]$ and $M \in \mathfrak{C}$. A generator m of M is an *outside corner generator* if $xm = ym = 0, m$ is *inner* if it is not an outside corner generator.

EXAMPLE 3.2. In Figure 1 those generators of M circled red are the outside corner generators of M and the rest are inner generators of M.

	Young diagram	M	$\mathfrak{R}(M)$	type
1.		$\frac{k[x]}{\langle x^7 \rangle}$	$\frac{\langle x^6 \rangle}{\langle x^7 \rangle}$	2
2.		$\frac{k[x,y]}{\langle x^6, xy, y^2 \rangle}$	$\frac{\langle x^5, y \rangle}{\langle x^6, xy, y^2 \rangle}$	4A

We classify $\mathfrak{R}(M) = J/I$ based on properties of its associated Young diagrams into four as follows:

- Type 1: The number of boxes along the rows and columns decreases by one (stair shape).
- Type 2: Every row (resp. column) contains an equal number of boxes (rectangular shape).
- Type 3: The Young diagram contains one row and one column each, containing at least three boxes (longer "L" shape).
- Type 4: Mixed (none of the above three types) Type 4 further classified as:
  - Type 4A: The number of boxes in some rows (resp. columns) decreases by one, and at least two columns (resp. rows) contain same number of boxes (partial stair shape).
  - Type 4B: There is at least one pair of columns and another pair of rows with the same number of boxes.

Characterization of type $1, 2$ and 3 are given in Theorem 3.2 and Theorem 3.3 depicts characterization of type $4A$ and $4B$.

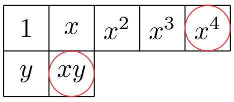
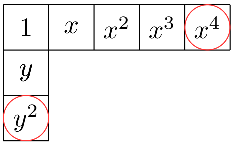
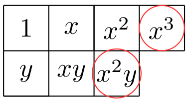
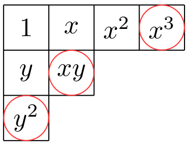
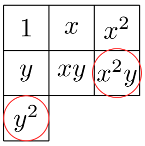
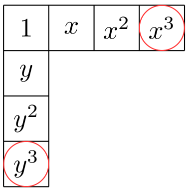
	Young diagram	M	$\mathfrak{R}(M)$	type
3.		$\frac{k[x,y]}{\langle x^5,x^2y,y^2\rangle}$	$\frac{\langle x^4,xy\rangle}{\langle x^5,x^2y,y^2\rangle}$	4A
4.		$\frac{k[x,y]}{\langle x^5,xy,y^3\rangle}$	$\frac{\langle x^4,y^2\rangle}{\langle x^5,xy,y^3\rangle}$	3
5.		$\frac{k[x,y]}{\langle x^4,x^3y,y^2\rangle}$	$\frac{\langle x^3,x^2y\rangle}{\langle x^4,x^3y,y^2\rangle}$	4A
6.		$\frac{k[x,y]}{\langle x^4,x^2y,xy^2,y^3\rangle}$	$\frac{\langle x^3,xy,y^2\rangle}{\langle x^4,x^2y,xy^2,y^3\rangle}$	4A
7.		$\frac{k[x,y]}{\langle x^3,xy^2,y^3\rangle}$	$\frac{\langle x^2y,y^2\rangle}{\langle x^3,xy^2,y^3\rangle}$	4B
8.		$\frac{k[x,y]}{\langle x^4,xy,y^4\rangle}$	$\frac{\langle x^3,y^3\rangle}{\langle x^4,xy,y^4\rangle}$	3

FIGURE 1. $\mathfrak{R}(M)$ for a 7-dimensional k-module $M \in \mathfrak{C}$, the circled are generators for $\mathfrak{R}(M)$.

THEOREM 3.2. Let $M = k[x,y]/I$ be an $R =: k[x,y]$-module such that $\dim M < \infty$. Then $\mathfrak{R}(M) = J/I$ is:

1. Type-1 if and only if J is both x-tight and y-tight ideal of R.
2. Type-2 if and only if J is a principal ideal of R.
3. Type-3 if and only if J is a pure power ideal of R.

PROOF. 1. Let $\mathfrak{R}(M)$ be a type-1 submodule of M and λ the associated Young diagram. Then, the number of boxes along rows and columns decreases by one, for which the monomials at the outside corner of λ are with the same degree. Thus, $\mathfrak{R}(M)$ has a general formula:

$$\mathfrak{R}(M) = J/I = \frac{\langle x^{n-1}, x^{n-2}y, \cdots, xy^{n-2}, y^{n-1}\rangle}{\langle x^n, x^{n-1}y, \cdots, xy^{n-1}, y^n\rangle}$$

and this shows that J is both x-tight and y-tight ideal of R, where n is a positive integer. Conversely, let J be an x and y-tight ideal of R then by the algorithm in [**15**] we generate an ideal I such that $\mathfrak{R}(M) = J/I$ and the associated Young diagram has the property that the number of boxes along rows and columns decreases by one and hence $\mathfrak{R}(M)$ is type 1.

2. Suppose $\mathfrak{R}(M)$ be a type-2 submodule of M and λ the associated Young diagram. Then every row (resp. columns) contains the same number of boxes, so λ is rectangular and the monomial at the outside corner of λ is the only one. Therefore, J is a principal ideal and the general formula is given as:

$$\mathfrak{R}(M) = J/I = \frac{\langle x^{a-1}y^{b-1}\rangle}{\langle x^a, y^b\rangle}$$

where $a, b \geqslant 2$. The converse is similar to the converse of proof of 1.

3. Suppose $\mathfrak{R}(M)$ be type 3 submodule of M and λ the associated Young diagram. Then λ contains one row and one column each containing at least three boxes. The outside corner elements are x^{a-1} and y^{b-1}, where $a, b \geqslant 3$ and the general formula is given as:

$$\mathfrak{R}(M) = J/I = \frac{\langle x^{a-1}, y^{b-1}\rangle}{\langle x^a, xy, y^b\rangle}$$

which shows that J is generated by a pure power ideal. The converse is similar to the converse of proof of 1.

□

LEMMA 3.1. If I is an x-tight or y-tight ideal of R, then so is J, where $\mathfrak{R}(M) = J/I$.

THEOREM 3.3. Let $\mathfrak{R}(M) = J/I$ and J doesn't have the form of type $1, 2$ and 3. Then,

1. $\mathfrak{R}(M)$ is type $4A$ if and only if I is an x-tight or y-tight (not both) ideal of R.
2. $\mathfrak{R}(M)$ is type $4B$ if and only if I is neither x-tight nor y-tight ideal of R.

PROOF. 1. Let $\mathfrak{R}(M)$ be type $4A$ and λ be the associated Young diagram. The number of boxes in some rows decreases by one, and at least two columns contain the same number of boxes. This shows that every power of the variable y appears in the generators of the ideal I. Thus, I is y-tight ideal. Conversely, suppose that I is y-tight ideal of R, then by Lemma 3.1 and Theorem 3.1 J is y-tight ideal of R such that $\mathfrak{R}(M) = J/I$ and this shows that the number of boxes in some rows decreases by one, and at least two columns contain the same number of boxes. Therefore, $\mathfrak{R}(M)$ is type $4A$.

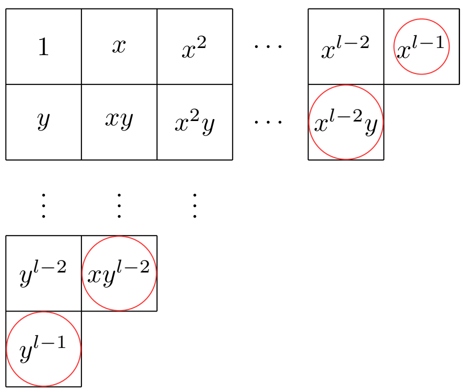
2. Since we have pairs of columns and rows with the same number of boxes. The Powers of each variable x and y don't strictly decrease within the monomial generators of I. Therefore, I is neither an x-tight nor y-tight ideal of R. Conversely, suppose I is neither an x-tight nor y-tight ideal of R. When we depict generators of $M = R/I$ in a Young diagram, at least one pair of columns and another pair of rows with the same number of boxes can be seen. This shows that the corresponding $\mathfrak{R}(M)$ is type $4B$.

□

Proposition 3.1. Let $M := \frac{k[x,y]}{I} \in \mathfrak{C}$ such that I is both x-tight and y-tight with $n+1$ distinct generators. Then,

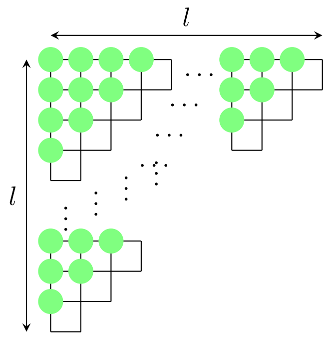
J is also an x-tight and y-tight ideal of R, where $\mathfrak{R}(M) = J/I$, $\dim_k \mathfrak{R}(M) = l$ and $\dim_k M = \frac{l(l+1)}{2}$.

Proof. 1. The Young diagram associated to M takes the following shape



The ideal J generated by elements at the outside corners of this Young diagram is $\langle x^{l-1}, x^{l-2}y, \cdots, xy^{l-2}, y^{l-1}\rangle$ which has l distinct linearly independent generators and also this ideal is both x-tight and y-tight. Then $\dim_k \mathfrak{R}(M) = l$ and by [**2**, Theorem 2.1], $\mathfrak{R}(M) = \frac{J}{I}$.

2. Identify each square in the Young diagram with a green dot at the top left corner of the square. We get



The dots when combined form shapes of triangles and their numbers form a sequence of triangle numbers, namely; $1, 3, 6, 10, 15, 21, \cdots$ whose sum of first l terms is given by $\frac{l(l+1)}{2}$, see [**8**]. Since these dots are in a one-to-one correspondence with the squares of the Young diagram, which are also in a one-to-one correspondence with the generators of M, $\dim_k M = \frac{l(l+1)}{2}$.

□

4. Characterization of $\mathfrak{R}(S)$, when $S = R/I$ is a ring

Definition 4.1. Let $S \in \mathfrak{C}$ (a zero-dimensional local ring). S is said to be Gorenstein if and only if $S \cong \mathrm{Hom}_k(S, k)$ (dual of S), [**5**].

Definition 4.2. The largest reduced ideal for a ring $S \in \mathfrak{C}$ is defined as the largest reduced submodule of S, when considering S as a right S-module or left S-module. Then $\mathfrak{R}(S) = (0 :_S \mathbf{n})$, where $\mathbf{n}$ is a maximal ideal for S. If $\dim_k(\mathfrak{R}(S)) = 1$, then S is Gorenstein.

Proposition 4.1. Let $S \in \mathfrak{C}$. The following are equivalent:

1 S is Gorenstein.
2 S is injective as an S-module.
3 $\mathfrak{R}(S)$ is simple and it is type 2.
4 $\mathrm{Hom}_k(S, k)$ can be generated by one element.

Proof. By [**2**], we have $\mathrm{soc}(S) = \mathfrak{R}(S)$, then the proof follows from [**5**, Proposition 21.5]. □

Proposition 4.2. Consider the x-tight ideal $I = \langle x^2, xy, y^n\rangle$, $n \geqslant 3$. Then we have an R-module $S = R/I$ and Then

1. $S/\mathfrak{R}(S)$ is Gorenstein.
2. $S/\mathfrak{R}(S)$ is injective module over itself.
3. $\mathfrak{R}(S/\mathfrak{R}(S))$ is type 2.
4. $\mathfrak{R}(S)$ is type $4A$.

PROOF. 1. $\mathfrak{R}(S/\mathfrak{R}(S)) = \langle y^{n-2}\rangle/\langle x, y^{n-1}\rangle$ and its dimension is 1. Hence, $S/\mathfrak{R}(S)$ is Gorenstein.

2. Since $S/\mathfrak{R}(S)$ is Gorenstein then by Proposition 3.2 $S/\mathfrak{R}(S)$ is injective module over itself.
3. It is clear from the proof of 1, 2 and Proposition 4.1.
4. Since I is generated by x-tight ideal, $\mathfrak{R}(S)$ is type $4A$.

□

An Artinian ring $S = \frac{k[x,y]}{I}$ is said to be *almost Gorenstein* of *type k* if $|\partial(I)| = k$, [**1**]. Note that $\mathfrak{R}(S)$ is generated by $\partial(I)$.

PROPOSITION 4.3. Consider the ring $S = k[x,y]/I$ then

1. If I is generated by n monomials such that $\mathfrak{R}(G)$ is type 1 then S is almost Gorenstein of type $(n-1)$.
2. If $\mathfrak{R}(G)$ is type 3 then S is almost Gorenstein of type 2.

PROOF. 1. Let $I = \langle x^n, x^{n-1}y, \cdots, xy^{n-1}, y^n\rangle$, $n \geqslant 2$ and then, $\mathfrak{R}(S) = \frac{\langle x^{n-1}, x^{n-2}y, \cdots, xy^{n-2}, y^{n-1}\rangle}{\langle x^n, x^{n-1}y, \cdots, xy^{n-1}, y^n\rangle}$, i.e., $\mathfrak{R}(S)$ is generated by $n-1$ monomials, thus S is almost Gorenstein of type $(n-1)$.

2. Since $\mathfrak{R}(S)$ is type 3 then, $\mathfrak{R}(S) = \frac{\langle x^{a-1}, y^{b-1}\rangle}{\langle x^a, xy, y^b\rangle}$, $a, b > 3$. This shows that $\mathfrak{R}(S)$ is generated by only two monomials and thus $\mathfrak{R}(S)$ is Gorenstein of type 2.

□

REMARK 4.1. The converses of the above Proposition isn't true in general. Consider the ring $S = k[x,y]/\langle x^3, xy, y^2\rangle$ and $\mathfrak{R}(S) = \langle x^2, y\rangle/\langle x^3, xy, y^2\rangle$, which is almost Gorenstein of type 2, but it is neither type 1 nor type 3.

QUESTION 4.1. Is there a possible way to find a general algebraic formula to all type 4 submodules, $\mathfrak{R}(M)$ of $M \in \mathfrak{C}$?

QUESTION 4.2. Can we find an algebraic characterization to all of the type 4 submodules $\mathfrak{R}(M)$ of $M \in \mathfrak{C}$?

QUESTION 4.3. Is there a method to find the generators of $\mathfrak{R}(M)$, where $M = \frac{k[x_1, \cdots, x_n]}{I}$, where $n \geqslant 3$? Is it still possible to classify $\mathfrak{R}(M)$ and characterize them?

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References

1. G. Agnarsson and N. Epstein, On monomial ideals and their socles, Order **37** (2) (2020), 341–369.
2. T. Abebaw, N. Arega, T. W. Bihonegn, and D. Ssevviiri, Reduced submodules of finite dimensional polynomial modules, arXiv e-prints, (2022).
3. M. P. Brodmann and R. Y. Sharp, Local cohomology: an algebraic introduction with geometric applications, Cambridge univ. press **136** (2012).
4. V. Crispin Quiñonez, Integral closure and other operations on monomial ideals, J. Commut. Algebra **2** (3) (2010), 359–386.
5. D. Eisenbud, Commutative algebra with a view towards algebraic geometry, Graduate Texts in Mathematics, Springer-verlag, New York, **150**, (1995).
6. S. B. Iyengar, G. J. Leuschke, A. Leykin, C. Miller, E. Miller, A. K. Singh, and U. Walther, Twenty-four hours of local cohomology, Amer. Math. Soc. **87** (2007).
7. W. Fulton (1997). Young tableaux: with applications to representation theory and geometry, Mathematical Society Student Texts, *Cambridge University Press*, Cambridge, **35**.
8. H. E. Ross and B. I. Knott, Dicuil (9th century) on triangular and square numbers, Br. J. Hist. Math. **34**(2) (2019), 79–94.
9. A. Kyomuhangi and D. Ssevviiri, The locally nilradical for modules over commutative rings, Beitr. Algebra Geom. **61**(4) (2020), 759–769.
10. T. K. Lee and Y. Zhou, Reduced modules, rings, modules, algebras and abelian groups, Lecture Notes in Pure and Appl. Math. **236** (2004), 365–377.
11. M. B. Rege and A. M. Buhphang, On reduced modules and rings, Int. Electron. J. Algebra **3** (2008), 58–74.
12. D. Ssevviiri, Applications of reduced and coreduced modules I, Int. Electron. J. Algebra (2022), 1–21.
13. R. Takahashi, Y. Yoshino, and T. Yoshizawa, Local cohomology based on a nonclosed support defined by a pair of ideals, J. Pure Appl.algebra. **213**(4) (2008), 582–600.
14. W. F. Moore, M. Rogers, and W. S. Sather, Monomial ideals and their decomposition, Springer, Cham., (2018).
15. A. R. G. Wolff, The survival complex, arXiv preprint arXiv:1602.08998, (2016).

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