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ON THE WORD PROBLEM FOR SCHUTZENBERGER-WREATH PRODUCT OF MONOIDS

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Abstract. In [**20**], the authors defined a new version of the Schützenberger product for any two monoids and gave a presentation of this new monoid construction. In this paper, by considering the presentation given in [**20**], we study on complete rewriting system for this monoid construction and obtain normal form structure of its elements. Then, we present solvability of the word problem for this construction. Finally, we illustrate our results with an example .

1. Introduction and Preliminaries

The origin of Combinatorial Group Theory can be traced back to 1911 when Max Dehn posed three questions concerning groups defined by finite presentations: the word, conjugacy and isomorphism problems [**1**, **14**]. The word problem is the problem of, given a presentation of the structure in terms of generators and relations, deciding whether or not two given words over the generators represent the same element of the structure. In this paper, we study on solvability of the word problem for a new monoid product, which is defined in [**20**] and called Schützenberger-wreath product of monoids. To have solvability word problem for this new monoid construction, we study on complete rewriting system.

At the rest of this section let us give the standard definitions and informations about complete rewriting system, Schützenberger product and wreath product of monoids.

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1.1. Complete rewriting system. Let X be a finite alphabet and let X^* be the free monoid consisting of all words obtained by the letters of X. A string rewriting system, or simply a rewriting system, on X^* is a subset $R \subseteq X^* \times X^*$ and an element $(x, y) \in R$, also written as $x \to y$, is called a rule of R. The idea for a rewriting system is an algorithm for substituting the right-hand side of a rule whenever the left-hand side appears in a word. In general, for a given rewriting system R, we write $x \to y$ for $x, y \in X^*$ if $x = uv_1w$, $y = uv_2w$ and $(v_1, v_2) \in R$. An element $x \in X^*$ is called irreducible with respect to R if there is no possible rewriting (or reduction) $x \to y$; otherwise x is called reducible. The rewriting system R is called

- *Noetherian* if there is no infinite chain of rewritings $x \to x_1 \to x_2 \to \cdots$ for any word $x \in X^*$,
- *Confluent* (diamond rule) if whenever $x \to^* y_1$ and $x \to^* y_2$, there is a $z \in X^*$ such that $y_1 \to^* z$ and $y_2 \to^* z$,
- *Complete* if R is both Noetherian and confluent.

If R is a complete rewriting system, then for every word x there is a unique irreducible word y such that $x \to^* y$; this word is called the normal form of x. Each element of the monoid presented by $\langle X; R\rangle$ has a unique normal form representive. For $u, v \in X^*$, if $|u| > |v|$ or if $|u| = |v|$ and v precedes u in the lexicograpfic ordering induced by a linear ordering on X, then we write $v < u$ and $<$ is called length-lexicographic ordering. A rewriting system is R is called a length-lexicographic rewriting system if $s < r$ for all $(r, s) \in R$. It is clear that length-lexicographic rewriting system is Noetherian.
If a rewriting system is complete then one can obtain normal form structure of elements. So we can say that the word problem for given rewriting system is solvable. We note that the reader is referred to [**6, 7, 25**] for a detailed survey on (complete) rewriting system and to [**10–13**] for word problem and complete rewriting systems of some group and semigroup constructions.

1.2. Schützenberger and wreath products of monoids. In [**24**], the author introduced a monoid construction, namely the Schützenberger product. This product was originally defined for two monoids in view of applications to Language Theory. The Schützenberger product plays an important role in the study of several problems of Automata Theory, such as, the Dot Depth Hierarchy of regular languages and studying concatenation product. In [**17**], the authors obtained a presentation for Schützenberger product of two monoids and gave the normal form structure of the elements of this product. In [**4**], the authors obtained normal form of elements of Schützenberger product of two monoids by using Gröbner-Shirshov bases theory. Many authors combined and extended Schützenberger product with different products to obtain a new monoid construction. As an example of these works, in [**2**], Ateş obtained a new monoid construction under semidirect and Schützenberger products. In [**15**], the authors defined a new monoid structure by taking into consideration crossed and Schützenberger products for any two monoids.

The Schützenberger product has been extended by Straubing for any number of monoids [**26**]. In [**16**], the authors gave a presentation of the Schützenberger product of n groups G_1, $G_2, \cdots, G_n$, given a monoid presentation $\langle X_i; R_i\rangle$ of each group G_i, by using Matrix Theory. Then in [**11**], by using the presentation given in [**16**], the authors computed a complete rewriting system and gave an algorithm for getting normal form of its elements.

Let us consider the monoids $A = \langle X_1; R_1\rangle$ and $B = \langle X_2; R_2\rangle$ and let $\mathcal{P}(A \times B)$ be the finitary power set of direct product $A \times B$. For $P \subseteq A \times B$ $(a \in A,\ b \in B)$, we define

$$aP = \{(ac, d) \mid (c, d) \in P\} \qquad \text{and} \qquad Pb = \{(c, db) \mid (c, d) \in P\}.$$

The Schützenberger product of monoids A and B is the set $A \times \mathcal{P}(A \times B) \times B$ with multiplication $(a_1, P_1, b_1)(a_2, P_2, b_2) = (a_1a_2, P_1b_2 \cup a_1P_2, b_1b_2)$. It is denoted by $A \Diamond B$. Here $A \Diamond B$ is a monoid with the identity $(1_A, \emptyset, 1_B)$ [**17**].
The reader is referred to [**3, 8, 9, 19, 21**] for some recent results on algebraic properties of Schützenberger product of groups, monoids and their derivations.

Now we present some information about wreath product of monoids. It is well known that the cartesian product of B copies of the monoid A is denoted by $A^{\times B}$ and the corresponding direct product is denoted by $A^{\oplus B}$. One may think of $A^{\times B}$ as the set of all such functions from B to A, and $A^{\oplus B}$ as the set all such functions f having finite support, that is to say, having the property that $(x)f = 1_A$ for all but finitely many $x \in B$. The unrestricted and restricted wreath products of the monoid A by the monoid B, are the sets $A^{\times B} \times B$ and $A^{\oplus B} \times B$, respectively, with the multiplication defined by

$$(f, b)(g, b') = (f\ {}^{b}g, bb'),$$

where ${}^{b}g : B \to A$ is defined by

$$(x)^{b}g = (xb)g \quad (x \in B). \tag{1.1}$$

It is also a well known fact that both these wreath products are monoids with the identity $(1, 1_B)$, where $x1 = 1_A$ for all $x \in B$.
For more details on the definition and applications of restricted (unrestricted) wreath products, we can refer the reader to [**5, 17, 18, 22, 23**].

2. Schützenberger − wreath product of monoids

In this section, our aim is to give the solvability of the word problem for Schützenberger - wreath product of two cyclic monoids. To do that, we obtain a complete rewriting system for this monoid construction by using the presentation of this product given in [**20**].

Definition 2.1. *[**20**] Let A and B be monoids and let $A^{\oplus B}$ be the set of all functions f from B into A having finite support. For $P \subseteq A^{\oplus B} \times B$ and $b \in B$, let us define a set*

$$Pb = \{(f, db) : (f, d) \in P\}.$$

The new version of the Schützenberger product of A by B (Schützenberger - wreath product), denoted by $A\Diamond_v B$, is the set $A^{\oplus B}\times\mathcal{P}(A^{\oplus B}\times B)\times B$ with the multiplication

$$(f,P_1,b_1)(g,P_2,b_2)=(f\ ^{b_1}g,P_1b_2\cup P_2,b_1b_2).$$

It is seen that $A\Diamond_v B$ is a monoid with the identity element $(\overline{1},\emptyset,1_B)$, where ^{b_1}g is defined as in (1.1).

Theorem 2.1. *[**20**] Let us suppose that the monoids A and B are defined by presentations*

$$A=\langle x;x^k=x^l\quad(k>l)\rangle\quad\text{and}\quad B=\langle y;y^s=y^t\quad(s>t)\rangle,$$

respectively. Then the Schützenberger - wreath product of monoids A and B, $A\Diamond_v B$, is defined by generators $x^{(i)}$, $z_{x^{(j)},y^m}$ and y, and relations

$$\begin{array}{ll}
(1) & y^s=y^t,\\
(2) & x^{(i)}x^{(j)}=x^{(j)}x^{(i)}\ \ (i<j),\\
(3) & (x^{(i)})^k=(x^{(i)})^l,\\
(4) & yx^{(t)}=x^{(s-1)}y,\\
(5) & yx^{(i)}=x^{(i-1)}y\ \ (0<i\leqslant s-1),\\
(6) & z^2_{x^{(j)},y^m}=z_{x^{(j)},y^m},\\
(7) & z_{x^{(j)},y^m}z_{x^{(i)},y^n}=z_{x^{(i)},y^n}z_{x^{(j)},y^m},\\
(8) & x^{(i)}z_{x^{(j)},y^m}=z_{x^{(j)},y^m}x^{(i)},\\
(9) & z_{x^{(j)},y^m}y=yz_{x^{(j)},y^{m+1}},
\end{array}$$

where $0\leqslant i,j,m,n\leqslant s-1$.

Regarding the numbers of relations given in the forms (1)-(9) in Theorem 2.1, the following numerical values and formulas are obtained.

Number of relations of the form (1):	1
Number of relations of the form (2):	$\frac{s(s-1)}{2}$
Number of relations of the form (3):	s
Number of relations of the form (4):	1
Number of relations of the form (5):	$s-1$
Number of relations of the form (6):	s^2
Number of relations of the form (7):	$\frac{s^2(s^2-1)}{2}$
Number of relations of the form (8):	s^3
Number of relations of the form (9):	s^2

With an easy calculation, the following result is obtained.

Corollary 2.1. *The total numbers of generators and relations in the presentation given in Theorem 2.1 are formulated as $(s+1)^2-s$ and $\frac{1}{2}\ (s^4+2s^3+4s^2+3s+2)$, respectively.*

Now we can give the main result of this section. To do that, let us consider the generators and relations given in Theorem 2.1, and order the generators as follows.

$$\begin{aligned}(2.1)\qquad & x^{(s-1)} > x^{(s-2)} > \cdots > x^{(1)} > x^{(0)} > \\ & z_{x^{(s-1)},y^{s-1}} > z_{x^{(s-1)},y^{s-2}} > \cdots > z_{x^{(s-1)},y^{2}} > z_{x^{(s-1)},y} > z_{x^{(s-1)},1} > \\ & z_{x^{(s-2)},y^{s-1}} > z_{x^{(s-2)},y^{s-2}} > \cdots > z_{x^{(s-2)},y^{2}} > z_{x^{(s-2)},y} > z_{x^{(s-2)},1} > \cdots > \\ & z_{x^{(1)},y^{s-1}} > z_{x^{(1)},y^{s-2}} > \cdots > z_{x^{(1)},y^{2}} > z_{x^{(1)},y} > z_{x^{(1)},1} > \\ & z_{x^{(0)},y^{s-1}} > z_{x^{(0)},y^{s-2}} > \cdots > z_{x^{(0)},y^{2}} > z_{x^{(0)},y} > z_{x^{(0)},1} > y.\end{aligned}$$

We note that we consider the reductions steps on words by taking into account length-lexicographic ordering on words. We also note that the notation $(r) \cap (p)$ denotes the overlapping word of left hand sides of relations (r) and (p).

THEOREM 2.2. *Let* $A = \langle x; x^k = x^l \quad (k > l)\rangle$ *and* $B = \langle y; y^s = y^t \quad (s > t)\rangle$. *A complete rewriting system for Schützenberger-wreath product of monoids* A *and* B, $A\diamond_v B$, *consists of the following rules by considering the order on generators given by (2.1):*

$$\begin{aligned}
&(1)\quad y^s \to y^t,\\
&(2)\quad x^{(j)}x^{(i)} \to x^{(i)}x^{(j)} \ (j > i),\\
&(3)\quad (x^{(i)})^k \to (x^{(i)})^l,\\
&(4)\quad x^{(s-1)}y \to yx^{(t)},\\
&(5)\quad x^{(i-1)}y \to yx^{(i)} \ (0 < i \leqslant s-1),\\
&(6)\quad z^2_{x^{(j)},y^m} \to z_{x^{(j)},y^m},\\
&(7)\quad z_{x^{(j)},y^m} z_{x^{(i)},y^n} \to z_{x^{(i)},y^n} z_{x^{(j)},y^m},\\
&(8)\quad x^{(i)} z_{x^{(j)},y^m} \to z_{x^{(j)},y^m} x^{(i)},\\
&(9)\quad z_{x^{(j)},y^m} y \to y z_{x^{(j)},y^{m+1}},
\end{aligned}$$

where $0 \leqslant i, j, m, n \leqslant s-1$.

PROOF. This rewriting system is Noetherian since there is no infinite chain of rewritings of overlapping words for the given length-lexicographic ordering given by (2.1). In order to show the second condition, the confluent property (diamond rule), the words obtained by appropriate overlappings of the words on the left-hand side of all rewriting rules are given below.

$$(1)\cap(1)\ : y^{s+1} \longrightarrow \begin{cases} y^t y = y^{t+1} \\ y y^t = y^{t+1} \end{cases}$$

For $x^{(j)} > x^{(i)} > x^{(p)}$,

$$(2)\cap(2)\ : x^{(j)}x^{(i)}x^{(p)} \longrightarrow \begin{cases} x^{(i)}x^{(j)}x^{(p)} \to x^{(i)}x^{(p)}x^{(j)} \to x^{(p)}x^{(i)}x^{(j)} \\ x^{(j)}x^{(p)}x^{(i)} \to x^{(p)}x^{(j)}x^{(i)} \to x^{(p)}x^{(i)}x^{(j)} \end{cases}$$

$$(2)\cap(3)\ : x^{(j)}(x^{(i)})^k \longrightarrow \begin{cases} x^{(i)}x^{(j)}(x^{(i)})^{k-1} \to \cdots \to (x^{(i)})^k x^{(j)} \to (x^{(i)})^l x^{(j)} \\ x^{(j)}(x^{(i)})^l \to x^{(i)}x^{(j)}(x^{(i)})^{l-1} \to \cdots \to (x^{(i)})^l x^{(j)} \end{cases}$$

$$(2)\cap(5)\ : x^{(j)}x^{(i-1)}y \longrightarrow \begin{cases} x^{(i-1)}x^{(j)}y \to x^{(i-1)}yx^{(j+1)} \to yx^{(i)}x^{(j+1)} \\ x^{(j)}yx^{(i)} \to yx^{(j+1)}x^{(i)} \to yx^{(i)}x^{(j+1)} \end{cases}$$

$(2)\cap(8)$:

$$x^{(j)}x^{(i)}z_{x^{(j)},y^m} \longrightarrow \begin{cases} x^{(i)}x^{(j)}z_{x^{(j)},y^m} \to x^{(i)}z_{x^{(j)},y^m}x^{(j)} \to z_{x^{(j)},y^m}x^{(i)}x^{(j)} \\ x^{(j)}z_{x^{(j)},y^m}x^{(i)} \to z_{x^{(j)},y^m}x^{(j)}x^{(i)} \to z_{x^{(j)},y^m}x^{(i)}x^{(j)} \end{cases}$$

$$(3)\cap(2)\ : (x^{(j)})^k x^{(i)} \longrightarrow \begin{cases} (x^{(j)})^l x^{(i)} \to (x^{(j)})^{l-1}x^{(i)}x^{(j)} \to \cdots \to x^{(i)}(x^{(j)})^l \\ (x^{(j)})^{k-1}x^{(i)}x^{(j)} \to \cdots \to x^{(i)}(x^{(j)})^k \to x^{(i)}(x^{(j)})^l \end{cases}$$

$$(3)\cap(3)\ : (x^{(i)})^{k+1} \longrightarrow \begin{cases} (x^{(i)})^l x^{(i)} = (x^{(i)})^{l+1} \\ x^{(i)}(x^{(i)})^l = (x^{(i)})^{l+1} \end{cases}$$

$$(3)\cap(4)\ : (x^{(s-1)})^k y \longrightarrow \begin{cases} (x^{(s-1)})^l y \to (x^{(s-1)})^{l-1}yx^{(t)} \to \cdots \to y(x^{(t)})^l \\ (x^{(s-1)})^{k-1}yx^{(t)} \to \cdots \to y(x^{(t)})^k \to y(x^{(t)})^l \end{cases}$$

$$(3)\cap(5)\ : (x^{(i-1)})^k y \longrightarrow \begin{cases} (x^{(i-1)})^l y \to (x^{(s-1)})^{l-1}yx^{(t)} \to \cdots \to y(x^{(t)})^l \\ (x^{(i-1)})^{k-1}yx^{(t)} \to \cdots \to y(x^{(t)})^k \to y(x^{(t)})^l \end{cases}$$

$(3)\cap(8)$:

$$(x^{(i)})^k z_{x^{(j)},y^m} \to \begin{cases} (x^{(i)})^l z_{x^{(j)},y^m} \to (x^{(i)})^{l-1}z_{x^{(j)},y^m}x^{(i)} \to \cdots \to z_{x^{(j)},y^m}(x^{(i)})^l \\ (x^{(i)})^{k-1}z_{x^{(j)},y^m}x^{(i)} \to \cdots \to z_{x^{(j)},y^m}(x^{(i)})^k \to z_{x^{(j)},y^m}(x^{(i)})^l \end{cases}$$

$$(4)\cap(1)\ : x^{(s-1)}y^s \longrightarrow \begin{cases} yx^{(t)}y^{s-1} \to y^2x^{(t+1)}y^{s-2} \to \cdots \to y^sx^{(t)} \to y^tx^{(t)} \\ x^{(s-1)}y^t \to yx^{(t)}y^{t-1} \to y^2x^{(t+1)}y^{t-2} \to \cdots \to y^tx^{(t)} \end{cases}$$

$$(5)\cap(1)\ : x^{(i-1)}y^s \longrightarrow \begin{cases} yx^{(i)}y^{s-1} \to y^2x^{(i)}y^{s-2} \to \cdots \to y^sx^{(i)} \to y^tx^{(i)} \\ x^{(i-1)}y^t \to yx^{(i)}y^{t-1} \to y^2x^{(i)}y^{t-2} \to \cdots \to y^tx^{(i)} \end{cases}$$

$$(6)\cap(6)\ : z^3_{x^{(j)},y^m} \longrightarrow \begin{cases} z^2_{x^{(j)},y^m} \to z_{x^{(j)},y^m} \\ z^2_{x^{(j)},y^m} \to z_{x^{(j)},y^m} \end{cases}$$

$(6)\cap(7)$:

$$z^2_{x^{(j)},y^m}z_{x^{(i)},y^n} \longrightarrow \begin{cases} z_{x^{(j)},y^m}z_{x^{(i)},y^n} \to z_{x^{(i)},y^n}z_{x^{(j)},y^m} \\ z_{x^{(j)},y^m}z_{x^{(i)},y^n}z_{x^{(j)},y^m} \to z_{x^{(i)},y^n}z^2_{x^{(j)},y^m} \to z_{x^{(i)},y^n}z_{x^{(j)},y^m} \end{cases}$$

$(6)\cap(9)$:

$$z^2_{x^{(j)},y^m}y \longrightarrow \begin{cases} z_{x^{(j)},y^m}y \to yz_{x^{(j)},y^{m+1}} \\ z_{x^{(j)},y^m}yz_{x^{(j)},y^{m+1}} \to yz^2_{x^{(j)},y^{m+1}} \to yz_{x^{(j)},y^{m+1}} \end{cases}$$

$(7)\cap(6)$:

$$z_{x^{(j)},y^m}z^2_{x^{(i)},y^n} \longrightarrow \begin{cases} z_{x^{(i)},y^n}z_{x^{(j)},y^m}z_{x^{(i)},y^n} \to z^2_{x^{(i)},y^n}z_{x^{(j)},y^m} \to z_{x^{(i)},y^n}z_{x^{(j)},y^m} \\ z_{x^{(j)},y^m}z_{x^{(i)},y^n} \to z_{x^{(i)},y^n}z_{x^{(j)},y^m} \end{cases}$$

$(7)\cap(7)$: Let us label the word $z_{x^{(j)},y^m}z_{x^{(i)},y^n}z_{x^{(p)},y^h}$ by A $(0\leqslant p,h\leqslant s-1)$.

$$A \to \begin{cases} z_{x^{(i)},y^n}z_{x^{(j)},y^m}z_{x^{(p)},y^h} \to z_{x^{(i)},y^n}z_{x^{(p)},y^h}z_{x^{(j)},y^m} \to z_{x^{(p)},y^h}z_{x^{(i)},y^n}z_{x^{(j)},y^m} \\ z_{x^{(j)},y^m}z_{x^{(p)},y^h}z_{x^{(i)},y^n} \to z_{x^{(p)},y^h}z_{x^{(j)},y^m}z_{x^{(i)},y^n} \to z_{x^{(p)},y^h}z_{x^{(i)},y^n}z_{x^{(j)},y^m} \end{cases}$$

$(7)\cap(9)$: Let us label the word $z_{x^{(j)},y^m}z_{x^{(i)},y^n}y$ by B.

$$B \to \begin{cases} z_{x^{(i)},y^n}z_{x^{(j)},y^m}y \to z_{x^{(i)},y^n}yz_{x^{(j)},y^{m+1}} \to yz_{x^{(i)},y^{n+1}}z_{x^{(j)},y^{m+1}} \\ z_{x^{(j)},y^m}yz_{x^{(i)},y^{n+1}} \to yz_{x^{(j)},y^{m+1}}z_{x^{(i)},y^{n+1}} \to yz_{x^{(i)},y^{n+1}}z_{x^{(j)},y^{m+1}} \end{cases}$$

$$(8)\cap(6)\ :\ x^{(i)}z^2_{x^{(j)},y^m} \longrightarrow \begin{cases} z_{x^{(j)},y^m}x^{(i)}z_{x^{(j)},y^m} \to z^2_{x^{(j)},y^m}x^{(i)} \to z_{x^{(j)},y^m}x^{(i)} \\ x^{(i)}z_{x^{(j)},y^m} \to z_{x^{(j)},y^m}x^{(i)} \end{cases}$$

$(8)\cap(7)$: Let us label the word $x^{(i)}z_{x^{(j)},y^m}z_{x^{(i)},y^n}$ by C.

$$C \to \begin{cases} z_{x^{(j)},y^m}x^{(i)}z_{x^{(i)},y^n} \to z_{x^{(j)},y^m}z_{x^{(i)},y^n}x^{(i)} \to z_{x^{(i)},y^n}z_{x^{(j)},y^m}x^{(i)} \\ x^{(i)}z_{x^{(i)},y^n}z_{x^{(j)},y^m} \to z_{x^{(i)},y^n}x^{(i)}z_{x^{(j)},y^m} \to z_{x^{(i)},y^n}z_{x^{(j)},y^m}x^{(i)} \end{cases}$$

$(8)\cap(9)$:

$$x^{(i)}z_{x^{(j)},y^m}y \longrightarrow \begin{cases} z_{x^{(j)},y^m}x^{(i)}y \to z_{x^{(j)},y^m}yx^{(i+1)} \to yz_{x^{(j)},y^{m+1}}x^{(i+1)} \\ x^{(i)}yz_{x^{(j)},y^{m+1}} \to yx^{(i+1)}z_{x^{(j)},y^{m+1}} \to yz_{x^{(j)},y^{m+1}}x^{(i+1)} \end{cases}$$

$$(9)\cap(1)\ :\ z_{x^{(j)},y^m}y^s \longrightarrow \begin{cases} yz_{x^{(j)},y^{m+1}}y^{s-1} \to \cdots \to y^s z_{x^{(j)},y^t} \to y^t z_{x^{(j)},y^t} \\ z_{x^{(j)},y^m}y^t \to yz_{x^{(j)},y^{m+1}}y^{t-1} \to \cdots \to y^t z_{x^{(j)},y^t} \end{cases}$$

It is seen that all overlapping words are reduced to the same words after appropriate reduction steps. Therefore, the confluent property for the given rewriting system is also satisfied. Consequently, since the presentation of $A\Diamond_v B$ is Noetherian and confluent, it is complete. Hence the result. □

By considering Theorem 2.2, we have the following other result of this section.

Corollary 2.2. *The normal form of a word w representing an element of $A\Diamond_v B$, is*

$$y^p W_{z_{x^{(i)},y^m}} {W'}_{(x^{(q)})^t} \quad (0\leqslant p,\ q\leqslant s-1 \quad \textit{and} \quad 0\leqslant t\leqslant k-1), \tag{2.2}$$

where $W_{z_{x^{(i)},y^m}}$ and ${W'}_{(x^{(q)})^t}$ are reduced words obtained by generators $z_{x^{(i)},y^m}$ and $\left(x^{(q)}\right)^t$, respectively.

By considering Theorem 2.2 and Corollary 2.2, we can give the following result.

COROLLARY 2.3. *Let A and B be finite cyclic monoids. Then the word problem for Schützenberger-wreath product of these monoids, $A\Diamond_v B$, is decidable.*

PROOF. By Theorem 2.2, we know that since the rewriting system of $A\Diamond_v B$ is Noetherian and confluent, this system is complete. By this complete rewriting system, each element of the monoid $A\Diamond_v B$ has a unique structure containing normal form given in (2.2). Thus, the word problem for $A\Diamond_v B$ is decidable. □

3. An example

In this section, by considering two finite cyclic monoids, we give applications of Theorem 2.2 and Corollary 2.2.

Let $A = \langle x; x^3 = x^2\rangle$ and $B = \langle y; y^3 = y\rangle$ be two finite cyclic monoids. The generator set of the Schützenberger-wreath product of A and B, $A\Diamond_v B$, is

$$\{x^{(2)}, x^{(1)}, x^{(0)}, z_{x^{(2)},y^2}, z_{x^{(2)},y}, z_{x^{(2)},1}, z_{x^{(1)},y^2}, z_{x^{(1)},y}, z_{x^{(1)},1}, z_{x^{(0)},y^2}, z_{x^{(0)},y}, z_{x^{(0)},1}, y\}.$$

Now we order these generators as follows.

$$\begin{aligned} (3.1) \qquad x^{(2)} > x^{(1)} > x^{(0)} > z_{x^{(2)},y^2} > z_{x^{(2)},y} > z_{x^{(2)},1} > z_{x^{(1)},y^2} >\\ z_{x^{(1)},y} > z_{x^{(1)},1} > z_{x^{(0)},y^2} > z_{x^{(0)},y} > z_{x^{(0)},1} > y. \end{aligned}$$

By the ordering given by (3.1), the monoid $A\Diamond_v B$ has the following complete rewriting system.

$$\begin{aligned}
&(1) \quad y^3 \to y,\\
&(2) \quad x^{(1)}x^{(0)} \to x^{(0)}x^{(1)}, \quad x^{(2)}x^{(0)} \to x^{(0)}x^{(2)}, \quad x^{(2)}x^{(1)} \to x^{(1)}x^{(2)},\\
&(3) \quad (x^{(0)})^3 \to (x^{(0)})^2, \quad (x^{(1)})^3 \to (x^{(1)})^2, \quad (x^{(2)})^3 \to (x^{(2)})^2,\\
&(4) \quad x^{(2)}y \to yx^{(1)},\\
&(5) \quad x^{(0)}y \to yx^{(1)}, \quad x^{(1)}y \to yx^{(2)},\\
&(6) \quad z^2_{x^{(0)},1} \to z_{x^{(0)},1}, \quad z^2_{x^{(0)},y} \to z_{x^{(0)},y}, \quad z^2_{x^{(0)},y^2} \to z_{x^{(0)},y^2},\\
&\qquad z^2_{x^{(1)},1} \to z_{x^{(1)},1}, \quad z^2_{x^{(1)},y} \to z_{x^{(1)},y}, \quad z^2_{x^{(1)},y^2} \to z_{x^{(1)},y^2},\\
&\qquad z^2_{x^{(2)},1} \to z_{x^{(2)},1}, \quad z^2_{x^{(2)},y} \to z_{x^{(2)},y}, \quad z^2_{x^{(2)},y^2} \to z_{x^{(2)},y^2},\\
&(7) \quad z_{x^{(0)},y}z_{x^{(0)},1} \to z_{x^{(0)},1}z_{x^{(0)},y}, \quad z_{x^{(0)},y^2}z_{x^{(0)},1} \to z_{x^{(0)},1}z_{x^{(0)},y^2},\\
&\qquad z_{x^{(1)},1}z_{x^{(0)},1} \to z_{x^{(0)},1}z_{x^{(1)},1}, \quad z_{x^{(1)},y}z_{x^{(0)},1} \to z_{x^{(0)},1}z_{x^{(1)},y},\\
&\qquad z_{x^{(1)},y^2}z_{x^{(0)},1} \to z_{x^{(0)},1}z_{x^{(1)},y^2}, \quad z_{x^{(2)},1}z_{x^{(0)},1} \to z_{x^{(0)},1}z_{x^{(2)},1},\\
&\qquad z_{x^{(2)},y}z_{x^{(0)},1} \to z_{x^{(0)},1}z_{x^{(2)},y}, \quad z_{x^{(2)},y^2}z_{x^{(0)},1} \to z_{x^{(0)},1}z_{x^{(2)},y^2},\\
&\qquad z_{x^{(0)},y^2}z_{x^{(0)},y} \to z_{x^{(0)},y}z_{x^{(0)},y^2}, \quad z_{x^{(1)},1}z_{x^{(0)},y} \to z_{x^{(0)},y}z_{x^{(1)},1},\\
&\qquad z_{x^{(1)},y}z_{x^{(0)},y} \to z_{x^{(0)},y}z_{x^{(1)},y}, \quad z_{x^{(1)},y^2}z_{x^{(0)},y} \to z_{x^{(0)},y}z_{x^{(1)},y^2},
\end{aligned}$$

$$z_{x^{(2)},1}z_{x^{(0)},y} \to z_{x^{(0)},y}z_{x^{(2)},1}, \quad z_{x^{(2)},y}z_{x^{(0)},y} \to z_{x^{(0)},y}z_{x^{(2)},y},$$
$$z_{x^{(2)},y^2}z_{x^{(0)},y} \to z_{x^{(0)},y}z_{x^{(2)},y^2}, \quad z_{x^{(1)},1}z_{x^{(0)},y^2} \to z_{x^{(0)},y^2}z_{x^{(1)},1},$$
$$z_{x^{(1)},y}z_{x^{(0)},y^2} \to z_{x^{(0)},y^2}z_{x^{(1)},y}, \quad z_{x^{(1)},y^2}z_{x^{(0)},y^2} \to z_{x^{(0)},y^2}z_{x^{(1)},y^2},$$
$$z_{x^{(2)},1}z_{x^{(0)},y^2} \to z_{x^{(0)},y^2}z_{x^{(2)},1}, \quad z_{x^{(2)},y}z_{x^{(0)},y^2} \to z_{x^{(0)},y^2}z_{x^{(2)},y},$$
$$z_{x^{(2)},y^2}z_{x^{(0)},y^2} \to z_{x^{(0)},y^2}z_{x^{(2)},y^2}, \quad z_{x^{(1)},y}z_{x^{(1)},1} \to z_{x^{(1)},1}z_{x^{(1)},y},$$
$$z_{x^{(1)},y^2}z_{x^{(1)},1} \to z_{x^{(1)},1}z_{x^{(1)},y^2}, \quad z_{x^{(2)},1}z_{x^{(1)},1} \to z_{x^{(1)},1}z_{x^{(2)},1},$$
$$z_{x^{(2)},y}z_{x^{(1)},1} \to z_{x^{(1)},1}z_{x^{(2)},y}, \quad z_{x^{(2)},y^2}z_{x^{(1)},1} \to z_{x^{(1)},1}z_{x^{(2)},y^2},$$
$$z_{x^{(1)},y^2}z_{x^{(1)},y} \to z_{x^{(1)},y}z_{x^{(1)},y^2}, \quad z_{x^{(2)},1}z_{x^{(1)},y} \to z_{x^{(1)},y}z_{x^{(2)},1},$$
$$z_{x^{(2)},y}z_{x^{(1)},y} \to z_{x^{(1)},y}z_{x^{(2)},y}, \quad z_{x^{(2)},y^2}z_{x^{(1)},y} \to z_{x^{(1)},y}z_{x^{(2)},y^2},$$
$$z_{x^{(2)},1}z_{x^{(1)},y^2} \to z_{x^{(1)},y^2}z_{x^{(2)},1}, \quad z_{x^{(2)},y}z_{x^{(1)},y^2} \to z_{x^{(1)},y^2}z_{x^{(2)},y},$$
$$z_{x^{(2)},y^2}z_{x^{(2)},y^2} \to z_{x^{(1)},y^2}z_{x^{(2)},y^2}, \quad z_{x^{(2)},y}z_{x^{(2)},1} \to z_{x^{(2)},1}z_{x^{(2)},y},$$
$$z_{x^{(2)},y^2}z_{x^{(2)},1} \to z_{x^{(2)},1}z_{x^{(2)},y^2}, \quad z_{x^{(2)},y^2}z_{x^{(2)},y} \to z_{x^{(2)},y}z_{x^{(2)},y^2},$$

(8) $$x^{(0)}z_{x^{(0)},1} \to z_{x^{(0)},1}x^{(0)}, \quad x^{(0)}z_{x^{(0)},y} \to z_{x^{(0)},y}x^{(0)}, \quad x^{(0)}z_{x^{(0)},y^2} \to z_{x^{(0)},y^2}x^{(0)},$$
$$x^{(0)}z_{x^{(1)},1} \to z_{x^{(1)},1}x^{(0)}, \quad x^{(0)}z_{x^{(1)},y} \to z_{x^{(1)},y}x^{(0)}, \quad x^{(0)}z_{x^{(1)},y^2} \to z_{x^{(1)},y^2}x^{(0)},$$
$$x^{(0)}z_{x^{(2)},1} \to z_{x^{(2)},1}x^{(0)}, \quad x^{(0)}z_{x^{(2)},y} \to z_{x^{(2)},y}x^{(0)}, \quad x^{(0)}z_{x^{(2)},y^2} \to z_{x^{(2)},y^2}x^{(0)},$$
$$x^{(1)}z_{x^{(0)},1} \to z_{x^{(0)},1}x^{(1)}, \quad x^{(1)}z_{x^{(0)},y} \to z_{x^{(0)},y}x^{(1)}, \quad x^{(1)}z_{x^{(0)},y^2} \to z_{x^{(0)},y^2}x^{(1)},$$
$$x^{(1)}z_{x^{(1)},1} \to z_{x^{(1)},1}x^{(1)}, \quad x^{(1)}z_{x^{(1)},y} \to z_{x^{(1)},y}x^{(1)}, \quad x^{(1)}z_{x^{(1)},y^2} \to z_{x^{(1)},y^2}x^{(1)},$$
$$x^{(1)}z_{x^{(2)},1} \to z_{x^{(2)},1}x^{(1)}, \quad x^{(1)}z_{x^{(2)},y} \to z_{x^{(2)},y}x^{(1)}, \quad x^{(1)}z_{x^{(2)},y^2} \to z_{x^{(2)},y^2}x^{(1)},$$
$$x^{(2)}z_{x^{(0)},1} \to z_{x^{(0)},1}x^{(2)}, \quad x^{(2)}z_{x^{(0)},y} \to_{x^{(0)},y} x^{(2)}, \quad x^{(2)}z_{x^{(0)},y^2} \to z_{x^{(0)},y^2}x^{(2)},$$
$$x^{(2)}z_{x^{(1)},1} \to z_{x^{(1)},1}x^{(2)}, \quad x^{(2)}z_{x^{(1)},y} \to z_{x^{(1)},y}x^{(2)}, \quad x^{(2)}z_{x^{(1)},y^2} \to z_{x^{(1)},y^2}x^{(2)},$$
$$x^{(2)}z_{x^{(2)},1} \to z_{x^{(2)},1}x^{(2)}, \quad x^{(2)}z_{x^{(2)},y} \to z_{x^{(2)},y}x^{(2)}, \quad x^{(2)}z_{x^{(2)},y^2} \to z_{x^{(2)},y^2}x^{(2)},$$

(9) $$z_{x^{(0)},1}y \to yz_{x^{(0)},y}, \quad z_{x^{(0)},y}y \to yz_{x^{(0)},y^2}, \quad z_{x^{(0)},y^2}y \to yz_{x^{(0)},y},$$
$$z_{x^{(1)},1}y \to yz_{x^{(1)},y}, \quad z_{x^{(1)},y}y \to yz_{x^{(1)},y^2}, \quad z_{x^{(1)},y^2}y \to yz_{x^{(1)},y},$$
$$z_{x^{(2)},1}y \to yz_{x^{(2)},y}, \quad z_{x^{(2)},y}y \to yz_{x^{(2)},y^2}, \quad z_{x^{(2)},y^2}y \to yz_{x^{(2)},y}.$$

To show that this system is confluent, we check all overlapping words as follows.
Overlapping of the relation (1) with itself:

$(1) \cap (1) : y^4$

Overlapping of the relation (2) with itself:

$(2) \cap (2) : x^{(2)}x^{(1)}x^{(0)}$

Overlapping of the relation (2) with relation (3):

$(2) \cap (3) : x^{(1)}(x^{(0)})^3$, $(2) \cap (3) : x^{(2)}(x^{(0)})^3$, $(2) \cap (3) : x^{(2)}(x^{(1)})^3$

Overlapping of the relation (2) with relation (5):

$(2) \cap (5) : x^{(1)}x^{(0)}y$, $(2) \cap (5) : x^{(2)}x^{(0)}y$, $(2) \cap (5) : x^{(2)}x^{(1)}y$

Overlapping of the relation (2) with relation (8): There are twenty seven overlapping words of this type. Here, we give six of them as application.

$(2)\cap(8) : x^{(1)}x^{(0)}z_{x^{(0)},1}$, $(2)\cap(8) : x^{(1)}x^{(0)}z_{x^{(0)},y}$, $(2)\cap(8) : x^{(1)}x^{(0)}z_{x^{(0)},y^2}$,

$(2)\cap(8): x^{(2)}x^{(1)}z_{x^{(2)},1}$, $(2)\cap(8): x^{(2)}x^{(1)}z_{x^{(2)},y}$, $(2)\cap(8): x^{(2)}x^{(1)}z_{x^{(2)},y^2}$

Overlapping of the relation (3) with relation (2):

$(3)\cap(2): {x^{(1)}}^3x^{(0)}$, $(3)\cap(2): {x^{(2)}}^3x^{(0)}$, $(3)\cap(2): {x^{(2)}}^3x^{(1)}$

Overlapping of the relation (3) with itself:

$(3)\cap(3): {x^{(0)}}^4$, $(3)\cap(3): {x^{(1)}}^4$, $(3)\cap(3): {x^{(2)}}^4$

Overlapping of the relation (3) with relation (4):

$(3)\cap(4): {x^{(2)}}^3y$

Overlapping of the relation (3) with relation (5):

$(3)\cap(5): {x^{(0)}}^3y$, $(3)\cap(5): {x^{(1)}}^3y$

Overlapping of the relation (3) with relation (8): There are twenty seven overlapping words of this type. Here, we give six of them as application.

$(3)\cap(8): {x^{(0)}}^3z_{x^{(0)},1}$, $(3)\cap(8): {x^{(0)}}^3z_{x^{(0)},y}$, $(3)\cap(8): {x^{(0)}}^3z_{x^{(0)},y^2}$,
$(3)\cap(8): {x^{(2)}}^3z_{x^{(2)},1}$, $(3)\cap(8): {x^{(2)}}^3z_{x^{(2)},y}$, $(3)\cap(8): {x^{(2)}}^3z_{x^{(2)},y^2}$

Overlapping of the relation (4) with relation (1):

$(4)\cap(1): x^{(2)}y^3$

Overlapping of the relation (5) with relation (1):

$(5)\cap(1): x^{(0)}y^3$, $(5)\cap(1): x^{(1)}y^3$

Overlapping of the relation (6) with itself: There are twenty nine overlapping words of this type. Here, we give three of them as application.

$(6)\cap(6): z^3_{x^{(0)},1}$, $(6)\cap(6): z^3_{x^{(0)},y}$, $(6)\cap(6): z^3_{x^{(0)},y^2}$

Overlapping of the relation (6) with relation (7): There are thirty six overlapping words of this type. Here, we give six of them as application.

$(6)\cap(7): z^2_{x^{(0)},y}z_{x^{(0)},1}$, $(6)\cap(7): z^2_{x^{(0)},y^2}z_{x^{(0)},1}$, $(6)\cap(7): z^2_{x^{(0)},y^2}z_{x^{(0)},y}$,
$(6)\cap(7): z^2_{x^{(2)},y^2}z_{x^{(1)},y^2}$, $(6)\cap(7): z^2_{x^{(1)},y}z_{x^{(0)},1}$, $(6)\cap(7): z^2_{x^{(2)},y^2}z_{x^{(2)},y}$

Overlapping of the relation (6) with relation (9): There are nine overlapping words of this type. Here, we give three of them as application.

$(6)\cap(9): z^2_{x^{(0)},1}y$, $(6)\cap(9): z^2_{x^{(1)},y}y$, $(6)\cap(9): z^2_{x^{(2)},y^2}y$

Overlapping of the relation (7) with relation (6): There are thirty six overlapping words of this type. Here, we give six of them as application.

$(7)\cap(6): z_{x^{(1)},y}z^2_{x^{(0)},1}$, $(7)\cap(6): z_{x^{(1)},y^2}z^2_{x^{(0)},1}$, $(7)\cap(6): z_{x^{(2)},1}z^2_{x^{(0)},1}$,
$(7)\cap(6): z_{x^{(2)},y}z^2_{x^{(2)},1}$, $(7)\cap(6): z_{x^{(2)},y^2}z^2_{x^{(2)},1}$, $(7)\cap(6): z_{x^{(2)},y^2}z^2_{x^{(2)},y}$

Overlapping of the relation (7) with itself: There are eighty four overlapping words of this type. Here, we give four of them as application.

$(7)\cap(7): z_{x^{(0)},y^2}z_{x^{(0)},y}z_{x^{(0)},1}$, $(7)\cap(7): z_{x^{(1)},1}z_{x^{(0)},y}z_{x^{(0)},1}$,
$(7)\cap(7): z_{x^{(2)},y^2}z_{x^{(2)},y}z_{x^{(1)},y}$, $(7)\cap(7): z_{x^{(2)},y^2}z_{x^{(2)},y}z_{x^{(1)},y^2}$

Overlapping of the relation (7) with relation (9): There are thirty six overlapping words of this type. Here, we give six of them as application.

$(7)\cap(9): z_{x^{(0)},y}z_{x^{(0)},1}y$, $(7)\cap(9): z_{x^{(0)},y^2}z_{x^{(0)},1}y$, $(7)\cap(9): z_{x^{(1)},1}z_{x^{(0)},1}y$,
$(7)\cap(9): z_{x^{(2)},y}z_{x^{(2)},1}y$, $(7)\cap(9): z_{x^{(2)},y^2}z_{x^{(2)},1}y$, $(7)\cap(9): z_{x^{(2)},y^2}z_{x^{(2)},y}y$

Overlapping of the relation (8) with relation (6): There are twenty seven overlapping words of this type. Here, we give six of them as application.

$(8)\cap(6): x^{(0)}z^2_{x^{(0)},1}$, $(8)\cap(6): x^{(0)}z^2_{x^{(2)},y^2}$, $(8)\cap(6): x^{(1)}z^2_{x^{(0)},1}$,

$(8)\cap(6): x^{(1)}z^2_{x^{(2)},y^2},\quad (8)\cap(6): x^{(2)}z^2_{x^{(0)},1},\quad (8)\cap(6): x^{(2)}z^2_{x^{(2)},y^2}$

Overlapping of the relation (8) with relation (7): There are one hundred eight overlapping words of this type. Here, we give four of them as application.

$(8)\cap(7): x^{(0)}z_{x^{(0)},y}z_{x^{(0)},1}\quad (8)\cap(7): x^{(0)}z_{x^{(1)},1}z_{x^{(0)},y},$

$(8)\cap(7): x^{(0)}z_{x^{(2)},1}z_{x^{(0)},1}\quad (8)\cap(7): x^{(0)}z_{x^{(2)},1}z_{x^{(0)},y^2}$

Overlapping of the relation (8) with relation (6): There are twenty seven overlapping words of this type. Here, we give six of them as application.

$(8)\cap(6): x^{(0)}z^2_{x^{(0)},1},\quad (8)\cap(6): x^{(0)}z^2_{x^{(1)},y},\quad (8)\cap(6): x^{(1)}z^2_{x^{(0)},y},$

$(8)\cap(6): x^{(1)}z^2_{x^{(0)},y^2},\quad (8)\cap(6): x^{(1)}z^2_{x^{(2)},y^2},\quad (8)\cap(6): x^{(2)}z^2_{x^{(2)},y^2}$

Overlapping of the relation (8) with relation (9): There are twenty seven overlapping words of this type. Here, we give six of them as application

$(8)\cap(9): x^{(0)}z_{x^{(0)},1}y,\quad (8)\cap(9): x^{(0)}z_{x^{(0)},y}y,\quad (8)\cap(9): x^{(0)}z_{x^{(2)},y^2}y,$

$(8)\cap(9): x^{(1)}z_{x^{(1)},y^2}y,\quad (8)\cap(9): x^{(2)}z_{x^{(0)},y}y,\quad (8)\cap(9): x^{(2)}z_{x^{(2)},y^2}y$

Overlapping of the relation (9) with relation (1): There are nine overlapping words of this type. Here, we give three of them as application

$(9)\cap(1): z_{x^{(0)},1}y^3,\quad (9)\cap(1): z_{x^{(1)},1}y^3,\quad (9)\cap(1): z_{x^{(2)},y^2}y^3.$

In fact, all these above overlappinng words are resolved by reduction steps. We show two of them as an example.

$$(2)\cap(8): x^{(1)}x^{(0)}z_{x^{(0)},1}\longrightarrow\begin{cases} x^{(1)}x^{(0)}z_{x^{(0)},1}\to x^{(0)}z_{x^{(0)},1}x^{(1)}\to z_{x^{(0)},1}x^{(0)}x^{(1)}\\ x^{(1)}z_{x^{(0)},1}x^{(0)}\to z_{x^{(0)},1}x^{(1)}x^{(0)}\to z_{x^{(0)},1}x^{(0)}x^{(1)}\end{cases}$$

$$(6)\cap(7): z^2_{x^{(1)},y}z_{x^{(0)},1}\longrightarrow\begin{cases} z_{x^{(1)},y}z_{x^{(0)},1}\to z_{x^{(0)},1}z_{x^{(1)},y}\\ z_{x^{(1)},y}z_{x^{(0)},1}z_{x^{(1)},y}\to z_{x^{(0)},1}z^2_{x^{(1)},y}\to z_{x^{(0)},1}z_{x^{(1)},y}\end{cases}$$

Since the rewriting system given with (1)-(9) is Noetherian and confluent, it is complete.

Now we consider normal form structure of an arbitrary word $u\in A\Diamond_v B$. It is easily seen that it is of the form;

$$y^pW_{z_{x^{(i)},y^m}}W'_{(x^{(q)})^t}\ (0\leqslant p,\ q,\ i,\ m,\ t\leqslant 2),\tag{3.2}$$

where $W_{z_{x^{(i)},y^m}}$ and $W'_{(x^{(q)})^t}$ are reduced words obtained by generators $z_{x^{(i)},y^m}$ and $\left(x^{(q)}\right)^t$, respectively. For example, the words $y\left(x^{(2)}\right)^2$ and $z_{x^{(i)},1}z_{x^{(0)},y^2}\left(x^{(1)}\right)^2x^{(2)}$ are of the forms given in (3.2).

Finally, we can say that the number of generators and relations of $A\Diamond_v B$ are 13 and 91, respectively. We can easily see these results by taking $s=3$ in the generator number formula $(s+1)^2-s$ and relator number formula $\frac{1}{2}(s^4+2s^3+4s^2+3s+2)$ in Corollary 2.1.

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