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INVERSE PAIRED DOMINATION IN SOME GRAPHS

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Abstract. Let $G = (V, E)$ be a graph and D be a minimum paired dominating set of G. If $V - D$ contains a paired dominating set $D^{'}$ of G, then $D^{'}$ is called an inverse paired dominating set with respect to D. The inverse paired domination number $\gamma_{pd}^{-1}(G)$ is the order of a smallest inverse paired dominating set. In this paper, we initiate the study of inverse paired domination in graphs and obtain the exact values for some standard graphs and some graph operations.

1. Introduction

Domination and its variants are well studied in graph theory. The study of dominating sets began around 1960 and it was first defined by O.Ore in 1962 [**5**]. There are many types of domination depending on the structures of dominating sets.

One of these variants, paired domination was introduced by T.Haynes and P.Slater [**3**]. The paired dominating set is a dominating set D in which induced sub-graph D contains perfect matching. The cardinality of the minimal paired domion set is called paired domination number and denoted by $\gamma_{pd}(G)$.

Inverse domination was introduced by V.R.Kulli and S.C.Sigarkanti [**4**] in 1991. If $V - D$ contains a dominating set $D^{'}$ of G, then $D^{'}$ is called an inverse dominating set of G with respect to D. The inverse domination number is the minimum cardinality of an inverse dominating set and is denoted by $\gamma^{-1}(G)$.

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In this paper, we introduce a new domination parameter in graphs called inverse paired domination number and denoted by $\gamma_{pd}^{-1}(G)$. We obtained inverse paired domination number for some certain graphs and some graph operations.

2. Results

In this section, we define the inverse paired domination $\gamma_{pd}^{-1}(G)$ of a graph G and study this parameter on some exact graphs. Also, some generalizations are given with their proofs.

DEFINITION 2.1. *If $V-D$ contains a paired dominating set D, then it is called an inverse paired dominating set and the inverse paired domination number denoted by $\gamma_{pd}^{-1}(G)$ is the minimum cardinality of D sets.*

THEOREM 2.1. *Let D be a paired dominating set of a connected graph G. If a $\gamma_{pd}^{-1}-$ set exists, then G at least 4 vertices.*

PROOF. Let D be a paired dominating set of G. Since G is a connected graph and has no isolated vertices, then $|D| \geqslant 2$. If a $\gamma_{pd}^{-1}-$ set exists, then $V-D$ contains a paired dominating set with respect to D. Thus, $|V-D| \geqslant 2$. Hence, G has at least 4 vertices. □

THEOREM 2.2. *Let D a paired dominating set of G. Then G has an inverse paired dominating set if and only if $|D| \leqslant |V-D|$.*

PROOF. Suppose that D is a paired dominating set and $|D| > |V-D|$. It is clear that every vertex of V is adjacent to some vertex of D. If every vertex in $V-D$ is adjacent to exactly one vertex of D, then there exist a vertex in D, which is not adjacent to any vertex in $V-D$. Thus, $V-D$ does not contain a paired dominating set and this is a contradiction. Hence, $|D| \leqslant |V-D|$.

□

The following Proposition is used to prove our later theorems.

PROPOSITION 2.1. *[**6**]*

(1) $\gamma_{pd}(P_n) = \gamma_{pd}(C_n) = 2\lceil \frac{n}{4} \rceil$.
(2) $\gamma_{pd}(K_n) = 2$.
(3) $\gamma_{pd}(K_{m,n}) = 2$ *where* $m, n \geqslant 2$.
(4) $\gamma_{pd}(K_{1,n}) = 2$.

Now we obtain the exact values of inverse paired domination number for some certain graphs.

THEOREM 2.3. *If K_n is a complete graph with $n \geqslant 4$, then $\gamma_{pd}^{-1}(K_n) = 2$.*

PROOF. Let D be a minimum paired dominating set of K_n. Then $|D| = 2$. The induced subgraph $< V-D >$ is K_{n-2}. The inverse paired dominating set of K_n is a minimum paired dominating set of K_{n-2}. Thus, $\gamma_{pd}^{-1}(K_n) = \gamma_{pd}(K_{n-2}) = 2$. □

THEOREM 2.4. *If P_n is a path graph with n vertices, then*

$$\gamma_{pd}^{-1}(P_n) = 2\lfloor \tfrac{n}{4} \rfloor \ , if n \equiv 2(mod 4).$$

PROOF. Suppose that $n \equiv 2(mod4)$ and D is a minimum paired dominating set of P_n. To provide an inverse paired dominating set in the induced subgraph $< V - D >$ of P_n, there must be a unique minimum paired dominating set $D = \{v_{4k+1}v_{4k+2} | 0 \leqslant k \leqslant \lceil \frac{n}{4} \rceil\}$ where each $v_i \in V(P_n)$. Hence, the induced subgraph $< V - D >$ contains at most $\lfloor \frac{n}{4} \rfloor$ edges and $2\lfloor \frac{n}{4} \rfloor$ number of vertices. Thus, $\gamma_{pd}^{-1}(P_n) = 2\lfloor \frac{n}{4} \rfloor$. □

THEOREM 2.5. *If C_n is a cycle graph with n vertices, then*

$$\gamma_{pd}^{-1}(C_n) = \begin{cases} 2\lfloor \frac{n}{4} \rfloor, & \text{if } n \equiv 2(mod4) \\ 2\lceil \frac{n}{4} \rceil, & \text{if } n \equiv 0(mod4) \end{cases}$$

PROOF. Here we consider two cases.

Case 1. Suppose that $n \equiv 2(mod4)$ and D be a minimum paired dominating set of C_n. As Proposition 2.1, to provide an inverse paired dominating set of C_n, the minimum paired dominating set must be $D = \{v_{4k+1}v_{4k+2} | 0 \leqslant k \leqslant \lceil \frac{n}{4} \rceil\}$ where $v_k \in V(C_n)$. The induced subgraph $< V - D >$ contains at most $\lfloor \frac{n}{4} \rfloor$ edges and $2\lfloor \frac{n}{4} \rfloor$ number of vertices. Thus, $\gamma_{pd}^{-1}(C_n) = 2\lfloor \frac{n}{4} \rfloor$.

Case 2. Suppose that $n \equiv 0(mod4)$. By Proposition 2.1, we have $\gamma_{pd}(C_n) = 2\lceil \frac{n}{4} \rceil$. Since the induced subgraph $< V - D >$ contains $\lceil \frac{n}{4} \rceil$ edges and $2\lceil \frac{n}{4} \rceil$ number of vertices, the inverse paired dominating set of C_n is a minimum paired dominating set of C_n. Thus, $\gamma_{pd}^{-1}(C_n) = \gamma_{pd}(C_n) = 2\lceil \frac{n}{4} \rceil$.

Therefore, from all the previous cases, the result is obtained. □

THEOREM 2.6. *If W_n is a wheel graph with $n \geqslant 4$ vertices, then*

$$\gamma_{pd}^{-1}(W_n) = 2\lceil \tfrac{n-2}{4} \rceil.$$

PROOF. Let $W_n = K_1 + C_{n-1}$ and $deg(v) = n - 1$. Let u be adjacent with v. Then $D = \{u, v\}$ is a minimum paired dominating set of W_n. The inverse paired dominating set of W_n is a minimum paired dominating set of C_{n-2}. Thus, $\gamma_{pd}^{-1}(W_n) = \gamma_{pd}(C_{n-2}) = 2\lceil \frac{n-2}{4} \rceil$. □

THEOREM 2.7. *If $K_{m,n}$ is a complete bipartite graph with $2 \leqslant m \leqslant n$, then*

$$\gamma_{pd}^{-1}(K_{m,n}) = 2$$

PROOF. Let $V(K_{m,n}) = V_1 \cup V_2$ and $u \in V_1$, $v \in V_2$. Then $D = \{u, v\}$ is a minimum paired dominating set of $K_{m,n}$ and the induced subgraph $< V - D >$ is $K_{m-1,n-1}$. Therefore, the inverse paired dominating set of $K_{m,n}$ is minimum paired dominating set of $K_{m-1,n-1}$. Thus, $\gamma_{pd}^{-1}(K_{m,n}) = \gamma_{pd}(K_{m-1,n-1}) = 2$. □

3. Some operation results

There are various graph products that have already been defined in the literature given two graphs G and H. In this section, we will focus on one such product namely corona product and we will calculate the γ_{pd}^{-1} of this product type and some graphs.

THEOREM 3.1. *For the Fan graph $F_n \equiv P_n + K_1$, then $\gamma_{pd}^{-1}(F_n) = 2\lceil \frac{n-1}{4} \rceil$.*

PROOF. Let $F_n \equiv P_n + K_1$ and $deg(v) = n$. Let $\{u_1, u_2, ..., u_n\}$ be the vertex set of the path P_n. Then, $D = \{v, u_1\}$ is a minimum paired dominating set and the induced subgraph $< V - D >$ is $< u_2, u_3, ..., u_n >$. The inverse paired dominating set of F_n is minimum paired dominating set of P_{n-1}. Thus, $\gamma_{pd}^{-1}(F_n) = \gamma_{pd}(P_{n-1}) = 2\lceil \frac{n-1}{4} \rceil$ □

THEOREM 3.2. *For a Windmill graph with $n \geqslant 4$, then $\gamma_{pd}^{-1}(Wd(n,m)) = 2m$.*

PROOF. From the definition [**2**], the windmill graph $Wd(n,m)$ can be constructed by joining m copies of the complete graph K_n with a common vertex v. Let u be any other vertex in graph. Then $D = \{v, u\}$ is a minimum paired dominating set of $Wd(n,m)$. The induced subgraph $< V - D >$ consists of m copies of K_{n-1}. By Proposition 2.1, we have $\gamma_{pd}(K_{n-1}) = 2$. Thus, the inverse paired dominating set is $\gamma_{pd}^{-1}(Wd(n,m)) = m.\gamma_{pd}(K_{n-1}) = 2m$. □

THEOREM 3.3. *For the corona graph $G \equiv C_n \odot P_m$, then*

$$\gamma_{pd}^{-1}(C_n \odot P_m) = \begin{cases} (n-1).2\lceil \frac{m}{4} \rceil + 2\lceil \frac{m-1}{4} \rceil, & \text{if } n \text{ is odd} \\ n.2\lceil \frac{m}{4} \rceil, & \text{if } n \text{ is even} \end{cases}$$

PROOF. There are two cases that depend on whether the value of n is odd or even.

Case 1. Let n be even. By the definition of corona graph [**1**], each vertex of C_n is adjacent to every vertex of P_m. Hence each vertex of C_n belongs to the paired dominating set and and it is also a paired dominating set containing perfect matching. The induced subgraph $< V - D >$ contains n copies of P_n. By Proposition 2.1, we have $\gamma_{pd}(P_m) = 2\lceil \frac{m}{4} \rceil$. Therefore, $\gamma_{pd}^{-1}(C_n \odot P_m) = n.2\lceil \frac{m}{4} \rceil$.

Case 2. Let n be odd and $\{v_1, v_2, ..., v_n\}$ be the vertex set of the graph C_n. Since n is odd, there exist an unpaired vertex in dominating set. Therefore, for any $u_{ij} \in P_m$ in $V - D$ where i represents the number of vertices of C_n and j represents P_m, $i = 1, 2, ..., n$; $j = 1, 2, ..., m$, $D \cup u_{ij}$ has a perfect matching and is a paired dominating set. Therefore, $\gamma_{pd}^{-1}(C_n \odot P_m) = (n-1).2\lceil \frac{m}{4} \rceil + 2\lceil \frac{m-1}{4} \rceil$.

Therefore, from all the previous cases, the result is obtained. □

THEOREM 3.4. *For the corona graph $G \equiv P_n \odot K_2$ with $n \geqslant 2k$ vertices and $k \geqslant 1$, then $\gamma_{pd}^{-1}(P_n \odot K_2) = 2n$.*

PROOF. Let n be even and $\{v_1, v_2, ..., v_n\}$ be the vertex set of the graph P_n. It is clear that $D = \{v_{2k+1}v_{2k+2} | 0 \leqslant k \leqslant \lfloor \frac{n-1}{2} \rfloor\}$ is a minimum paired dominating set of $P_n \odot K_2$. The induced subgraph $< V - D >$ contains n copies of K_2. Hence, the inverse paired dominating set is $n.|V(K_2)|$. Thus, $\gamma_{pd}^{-1}(P_n \odot K_2) = n.|V(K_2)| = 2n$. □

THEOREM 3.5. *For the corona graph $G \equiv C_n \odot K_2$, then $\gamma_{pd}^{-1}(C_n \odot K_2) = 2n$.*

PROOF. It is clear that the result is obtained by using Theorem 3.4. □

THEOREM 3.6. *For the corona graph $G \equiv C_n \odot K_{1,m}$, then $\gamma_{pd}^{-1}(C_n \odot K_{1,m}) = 2n$.*

PROOF. Let $\{v_1, v_2, ..., v_n\}$ be the vertex set of the graph C_n and $m \geqslant 2$. Let's examine it in two cases, depending on whether the value of n is odd or even.

When n is even, the minimum paired dominating set is $D = \{v_{2k+1}v_{2k+2} | 0 \leqslant k \leqslant \lfloor \frac{n-1}{2} \rfloor\}$. The induced subgraph $< V - D >$ contains n copies of $K_{1,m}$. By Proposition 2.1, we have $\gamma_{pd}(K_{1,m}) = 2$. Thus, the inverse paired dominating set is $\gamma_{pd}^{-1}(C_n \odot K_{1,m}) = n.\gamma_{pd}(K_{1,m}) = 2n$.

When n is odd, D is not a paired dominating set. There exist an unpaired vertex in dominating set. Therefore, for any $u_{ij} \in K_{1,m}$ in $V - D$ where i represents the number of vertices of C_n and j represents $K_{1,m}$, $i = 1, 2, ..., n$ and $j = 1, 2, ..., m$ where $m \geqslant 3$, $D \cup u_{ij}$ has a perfect matching and is a paired dominating set. Hereby, even if any of the $K_{1,m}$ in graph be $K_{1,m-1}$, the result does not change. In the same way, the inverse paired domination number is $\gamma_{pd}^{-1}(C_n \odot K_{1,m}) = 2n$.

Therefore, from all the previous cases, the result is obtained. □

THEOREM 3.7. *For the corona graph $G \equiv P_n \odot K_m$, then $\gamma_{pd}^{-1}(P_n \odot K_m) = 2n$.*

PROOF. The result can be obtained similarly by using Theorem 3.6. □

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