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ON SOME FIXED POINT RESULTS FOR $(\alpha, \beta, \mathcal{Z})$-CONTRACTION MAPPINGS IN PARTIAL b-METRIC SPACES VIA SIMULATION FUNCTIONS

Gurucharan Singh Saluja

Abstract. In the present work, our aim is to examine some fixed point theorems using $(\alpha, \beta, \mathcal{Z})$-contraction mappings in the framework of complete partial b-metric spaces under cyclic (α, β)-admissible mapping imbedded in simulation function. Some consequences of main results are also deduced. We present some examples to illustrate and support our results. The results obtained in this work provide extension as well as substantial generalization and improvement of several well-known fixed point results from the existing literature.

1. Introduction

Bakhtin [**5**] and *Czerwik* [**9**] introduced b-metric spaces as a generalization of metric spaces (see, also [**10**]). They proved the contraction mapping principle in b-metric spaces that generalized the well-known Banach contraction principle in such spaces. *Matthews* (see, [**14, 15**]) introduced the notion of partial metric spaces as a part of the study of denotational semantics of data flow networks. In this space, the usual metric is replaced by partial metric with an interesting property that the self-distance of any pint of space may not be zero. Further, *Matthews* showed that the Banach contraction principle [**6**] is valid in partial metric space. *Shukla* [**26**] generalized both the notions of b-metric and partial metric spaces by introducing partial b-metric spaces. He proved Banach contraction principle as well as the Kannan type fixed point theorem in partial b-metric spaces. Also some examples are given which illustrate the results obtained in this new space.

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Recently, *Samet et al.* [**24**] proved a generalization of Banach contraction principle by introducing the concept of $(\alpha - \psi)$ contractive type mappings and α-admissible mappings. The notion of cyclic (α, β)-admissible mapping was introduced by *Alizadeh et al.* [**1**] by generalizing the concept of α-admissible mapping of *Samet et al.* [**24**]. They proved various fixed point theorems in the framework of metric spaces. Also, *Khojasteh et al.* [**12**] introduced the notion of $\mathcal{Z}$-contraction by defining the concept of simulation function. Consequently, they proved the existence and uniqueness of fixed point for $\mathcal{Z}$-contractive mappings (see [**12**], Theorem 2.8). The concept of *Khojasteh et al.* [**12**] is further modified by *Argoubi et al.* [**2**]. They proved the existence of common fixed point results of a pair of nonlinear operators satisfying a certain contractive condition involving simulation functions in the setting of ordered metric spaces. Later on, several authors studied the existence of fixed point by using the simulation function, for example see [**2**–**4**, **7**, **8**, **11**, **13**, **17**–**23**, **25**, **27**] and many others).

In this work, we consider $(\alpha, \beta, \mathcal{Z})$-contraction mappings under simulation functions involving cyclic (α, β)-admissibility in partial b-metric space. Using the above said contractions, we establish some fixed point results. The results obtained in this work generalize and extend the corresponding results of [**8**] and [**19**] from metric space to the setting of partial b-metric space.

2. Preliminaries

In this section, we need the following concepts to prove our main results.

Definition 2.1. *([5]) Let $\Lambda_b \neq \emptyset$ be a set and the mapping $\rho_b\colon \Lambda_b \times \Lambda_b \to \mathbb{R}^+$ ($\mathbb{R}^+$ stands for nonnegative reals) satisfies:*

(ρ_{b1}) *$\rho_b(\lambda_b, \mu_b) = 0$ if and only if $\lambda_b = \mu_b$ for all $\lambda_b, \mu_b \in \Lambda_b$;*

(ρ_{b2}) *$\rho_b(\lambda_b, \mu_b) = \rho_b(\mu_b, \lambda_b)$ for all $\lambda_b, \mu_b \in \Lambda_b$;*

(ρ_{b3}) *there exists a real number $s \geqslant 1$ such that $\rho_b(\lambda_b, \mu_b) \leqslant s[\rho_b(\lambda_b, \nu_b) + \rho_b(\nu_b, \mu_b)]$ for all $\lambda_b, \mu_b, \nu_b \in \Lambda_b$.*

Then ρ_b is called a b-metric on Λ_b and the pair (Λ_b, ρ_b) is called a b-metric space with coefficient s.

Definition 2.2. *([**15**]) A partial metric on a nonempty set Υ_p is a function $p\colon \Upsilon p \times \Upsilon_p \to \mathbb{R}^+$ ($\mathbb{R}^+$ stands for nonnegative reals), such that for all $\lambda_p, \mu_p, \nu_p \in \Upsilon_p$:*

(p_1) *$\lambda_p = \mu_p$ if and only if $p(\lambda_p, \lambda_p) = p(\lambda_p, \mu_p) = p(\mu_p, \mu_p)$;*

(p_2) *$p(\lambda_p, \lambda_p) \leqslant p(\lambda_p, \mu_p)$;*

(p_3) *$p(\lambda_p, \mu_p) = p(\mu_p, \lambda_p)$;*

(p_4) *$p(\lambda_p, \mu_p) \leqslant p(\lambda_p, \nu_p) + p(\nu_p, \mu_p) - p(\nu_p, \nu_p)$.*

A partial metric space is a pair (Υ_p, p) such that Υ_p is a nonempty set and p is a partial metric on Υ_p.

Definition 2.3. *([**26**]) A partial b-metric on a nonempty set Φ_{p_b} is a function $p_b\colon \Phi_{p_b} \times \Phi_{p_b} \to \mathbb{R}^+$ such that for all $\zeta, \eta, \theta \in \Phi_{p_b}$:*

(p_{b1}) *$\zeta = \eta$ if and only if $p_b(\zeta, \zeta) = p_b(\zeta, \eta) = p_b(\eta, \eta)$;*

(p_{b2}) *$p_b(\zeta, \zeta) \leqslant p_b(\zeta, \eta)$;*

(p_{b3}) *$p_b(\zeta,\eta)=p_b(\eta,\zeta)$;*

(p_{b4}) *there exists a real number $s\geqslant 1$ such that $p_b(\zeta,\eta)\leqslant s[p_b(\zeta,\theta)+p_b(\theta,\eta)]-p_b(\theta,\theta)$.*

A partial b-metric space is a pair (Φ_{p_b},p_b) such that Φ_{p_b} is a nonempty set and p_b is a partial b-metric on Φ_{p_b}. The number s is called the coefficient of (Φ_{p_b},p_b).

REMARK 2.1. In a partial b-metric space (Φ_{p_b},p_b) if $\zeta,\eta\in\Phi_{p_b}$ and $p_b(\zeta,\eta)=0$, then $\zeta=\eta$, but the converse may not be true.

REMARK 2.2. It is clear that every partial metric space is a partial b-metric space with coefficient $s=1$ and every b-metric space is a partial b-metric space with the same coefficient and zero self-distance. However, the converse of this fact need not hold.

EXAMPLE 2.1. *([**26**]) Let $\Phi_{p_b}=\mathbb{R}^+$, where $\mathbb{R}^+=[0,+\infty)$, $m>1$ a constant and $p_b\colon \Phi_{p_b}\times\Phi_{p_b}\to\mathbb{R}^+$ be defined by*

$$p_b(\zeta,\eta)=[\max\{\zeta,\eta\}]^m+|\zeta-\eta|^m \quad \text{for all } \zeta,\eta\in\Phi_{p_b}.$$

Then (Φ_{p_b},p_b) is a partial b-metric space with coefficient $s=2^m>1$, but it is neither a b-metric nor a partial metric space. Indeed, for any $\zeta>0$, we have $p_b(\zeta,\zeta)=\zeta^m\neq 0$; therefore, p_b is not a b-metric on Φ_{p_b}. Also, for $\zeta=5$, $\eta=1$, $\theta=4$ we have $p_b(\zeta,\eta)=5^m+4^m$ and $p_b(\zeta,\theta)+p_b(\theta,\eta)-p_b(\theta,\theta)=5^m+1+4^m+3^m-4^m=5^m+1+3^m$, so $p_b(\zeta,\eta)>p_b(\zeta,\theta)+p_b(\theta,\eta)-p_b(\theta,\theta)$ for all $m>1$; therefore, p_b is not a partial metric on Φ_{p_b}.

Every partial b-metric "p_b" on a nonempty set Φ_{p_b} generates a topology τ_{p_b} on Φ_{p_b} whose base is the family of open p_b-balls where $\tau_{p_b}=\{B_{p_b}(\zeta,\varepsilon):\zeta\in\Phi_{p_b},\varepsilon>0\}$ and

$$B_{p_b}(\zeta,\varepsilon)=\{\eta\in\Phi_{p_b}:p_b(\zeta,\eta)<p_b(\zeta,\zeta)+\varepsilon\},$$

for all $\zeta\in\Phi_{p_b}$ and $\varepsilon>0$. Obviously, the topological space (Φ_{p_b},τ_{p_b}) is T_0, but need not be T_1.

Now we recall the definition of Cauchy sequence and convergent sequence in partial b-metric spaces.

DEFINITION 2.4. *([**26**]) Let (Φ_{p_b},p_b) be a partial b-metric space with coefficient s. Then:*

(1) *a sequence $\{\zeta_n\}$ in (Φ_{p_b},p_b) is said to be convergent with respect to τ_{p_b} and converges to a point $\zeta\in\Phi_{p_b}$, if $\lim_{n\to\infty}p_b(\zeta_n,\zeta)=p_b(\zeta,\zeta)$;*

(2) *a sequence $\{\zeta_n\}$ is said to be Cauchy sequence in (Φ_{p_b},p_b) if $\lim_{n,m\to\infty} p_b(\zeta_n,\zeta_m)$ exists and is finite.*

(3) *(Φ_{p_b},p_b) is said to be a complete partial b-metric space if for every Cauchy sequence $\{\zeta_n\}$ in Φ_{p_b} there exists $\zeta\in\Phi_{p_b}$ such that*

$$\lim_{n,m\to\infty}p_b(\zeta_n,\zeta_m)=\lim_{n\to\infty}p_b(\zeta_n,\zeta)=p_b(\zeta,\zeta).$$

(4) *A mapping $\mathcal{F}\colon\Phi_{p_b}\to\Phi_{p_b}$ is said to be continuous at $\zeta_0\in\Phi_{p_b}$ if for every $\varepsilon>0$, there exists $\delta>0$ such that $\mathcal{F}\Big(B_{p_b}(\zeta_0,\delta)\Big)\subset B_{p_b}\Big(\mathcal{F}(\zeta_0),\varepsilon\Big)$.*

Note: In a partial b-metric space the limit of convergent sequence may not be unique.

EXAMPLE 2.2. *([**26**]) Let $\Phi_{p_b} = \mathbb{R}^+$, where $\mathbb{R}^+ = [0, +\infty)$, $k > 0$ be any constant and define $p_b \colon \Phi_{p_b} \times \Phi_{p_b} \to \mathbb{R}^+$ by*

$$p_b(\zeta, \eta) = \max\{\zeta, \eta\} + k \quad \textit{for all}\, \zeta, \eta \in \Phi_{p_b}.$$

Then (Φ_{p_b}, p_b) is a partial b-metric space with arbitrary coefficient $s \geqslant 1$. Now, define a sequence $\{\zeta_n\}$ in Φ_{p_b} by $\zeta_n = 1$ for all $n \in \mathbb{N}$. Note that, if $\eta \geqslant 1$, we have $p_b(\zeta_n, \eta) = \eta + k = p_b(\eta, \eta)$; therefore, $\lim_{n\to\infty} p_b(\zeta_n, \eta) = p_b(\eta, \eta)$ for all $\eta \geqslant 1$. Thus, the limit of convergent sequence in partial b-metric space need not be unique.

Samet et al. [**24**] introduced the concept of α-admissible mappings.

DEFINITION 2.5. *([**24**]) Let $\Phi \neq \emptyset$ be a set. Let $T \colon \Phi \to \Phi$ and $\alpha \colon \Phi \times \Phi \to [0, +\infty)$ be given mappings. We say that T is α-admissible if for all $\zeta, \eta \in \Phi$, we have*

$$\alpha(\zeta, \eta) \geqslant 1 \Rightarrow \alpha(T(\zeta), T(\eta)) \geqslant 1. \tag{2.1}$$

Alizadeh et al. [**1**] introduced the concept of cyclic (α, β)-admissible mappings.

DEFINITION 2.6. *([**1**]) Let $\Phi \neq \emptyset$ be a set, let $T \colon \Phi \to \Phi$ be a mapping and $\alpha, \beta \colon \Phi \to [0, +\infty)$ be given mappings. We say that T is a cyclic (α, β)-admissible mapping if for all $\zeta \in \Phi$, we have*

$$\alpha(\zeta) \geqslant 1 \Rightarrow \beta(T(\zeta)) \geqslant 1,$$

and

$$\beta(\zeta) \geqslant 1 \Rightarrow \alpha(T(\zeta)) \geqslant 1. \tag{2.2}$$

In 2015, *Khojasteh et al.* [**12**] introduced simulation functions and defined $\mathcal{Z}$-contraction with respect to simulation function and it includes large class of contractive conditions as follows.

DEFINITION 2.7. *([**12**]) A simulation function is a mapping $\Omega \colon [0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfying the following conditions:*
(Ω_1) *$\Omega(0, 0) = 0$;*
(Ω_2) *$\Omega(t, s) < s - t$ for all $t, s > 0$;*
(Ω_3) *if $\{t_n\}$ and $\{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l > 0$, then*

$$\limsup_{n\to\infty} \Omega(t_n, s_n) < 0.$$

The following are examples of simulation functions.

EXAMPLE 2.3. *Let $\Omega \colon [0, \infty) \times [0, \infty) \to \mathbb{R}$ be defined by:*
(1) $\Omega(t, s) = \frac{s}{1+s} - t$ *for all $s, t \in [0, \infty)$.*
(2) $\Omega(t, s) = \lambda\, s - t$ *for all $s, t \in [0, \infty)$, where $\lambda \in [0, 1)$.*
(3) $\Omega(t, s) = s - \lambda\, t$ *for all $s, t \in [0, \infty)$, where $\lambda > 1$.*
(4) $\Omega(t, s) = \frac{1}{s+1} - (t + 1)$ *for all $s, t \in [0, \infty)$.*
(5) $\Omega(t, s) = \frac{s}{s+1} - te^t$ *for all $s, t \in [0, \infty)$.*

(6) $\Omega(t, s) = \frac{k}{r}s - t$ *for all* $s, t \in [0, \infty)$ *where* $k \in [0, 1)$ *and* $r \in (1, \infty)$.

(7) $\Omega(t, s) = \psi(s) - \phi(t)$ *for all* $t, s \in [0, \infty)$ *where* $\psi, \phi\colon [0, \infty) \to [0, \infty)$ *such that* $\psi(r) = \phi(r) = 0$ *if and only if* $r = 0$ *and* $\psi(t) < t < \phi(t)$ *for all* $t > 0$.

Definition 2.8. *([**12**]) Let* (Φ, σ) *be a metric space and let* $\mathcal{T}\colon \Phi \to \Phi$ *be a self-mapping of* Φ. *We say that* $\mathcal{T}$ *is a* $\mathcal{Z}$*-contraction with respect to* Ω, *if there exists a simulation function* $\mathcal{Z}$ *such that*

$$\Omega\Big(\sigma(\mathcal{T}\zeta, \mathcal{T}\eta), \sigma(\zeta, \eta)\Big) \geqslant 0, \tag{2.3}$$

for all $\zeta, \eta \in \Phi$.

Remark 2.3. ([**12**]) Every $\mathcal{Z}$-contraction mapping is contractive and therefore it is continuous.

It is worth mentioning that the Banach contraction is an example of $\mathcal{Z}$-contraction by defining $\zeta\colon [0, \infty) \times [0, \infty) \to \mathbb{R}$ via $\zeta(t, s) = \lambda s - t$ for all $t, s \in [0, \infty)$, where $\lambda \in [0, 1)$.

Argoubi et al. [**2**] slightly modified the definition of [**12**] as follows.

Definition 2.9. *([2]) A simulation function is a mapping* $\Omega\colon [0, \infty) \times [0, \infty) \to \mathbb{R}$ *satisfying the following conditions:*

(Ω1) $\Omega(t, s) < s - t$ *for all* $t, s > 0$*;*

(Ω2) *if* $\{t_n\}$ *and* $\{s_n\}$ *are sequences in* $(0, \infty)$ *such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l > 0$, *then*

$$\limsup_{n\to\infty} \Omega(t_n, s_n) < 0.$$

It is clear that any simulation function in the sense of *Khojasteh et al.* [**12**] is also a simulation function in the sense of *Argoubi et al.* [**2**]. The following example is a simulation function in the sense of *Argoubi et al.* [**2**].

example 2.4. *Let* $\Omega_\lambda\colon [0, \infty) \times [0, \infty) \to \mathbb{R}$ *be the function defined by*

$$\Omega_\lambda(t, s) = \begin{cases} 1, & \text{if } (t, s) = (0, 0), \\ \lambda s - t, & \text{otherwise}, \end{cases}$$

where $\lambda \in (0, 1)$. *Then* Ω_λ *is a simulation function.*

Lemma 2.1. *([**12**], Lemma 2.5) Let* (Φ, σ) *be a metric space and* $\mathcal{T}\colon \Phi \to \Phi$ *be a* $\mathcal{Z}$*-contraction with respect to* $\Omega \in \mathcal{Z}$. *Then the fixed point of* $\mathcal{T}$ *is unique, if it exists.*

Lemma 2.2. *([**16**], Lemma 1)*

(1) *A sequence* $\{\zeta_n\}$ *is a* p_b*-Cauchy sequence in a partial b-metric space* (Φ_{p_b}, p_b) *if only if it is a b-Cauchy sequence in the b-metric space* $(\Phi_{p_b}, \sigma_{p_b})$.

(2) *A partial b-metric space* (Φ_{p_b}, p_b) *is complete if and only if the b-metric space* $(\Phi_{p_b}, \sigma_{p_b})$ *is b-complete. Moreover,* $\lim_{n\to\infty} \sigma_{p_b}(\zeta, \zeta_n) = 0$ *if and only if*

$$\lim_{n\to\infty} p_b(\zeta, \zeta_n) = \lim_{n,m\to\infty} p_b(\zeta_n, \zeta_m) = p_b(\zeta, \zeta).$$

Proposition 2.1. *([**16**], Proposition 3) Every partial b-metric space* (Φ_{p_b}, p_b) *defines a b-metric* σ_{p_b}*, where*

$$\sigma_{p_b}(\zeta,\eta) = 2p_b(\zeta,\eta) - p_b(\zeta,\zeta) - p_b(\eta,\eta),$$

for all $\zeta,\eta \in \Phi_{p_b}$*.*

3. Main results

Firstly, we introduce the following definition.

Definition 3.1. *Let* (Φ_{p_b}, p_b) *be a complete partial b-metric space with arbitrary coefficient* $s \geqslant 1$*, let* $\mathcal{K}\colon \Phi_{p_b} \to \Phi_{p_b}$ *be a mapping and* $\alpha,\beta\colon \Phi_{p_b} \to [0,+\infty)$ *be two functions. Then* $\mathcal{K}$ *is said to be a* $(\alpha,\beta,\mathcal{Z})$*-contraction mapping, if it satisfies the following conditions:*

(1) $\mathcal{K}$ *is cyclic* (α,β)*-admissible,*

(2) *there exists a simulation function* $\Omega \in \mathcal{Z}$ *such that*

$$\Omega\Big(p_b(\mathcal{K}(\zeta),\mathcal{K}(\eta)),\Delta_p^b(\zeta,\eta)\Big) \geqslant 0, \tag{3.1}$$

for all $\zeta,\eta \in \Phi_{p_b}$*, where*

$$\Delta_p^b(\zeta,\eta) = \max\Big\{p_b(\zeta,\eta), p_b(\zeta,\mathcal{K}\zeta), p_b(\eta,\mathcal{K}\eta), \frac{p_b(\zeta,\mathcal{K}\zeta)+p_b(\eta,\mathcal{K}\zeta)}{2s}\Big\}.$$

Now, we are ready to prove our main result.

Theorem 3.1. *Let* (Φ_{p_b}, p_b) *be a complete partial b-metric space with coefficient* $s \geqslant 1$*,* $\mathcal{K}\colon \Phi_{p_b} \to \Phi_{p_b}$ *be a mapping and* $\alpha,\beta\colon \Phi_{p_b} \to [0,+\infty)$ *be two functions. Suppose that the following conditions hold:*

(1) $\mathcal{K}$ *is a* $(\alpha,\beta,\mathcal{Z})$*-contraction mapping.*

(2) *There exists an element* $\zeta_0 \in \Phi_{p_b}$ *such that* $\alpha(\zeta_0) \geqslant 1$ *and* $\beta(\zeta_0) \geqslant 1$*.*

(3) $\mathcal{K}$ *is continuous.*

Then $\mathcal{K}$ *has a fixed point* $\tau \in \Phi_{p_b}$ *and* $p_b(\tau,\tau) = 0$*.*

Proof. Assume that there exists $\zeta_0 \in \Phi_{p_b}$ such that $\alpha(\zeta_0) \geqslant 1$. We divide the proof of Theorem 3.1 into the following three steps:

Step 1. Define a sequence $\{\zeta_n\}$ in Φ_{p_b} such that $\zeta_{n+1} = \mathcal{K}\zeta_n$ for all $n \in \mathbb{N}\cup\{0\}$. If $\zeta_n = \zeta_{n+1}$ for all $n \in \mathbb{N}\cup\{0\}$, then $\mathcal{K}$ has a fixed point and the proof is finished. So, we assume that $\zeta_n \neq \zeta_{n+1}$ for some $n \in \mathbb{N}\cup\{0\}$, that is, $p_b(\zeta_n,\zeta_{n+1}) \neq 0$ for $n \in \mathbb{N}\cup\{0\}$. Since $\mathcal{K}$ is cyclic (α,β)-admissible mapping, $\alpha(\zeta_0) \geqslant 1$ and $\beta(\zeta_0) \geqslant 1$,

$$\beta(\zeta_1) = \beta(\mathcal{K}\zeta_0) \geqslant 1.$$

It implies that

$$\alpha(\zeta_2) = \alpha(\mathcal{K}\zeta_1) \geqslant 1.$$

And also, we have

$$\alpha(\zeta_1) = \alpha(\mathcal{K}\zeta_0) \geqslant 1.$$

It implies that

$$\beta(\zeta_2) = \beta(\mathcal{K}\zeta_1) \geqslant 1.$$

By continuing the same process as above, we have $\alpha(\zeta_n) \geqslant 1$ and $\beta(\zeta_n) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$. Thus $\alpha(\zeta_n)\beta(\zeta_{n+1}) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$. Therefore, we obtain

$$\Omega\Big(p_b(\mathcal{K}(\zeta_n), \mathcal{K}(\zeta_{n+1})), \Delta_p^b(\zeta_n, \zeta_{n+1})\Big) \geqslant 0, \tag{3.2}$$

for all $n \in \mathbb{N}$, where

$$\begin{aligned}
\Delta_p^b(\zeta_n, \zeta_{n+1}) &= \max\Big\{p_b(\zeta_n, \zeta_{n+1}), p_b(\zeta_n, \mathcal{K}\zeta_n), p_b(\zeta_{n+1}, \mathcal{K}\zeta_{n+1}), \\
&\qquad \frac{p_b(\zeta_n, \mathcal{K}\zeta_n) + p_b(\zeta_{n+1}, \mathcal{K}\zeta_n)}{2s}\Big\} \\
&= \max\Big\{p_b(\zeta_n, \zeta_{n+1}), p_b(\zeta_n, \zeta_{n+1}), p_b(\zeta_{n+1}, \zeta_{n+2}), \\
&\qquad \frac{p_b(\zeta_n, \zeta_{n+1}) + p_b(\zeta_{n+1}, \zeta_{n+1})}{2s}\Big\} \\
&= \max\Big\{p_b(\zeta_n, \zeta_{n+1}), p_b(\zeta_n, \zeta_{n+1}), p_b(\zeta_{n+1}, \zeta_{n+2}), \\
&\qquad \frac{p_b(\zeta_n, \zeta_{n+1}) + p_b(\zeta_{n+1}, \zeta_{n+1})}{2s}\Big\} \\
&= \max\Big\{p_b(\zeta_n, \zeta_{n+1}), p_b(\zeta_{n+1}, \zeta_{n+2})\Big\}.
\end{aligned}$$

It follows that

$$\Omega\Big(p_b(\zeta_{n+1}, \zeta_{n+2}), \max\Big\{p_b(\zeta_n, \zeta_{n+1}), p_b(\zeta_{n+1}, \zeta_{n+2})\Big\}\Big) \geqslant 0. \tag{3.3}$$

Condition (Ω_2) of Definition 2.7 implies that

$$\begin{aligned}
0 &\leqslant \Omega\Big(p_b(\zeta_{n+1}, \zeta_{n+2}), \max\Big\{p_b(\zeta_n, \zeta_{n+1}), p_b(\zeta_{n+1}, \zeta_{n+2})\Big\}\Big) \\
&< \max\Big\{p_b(\zeta_n, \zeta_{n+1}), p_b(\zeta_{n+1}, \zeta_{n+2})\Big\} - p_b(\zeta_{n+1}, \zeta_{n+2}).
\end{aligned} \tag{3.4}$$

Thus, we conclude that

$$p_b(\zeta_{n+1}, \zeta_{n+2}) < \max\Big\{p_b(\zeta_n, \zeta_{n+1}), p_b(\zeta_{n+1}, \zeta_{n+2})\Big\}, \tag{3.5}$$

for all $n \geqslant 1$. From equation (3.5), we have

$$p_b(\zeta_{n+1}, \zeta_{n+2}) < p_b(\zeta_n, \zeta_{n+1}) \quad \text{for all } n \geqslant 1. \tag{3.6}$$

It follows that the sequence $\{p_b(\zeta_n, \zeta_{n+1})\}$ is non-increasing. Therefore, there exists a constant $c \geqslant 0$ such that

$$\lim_{n\to\infty} p_b(\zeta_n, \zeta_{n+1}) = c. \tag{3.7}$$

If $c \neq 0$, that is, if $c > 0$, then by condition (Ω_2) of Definition 2.7, we obtain

$$0 \leqslant \limsup_{n\to\infty} \Omega\Big(p_b(\zeta_n, \zeta_{n+1}), p_b(\zeta_{n+1}, \zeta_{n+2})\Big) < 0, \tag{3.8}$$

which is a contradiction. This implies that $c = 0$, that is,

$$\lim_{n\to\infty} p_b(\zeta_n, \zeta_{n+1}) = 0. \tag{3.9}$$

Step 2. Now, we show that $\{\zeta_n\}$ is a partial b-Cauchy sequence in Φ_{p_b}. For this, we have to show that $\{\zeta_n\}$ is a b-Cauchy sequence in $(\Phi_{p_b}, \sigma_{p_b})$. Suppose the contrary; that is, $\{\zeta_n\}$ is not a b-Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{\zeta_{n_{(k)}}\}$ and $\{\zeta_{m_{(k)}}\}$ of $\{\zeta_n\}$ such that $m_{(k)}$ is the smallest index for which

$$m_{(k)} > n_{(k)} > k, \quad \sigma_{p_b}(\zeta_{n_{(k)}}, \zeta_{m_{(k)}}) \geqslant \varepsilon. \tag{3.10}$$

This means that

$$\sigma_{p_b}(\zeta_{n_{(k)}}, \zeta_{m_{(k)}-1}) < \varepsilon. \tag{3.11}$$

From equation (3.10) and using the triangular inequality, we have

$$\begin{aligned} \varepsilon &\leqslant \sigma_{p_b}(\zeta_{n_{(k)}}, \zeta_{m_{(k)}}) \\ &\leqslant s[\sigma_{p_b}(\zeta_{n_{(k)}}, \zeta_{m_{(k)}-1}) + \sigma_{p_b}(\zeta_{m_{(k)}-1}, \zeta_{m_{(k)}})] \\ &\quad -\sigma_{p_b}(\zeta_{m_{(k)}-1}, \zeta_{m_{(k)}-1}) \\ &\leqslant s[\sigma_{p_b}(\zeta_{n_{(k)}}, \zeta_{m_{(k)}-1}) + \sigma_{p_b}(\zeta_{m_{(k)}-1}, \zeta_{m_{(k)}})]. \end{aligned} \tag{3.12}$$

Taking the limit as $k \to \infty$ in the above inequality and using equations (3.9) and (3.12), we get

$$\frac{\varepsilon}{s} \leqslant \liminf_{k\to\infty} \sigma_{p_b}(\zeta_{n_{(k)}}, \zeta_{m_{(k)}-1}) \leqslant \limsup_{k\to\infty} \sigma_{p_b}(\zeta_{n_{(k)}}, \zeta_{m_{(k)}-1}) \leqslant \varepsilon. \tag{3.13}$$

Also from (3.12) and (3.13), we have

$$\varepsilon \leqslant \limsup_{k\to\infty} \sigma_{p_b}(\zeta_{n_{(k)}}, \zeta_{m_{(k)}}) \leqslant s\varepsilon. \tag{3.14}$$

Further,

$$\sigma_{p_b}(\zeta_{n_{(k)}}, \zeta_{m_{(k)}+1}) \leqslant s\sigma_{p_b}(\zeta_{n_{(k)}}, \zeta_{m_{(k)}}) + s\sigma_{p_b}(\zeta_{m_{(k)}}, \zeta_{m_{(k)}+1}),$$

and hence

$$\limsup_{k\to\infty} \sigma_{p_b}(\zeta_{n_{(k)}}, \zeta_{m_{(k)}+1}) \leqslant s^2\varepsilon. \tag{3.15}$$

Again,

$$\sigma_{p_b}(\zeta_{n_{(k)}-1}, \zeta_{m_{(k)}+1}) \leqslant s\sigma_{p_b}(\zeta_{n_{(k)}-1}, \zeta_{m_{(k)}}) + s\sigma_{p_b}(\zeta_{m_{(k)}}, \zeta_{m_{(k)}+1}),$$

and hence

$$\limsup_{k\to\infty} \sigma_{p_b}(\zeta_{n_{(k)}-1}, \zeta_{m_{(k)}+1}) \leqslant s\varepsilon. \tag{3.16}$$

Again, we have

$$\sigma_{p_b}(\zeta_{n_{(k)}-1}, \zeta_{m_{(k)}-1}) \leqslant s\sigma_{p_b}(\zeta_{n_{(k)}-1}, \zeta_{n_{(k)}}) + s\sigma_{p_b}(\zeta_{n_{(k)}}, \zeta_{m_{(k)}-1}),$$

and hence

$$\limsup_{k\to\infty} \sigma_{p_b}(\zeta_{n_{(k)}-1}, \zeta_{m_{(k)}-1}) \leqslant s\varepsilon. \tag{3.17}$$

Finally, we have

$$\sigma_{p_b}(\zeta_{n_{(k)}},\zeta_{m_{(k)}-1}) \leqslant s\sigma_{p_b}(\zeta_{n_{(k)}},\zeta_{n_{(k)}-1}) + s\sigma_{p_b}(\zeta_{n_{(k)}-1},\zeta_{m_{(k)}-1}),$$

and hence

$$\limsup_{k\to\infty} \sigma_{p_b}(\zeta_{n_{(k)}},\zeta_{m_{(k)}-1}) \leqslant s^2\varepsilon. \tag{3.18}$$

On the other hand, by the definition of σ_{p_b} and (3.9),

$$\limsup_{k\to\infty} \sigma_{p_b}(\zeta_{n_{(k)}-1},\zeta_{m_{(k)}}) = 2\limsup_{k\to\infty} p_b(\zeta_{n_{(k)}-1},\zeta_{m_{(k)}}).$$

Hence by equation (3.13), we have

$$\frac{\varepsilon}{2s} \leqslant \liminf_{k\to\infty} p_b(\zeta_{n_{(k)}-1},\zeta_{m_{(k)}}) \leqslant \limsup_{k\to\infty} p_b(\zeta_{n_{(k)}-1},\zeta_{m_{(k)}}) \leqslant \frac{\varepsilon}{2}. \tag{3.19}$$

Likewise,

$$\limsup_{k\to\infty} p_b(\zeta_{n_{(k)}},\zeta_{m_{(k)}}) \leqslant \frac{s\varepsilon}{2}, \tag{3.20}$$

$$\frac{\varepsilon}{2s} \leqslant \limsup_{k\to\infty} p_b(\zeta_{n_{(k)}},\zeta_{m_{(k)}+1}), \tag{3.21}$$

$$\limsup_{k\to\infty} p_b(\zeta_{n_{(k)}-1},\zeta_{m_{(k)}+1}) \leqslant \frac{s\varepsilon}{2}, \tag{3.22}$$

$$\limsup_{k\to\infty} p_b(\zeta_{n_{(k)}-1},\zeta_{m_{(k)}-1}) \leqslant \frac{s\varepsilon}{2}, \tag{3.23}$$

$$\limsup_{k\to\infty} p_b(\zeta_{n_{(k)}},\zeta_{m_{(k)}-1}) \leqslant \frac{s^2\varepsilon}{2}. \tag{3.24}$$

Since $\alpha(\zeta_n) \geqslant 1$ and $\beta(\zeta_n) \geqslant 1$ for all $n = 1,2,3,\ldots$, we conclude that

$$\alpha(\zeta_{n_{(k)}-1})\beta(\zeta_{m_{(k)}-1}) \geqslant 1. \tag{3.25}$$

Since $\mathcal{K}$ is a $(\alpha,\beta,\mathcal{Z})$-contraction. Hence from equation (3.1), we have

$$\Omega\Big(p_b(\mathcal{K}(\zeta_{n_{(k)}-1}),\mathcal{K}(\zeta_{m_{(k)}-1})),\Delta_p^b(\zeta_{n_{(k)}-1},\zeta_{m_{(k)}-1})\Big) \geqslant 0, \tag{3.26}$$

where

$$\begin{aligned}
\Delta_p^b(\zeta_{n_{(k)}-1},\zeta_{m_{(k)}-1}) &= \max\Big\{p_b(\zeta_{n_{(k)}-1},\zeta_{m_{(k)}-1}), p_b(\zeta_{n_{(k)}-1},\mathcal{K}\zeta_{n_{(k)}-1}),\\
&\qquad p_b(\zeta_{m_{(k)}-1},\mathcal{K}\zeta_{m_{(k)}-1}),\\
&\qquad \frac{p_b(\zeta_{n_{(k)}-1},\mathcal{K}\zeta_{n_{(k)}-1}) + p_b(\zeta_{m_{(k)}-1},\mathcal{K}\zeta_{n_{(k)}-1})}{2s}\Big\}\\
&= \max\Big\{p_b(\zeta_{n_{(k)}-1},\zeta_{m_{(k)}-1}), p_b(\zeta_{n_{(k)}-1},\zeta_{n_{(k)}}),\\
&\qquad p_b(\zeta_{m_{(k)}-1},\zeta_{m_{(k)}}),\\
&\qquad \frac{p_b(\zeta_{n_{(k)}-1},\zeta_{n_{(k)}}) + p_b(\zeta_{m_{(k)}-1},\zeta_{n_{(k)}})}{2s}\Big\}.
\end{aligned} \tag{3.27}$$

Taking the upper limit as $k \to \infty$ in equation (3.27) and using equations (3.9), (3.23) and (3.24), we get

$$\begin{aligned}\limsup_{k\to\infty} \Delta_p^b(\zeta_{n_{(k)}-1}, \zeta_{m_{(k)}-1}) &= \max\Big\{ \limsup_{k\to\infty} p_b(\zeta_{n_{(k)}-1}, \zeta_{m_{(k)}-1}), 0, 0, \\ &\qquad \frac{0 + \limsup_{k\to\infty} p_b(\zeta_{m_{(k)}-1}, \zeta_{n_{(k)}})}{2s}\Big\} \\ &\leqslant \max\Big\{ \frac{s\varepsilon}{2}, 0, 0, \frac{\frac{s^2\varepsilon}{2}}{2s} \Big\} \\ &= \max\Big\{ \frac{s\varepsilon}{2}, 0, 0, \frac{s\varepsilon}{4} \Big\} = \frac{s\varepsilon}{2}. \qquad (3.28)\end{aligned}$$

Now taking the upper limit as $k \to \infty$ in equation (3.26) and by conditions (Ω_2), (Ω_3) of Definition 2.7 and using equations (3.20) and (3.28), we get

$$0 \leqslant \limsup_{k\to\infty} \zeta\Big(p_b(\zeta_{n_{(k)}}, \zeta_{m_{(k)}}), \Delta_p^b(\zeta_{n_{(k)}-1}, \zeta_{m_{(k)}-1}) \Big) < 0,$$

which is a contradiction. Thus we have proved that $\{\zeta_n\}$ is a b-Cauchy sequence in b-metric space $(\Phi_{p_b}, \sigma_{p_b})$.

Step 3. Finally, we prove that $\mathcal{K}$ has a fixed point. Since (Φ_{p_b}, p_b) is a complete partial b-metric space, then from Lemma 2.2, $(\Phi_{p_b}, \sigma_{p_b})$ is complete b-metric space. Therefore, the sequence converges to some $\tau \in \Phi_{p_b}$, that is, $\lim_{n\to\infty} \sigma_{p_b}(\zeta_n, \tau) = 0$. Again, from Lemma 2.2,

$$\lim_{n\to\infty} p_b(\tau, \zeta_n) = \lim_{n\to\infty} p_b(\zeta_n, \zeta_n) = p_b(\tau, \tau). \qquad (3.29)$$

On the other hand, from equation (3.9) and (p_{b2}), $\lim_{n\to\infty} p_b(\zeta_n, \zeta_n) = 0$, which yields that

$$\lim_{n\to\infty} p_b(\tau, \zeta_n) = \lim_{n\to\infty} p_b(\zeta_n, \zeta_n) = p_b(\tau, \tau) = 0. \qquad (3.30)$$

Now, since $\lim_{n\to\infty} p_b(\tau, \zeta_n) = 0$ or $\zeta_n \to \tau$ as $n \to \infty$, the continuity of $\mathcal{K}$ implies that $\mathcal{K}\zeta_{2n} \to \mathcal{K}\tau$. Since $\zeta_{2n+1} = \mathcal{K}\zeta_{2n}$ and $\zeta_{2n+1} \to \tau$, by uniqueness of limit, we get $\mathcal{K}\tau = \tau$. So, τ is a fixed point of $\mathcal{K}$ and $p_b(\tau, \tau) = 0$. This completes the proof. □

4. Consequences of Theorem 3.1

In this section, we give some very interesting fixed point results which can be derived from the condition (3.1) of Theorem 3.1, on several form of functions $\Omega \in \mathcal{Z}$. We give a few examples as corollaries which extend and unify the existing results of the literature.

COROLLARY 4.1. *Let (Φ_{p_b}, p_b) be a complete partial b-metric space with coefficient $s \geqslant 1$, $\mathcal{K}\colon \Phi_{p_b} \to \Phi_{p_b}$ be a mapping and $\alpha, \beta\colon \Phi_{p_b} \to [0, +\infty)$ be two functions. Suppose that the following conditions hold:*

(1)

$$p_b(\mathcal{K}(\zeta), \mathcal{K}(\eta)) \leqslant \gamma\, \Delta_p^b(\zeta, \eta), \qquad (4.1)$$

for all $\zeta, \eta \in \Phi_{p_b}$, *where* $\gamma \in [0, 1)$ *and*

$$\Delta_p^b(\zeta, \eta) = \max\Big\{p_b(\zeta, \eta), p_b(\zeta, \mathcal{K}\zeta), p_b(\eta, \mathcal{K}\eta), \frac{p_b(\zeta, \mathcal{K}\zeta) + p_b(\eta, \mathcal{K}\zeta)}{2s}\Big\}.$$

(2) *There exists an element* $\zeta_0 \in \Phi_{p_b}$ *such that* $\alpha(\zeta_0) \geqslant 1$ *and* $\beta(\zeta_0) \geqslant 1$.
(3) $\mathcal{K}$ *is continuous.*
Then $\mathcal{K}$ *has a fixed point* $\tau \in \Phi_{p_b}$ *and* $p_b(\tau, \tau) = 0$.

Proof. The result follows from Theorem 3.1, by taking as Ω-simulation function

$$\Omega(t, s) = \gamma s - t,$$

for all $t, s \geqslant 0$ with $0 \leqslant \gamma < 1$ in (4.1), we have the conclusion. □

The following result is a particular case of Corollary 4.1.

Corollary 4.2. *([**26**], Theorem 1) Let* (Φ_{p_b}, p_b) *be a complete partial b-metric space with coefficient* $s \geqslant 1$. *Suppose that the mapping* $\mathcal{K}\colon \Phi_{p_b} \to \Phi_{p_b}$ *satisfying the following contractive condition:*

$$p_b(\mathcal{K}(\zeta), \mathcal{K}(\eta)) \leqslant \gamma\, p_b(\zeta, \eta), \tag{4.2}$$

for all $\zeta, \eta \in \Phi_{p_b}$, *where* $\gamma \in [0, 1)$ *is a constant. Then* $\mathcal{K}$ *has a unique fixed point* $\tau \in \Phi_{p_b}$ *and* $p_b(\tau, \tau) = 0$.

Corollary 4.3. *Let* (Φ_{p_b}, p_b) *be a complete partial b-metric space with coefficient* $s \geqslant 1$ *such that for some positive integer n,* $\mathcal{K}^n$ *satisfies the contraction condition (4.2) for all* $\zeta, \eta \in \Phi_{p_b}$, *where* γ *is as in Corollary 4.2. Then* $\mathcal{K}$ *has a unique fixed point* $u \in \Phi_{p_b}$ *and* $p_b(u, u) = 0$.

Proof. From Corollary 4.2, let p_0 be the unique fixed point of $\mathcal{K}^n$, that is, $\mathcal{K}^n(p_0) = p_0$. Then

$$\mathcal{K}(\mathcal{K}^n p_0) = \mathcal{K}p_0 \quad \text{or} \quad \mathcal{K}^n(\mathcal{K}p_0) = \mathcal{K}p_0.$$

This gives $\mathcal{K}p_0 = p_0$. This shows that p_0 is a unique fixed point of $\mathcal{K}$. This completes the proof. □

Remark 4.1. Corollary 4.2 extends the well-known Banach fixed point theorem [**6**] from complete metric space to the setting of complete partial b-metric space.

Remark 4.2. Corollary 4.2 is a special case of Corollary 4.3 for $n = 1$.

Note that the continuity of the mapping $\mathcal{K}$ in Theorem 3.1 can be dropped if we replace condition (3) by a suitable one as in the following result.

Corollary 4.4. *Let* (Φ_{p_b}, p_b) *be a complete partial b-metric space with coefficient* $s \geqslant 1$, $\mathcal{K}\colon \Phi_{p_b} \to \Phi_{p_b}$ *be a mapping and* $\alpha, \beta\colon \Phi_{p_b} \to [0, +\infty)$ *be two functions. Suppose that the following conditions hold:*
(1) $\mathcal{K}$ *is a* $(\alpha, \beta, \mathcal{Z})$*-contraction mapping.*
(2) *There exists an element* $\zeta_0 \in \Phi_{p_b}$ *such that* $\alpha(\zeta_0) \geqslant 1$ *and* $\beta(\zeta_0) \geqslant 1$.

(3) *If* $\{\zeta_n\}$ *is a sequence in* Φ_{p_b} *converges to* $v \in \Phi_{p_b}$ *with* $\alpha(\zeta_n) \geqslant 1$ *(or* $\beta(\zeta_n) \geqslant 1$*) for all* $n \in \mathbb{N}$*, then* $\alpha(v) \geqslant 1$ *(or* $\beta(v) \geqslant 1$*).*

Then $\mathcal{K}$ *has a fixed point* $v \in \Phi_{p_b}$ *and* $p_b(v, v) = 0$.

By setting the function $\beta\colon \Phi_{p_b} \to [0, +\infty)$ to be α in Theorem 3.1, then we obtain the following result.

Corollary 4.5. *Let* (Φ_{p_b}, p_b) *be a complete partial b-metric space with coefficient* $s \geqslant 1$*,* $\mathcal{K}\colon \Phi_{p_b} \to \Phi_{p_b}$ *be a mapping and* $\alpha\colon \Phi_{p_b} \to [0, +\infty)$ *be a function. Suppose that the following conditions hold:*

(1) *There exists* $\Omega \in \mathcal{Z}$ *such that if* $\zeta, \eta \in \Phi_{p_b}$ *with* $\alpha(\zeta) \geqslant 1$*, then*

$$\Omega\Big(p_b(\mathcal{K}(\zeta), \mathcal{K}(\eta)), \Delta_p^b(\zeta, \eta)\Big) \geqslant 0, \tag{4.3}$$

where

$$\Delta_p^b(\zeta, \eta) = \max\Big\{p_b(\zeta, \eta), p_b(\zeta, \mathcal{K}\zeta), p_b(\eta, \mathcal{K}\eta), \frac{p_b(\zeta, \mathcal{K}\zeta) + p_b(\eta, \mathcal{K}\zeta)}{2s}\Big\}.$$

(2) *If* $\zeta \in \Phi_{p_b}$ *with* $\alpha(\zeta) \geqslant 1$*, then* $\alpha(\mathcal{K}(\zeta)) \geqslant 1$.

(3) *There exists an element* $\zeta_0 \in \Phi_{p_b}$ *such that* $\alpha(\zeta_0) \geqslant 1$.

(4) *If* $\{\zeta_n\}$ *is a sequence in* Φ_{p_b} *converges to* $q \in \Phi_{p_b}$ *with* $\alpha(\zeta_n) \geqslant 1$ *for all* $n \in \mathbb{N}$*, then* $\alpha(q) \geqslant 1$.

Then $\mathcal{K}$ *has a fixed point* $q \in \Phi_{p_b}$ *and* $p_b(q, q) = 0$.

Now, we give an example in support of Corollary 4.2.

example 4.1. *Let* $\Phi_{p_b} = \{1, 2, 3, 4\}$ *and* $p_b\colon \Phi_{p_b} \times \Phi_{p_b} \to \mathbb{R}$ *be defined by*

$$p_b(\zeta, \eta) = \begin{cases} |\zeta - \eta|^2 + \max\{\zeta, \eta\}, & \text{if } \zeta \neq \eta, \\ \zeta, & \text{if } \zeta = \eta \neq 1, \\ 0, & \text{if } \zeta = \eta = 1, \end{cases}$$

for all $\zeta, \eta \in \Phi_{p_b}$*. Then* (Φ_{p_b}, p_b) *is a complete partial b-metric space with the coefficient* $s = 4 > 1$.

Define the mapping $\mathcal{K}\colon \Phi_{p_b} \to \Phi_{p_b}$ *by*

$$\mathcal{K}(1) = 1, \mathcal{K}(2) = 1, \mathcal{K}(3) = 2, \mathcal{K}(4) = 2.$$

Now, we have

$$p_b(\mathcal{K}(1), \mathcal{K}(2)) = p_b(1, 1) = 0 \leqslant \frac{3}{4}.3 = \frac{3}{4}p_b(1, 2),$$

$$p_b(\mathcal{K}(1), \mathcal{K}(3)) = p_b(1, 2) = 3 \leqslant \frac{3}{4}.7 = \frac{3}{4}p_b(1, 3),$$

$$p_b(\mathcal{K}(1), \mathcal{K}(4)) = p_b(1, 2) = 3 \leqslant \frac{3}{4}.13 = \frac{3}{4}p_b(1, 4),$$

$$p_b(\mathcal{K}(2), \mathcal{K}(3)) = p_b(1, 2) = 3 \leqslant \frac{3}{4}.4 = \frac{3}{4}p_b(2, 3),$$

$$p_b(\mathcal{K}(2), \mathcal{K}(4)) = p_b(1, 2) = 3 \leqslant \frac{3}{4}.8 = \frac{3}{4}p_b(2, 4),$$

$$p_b(\mathcal{K}(3),\mathcal{K}(4)) = p_b(2,2) = 2 \leqslant \frac{3}{4}.5 = \frac{3}{4}p_b(3,4).$$

*Thus, $\mathcal{K}$ satisfies all the conditions of Corollary 4.2 with $\gamma = \frac{3}{4} < 1$. Now by applying Corollary 4.2, $\mathcal{K}$ has a unique fixed point namely 1 is the unique fixed point of $\mathcal{K}$. We note that, since $p_b(2,2) = 2 \neq 0$ it follows that p_b is not a b-metric. Also p_b is not a partial metric. Indeed, $p_b(4,1) = 13 > 9 = p_b(4,3) + p_b(3,1) - p_b(3,3)$. Therefore, results from [**9**] and [**15**] are not applicable while Corollary 4.2 is applicable.*

The following example is in the support of Theorem 3.1.

EXAMPLE 4.2. *Let $\Phi_{p_b} = [-1,1]$ and $p_b\colon \Phi_{p_b}\times\Phi_{p_b}\to\mathbb{R}$ be defined by*

$$p_b(\zeta,\eta) = \begin{cases} |\zeta-\eta|^2 + \max\{\zeta,\eta\}, & \text{if } \zeta\neq\eta,\\ 0, & \text{otherwise,}\end{cases}$$

for all $\zeta,\eta\in\Phi_{p_b}$.

Also, define the mapping $\mathcal{K}\colon \Phi_{p_b}\to\Phi_{p_b}$, the two functions $\alpha,\beta\colon \Phi_{p_b}\to[0,+\infty)$ and the function $\Omega\colon [0,+\infty)\times[0,+\infty)\to\mathbb{R}$ as follows:

$$\mathcal{K}(\zeta) = \begin{cases} \frac{\zeta}{4}, & \text{if } \zeta\in[0,1],\\ \frac{1}{4}, & \text{otherwise,}\end{cases} \qquad \alpha(\zeta) = \begin{cases} \frac{\zeta+3}{2}, & \text{if } \zeta\in[0,1],\\ 0, & \text{otherwise,}\end{cases}$$

$$\beta(\zeta) = \begin{cases} \frac{\zeta+5}{4}, & \text{if } \zeta\in[0,1],\\ 0, & \text{otherwise,}\end{cases} \qquad \Omega(t,s) = \frac{s}{s+1} - t.$$

Then, we have the following

(1) *(Φ_{p_b}, p_b) is a complete partial b-metric space with the coefficient $s = 4 > 1$.*
(2) *Ω is a simulation function.*
(3) *There exists $\zeta_0\in\Phi_{p_b}$ such that $\alpha(\zeta_0)\geqslant 1$ and $\beta(\zeta_0)\geqslant 1$.*
(4) *$\mathcal{K}$ is continuous.*
(5) *$\mathcal{K}$ is cyclic (α,β)-admissible mapping.*
(6) *For $\zeta,\eta\in\Phi_{p_b}$ with $\alpha(\zeta)\beta(\eta)\geqslant 1$, we have*

$$\Omega\Big(p_b(\mathcal{K}(\zeta),\mathcal{K}(\eta)), \Delta_p^b(\zeta,\eta)\Big) \geqslant 0,$$

where

$$\Delta_p^b(\zeta,\eta) = \max\Big\{p_b(\zeta,\eta), p_b(\zeta,\mathcal{K}\zeta), p_b(\eta,\mathcal{K}\eta), \frac{p_b(\zeta,\mathcal{K}\zeta)+p_b(\eta,\mathcal{K}\zeta)}{2s}\Big\}.$$

It is easy to check that, the proof of (1), (2), (3) and (4) are clear.

To prove (5), let $\zeta\in\Phi_{p_b}$. If $\alpha(\zeta)\geqslant 1$, then $\zeta\in[0,1]$. So,

$$\beta(\mathcal{K}\zeta) = \beta\Big(\frac{\zeta}{4}\Big) = \frac{\frac{\zeta}{4}+5}{4} = \frac{\zeta+20}{16} \geqslant 1.$$

If $\beta(\zeta)\geqslant 1$, then $\zeta\in[0,1]$. So,

$$\alpha(\mathcal{K}\zeta) = \alpha\Big(\frac{\zeta}{4}\Big) = \frac{\frac{\zeta}{4}+3}{2} = \frac{\zeta+12}{8} \geqslant 1.$$

Thus, $\mathcal{K}$ is cyclic (α,β)-admissible.

To prove (6)*, let $\zeta,\eta\in\Phi_{p_b}$ with $\alpha(\zeta)\beta(\eta)\geqslant 1$. Then $\zeta,\eta\in[0,1]$. Thus, we have*

$$\begin{aligned}
&\Omega\Big(p_b(\mathcal{K}(\zeta),\mathcal{K}(\eta)),\Delta_p^b(\zeta,\eta)\Big)\\
&= \frac{\Delta_p^b(\zeta,\eta)}{1+\Delta_p^b(\zeta,\eta)}-p_b(\mathcal{K}(\zeta),\mathcal{K}(\eta))\\
&\geqslant \frac{p_b(\zeta,\eta)}{1+p_b(\zeta,\eta)}-p_b(\mathcal{K}(\zeta),\mathcal{K}(\eta))\\
&= \frac{p_b(\zeta,\eta)}{1+p_b(\zeta,\eta)}-|\mathcal{K}\zeta-\mathcal{K}\eta|^2-\max\{\mathcal{K}\zeta,\mathcal{K}\eta\}\\
&= \frac{|\zeta-\eta|^2+\max\{\zeta,\eta\}}{1+|\zeta-\eta|^2+\max\{\zeta,\eta\}}-\left|\frac{\zeta}{4}-\frac{\eta}{4}\right|^2-\max\left\{\frac{\zeta}{4},\frac{\eta}{4}\right\}\\
&= \frac{15|\zeta-\eta|^2+12\max\{\zeta,\eta\}}{16\Big(1+|\zeta-\eta|^2+\max\{\zeta,\eta\}\Big)}\geqslant 0.
\end{aligned}$$

Consequently, $\mathcal{K}$ is $(\alpha,\beta,\mathcal{Z})$-contraction mapping. Hence this satisfies all the conditions of Theorem 3.1. So, $\mathcal{K}$ has a fixed point in Φ_{p_b}. Here, 0 *is the required fixed point of $\mathcal{K}$.*

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Gurucharan Singh Saluja, H.N. 3/1005, Geeta Nagar, Raipur, Raipur - 492001 (C.G.), India.

Email address: saluja1963@gmail.com