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EIGENVALUES OF PARAMETER-DEPENDENT STURM-LIOUVILLE PROBLEMS WITH A FROZEN ARGUMENT ON TIME SCALES

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Abstract. In this paper, a boundary value problem established with the Sturm-Liouville equation which has a frozen argument, and with parameter-dependent boundary conditions is considered on time scales. Some properties of the eigenvalues of the problem are investigated on a finite time scale as well as on a union of two intervals.

1. Introduction and Preliminaries

Let us consider the following boundary value problem

$$-y^{\Delta\Delta}(t) + q(t)y(a) = \lambda y^{\sigma}(t),\ t \in \mathbb{T}^{\kappa^2} \tag{1.1}$$

$$U(y) := a_1(\lambda)\, y(\alpha) + a_2(\lambda)\, y^{\Delta}(\alpha) = 0, \tag{1.2}$$

$$V(y) := b_1(\lambda)\, y(\beta) + b_2(\lambda)\, y^{\Delta}(\beta) = 0. \tag{1.3}$$

where $q(t)$ is a real-valued continuous function, $a \in \mathbb{T}^{\kappa} := \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}]$ is the frozen argument, $y^{\sigma}(t) = y(\sigma(t))$, $\alpha = \inf \mathbb{T}$, $\beta = \rho(\sup \mathbb{T})$, $\alpha \neq \beta$, $a_i(\lambda)$ and $b_i(\lambda)$ are real polynomials for $i, j = 1, 2$, and λ is the complex spectral parameter.

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Asuming $d_a := \deg(a_1) = \deg(a_2)$ and $d_b := \max\{\deg(b_1), \deg(b_2)\}$, we can take

$$\begin{aligned} a_1(\lambda) &= \sum_{k=0}^{d_a} a_{1k}\lambda^k, \quad a_2(\lambda) = \sum_{k=0}^{d_a} a_{2k}\lambda^k, \\ b_1(\lambda) &= \sum_{k=0}^{d_b} b_{1k}\lambda^k, \quad b_2(\lambda) = \sum_{k=0}^{d_b} b_{2k}\lambda^k. \end{aligned}$$

In a contiuous interval, the spectral analysis of boundary value problems for the Sturm-Liouville equation with a frozen argument are studied in [**3**], [**10**], [**11**], [**17**], [**21**], [**24**], [**28**], and references therein. These kinds of problems which appear in various applications are related to some non-local boundary value problems. (see [**4**], [**7**], [**20**], and [**29**])

The Sturm-Liouville problems without a frozen argument on time scales have been investigated in several studies (see e.g. [**1**], [**2**], [**5**], [**6**], [**13**]- [**16**], [**18**], [**19**], [**25**]- [**27**], [**30**]). However, there is only two publications about the Sturm-Liouville equation with a frozen argument on a time scale ([**12**] and [**22**])

The aim of this paper is to investigate some important properties of solutions and eigenvalues of problem (1)-(3). Before begining, it must be noted that we refer the publications [**8**], [**9**] and [**23**] for the basic notation and terminology of the time scales theory.

2. Main results

Let $S(t,\lambda)$ and $C(t,\lambda)$ be the solutions of (1) with the initial conditions

$$S(a,\lambda) = 0,\ S^{\Delta}(a,\lambda) = 1, \tag{2.1}$$

$$C(a,\lambda) = 1,\ C^{\Delta}(a,\lambda) = 0, \tag{2.2}$$

respectively. Clearly, $S(t,\lambda)$ and $C(t,\lambda)$ satisfy

$$\begin{aligned} S^{\Delta\Delta}(t,\lambda) + \lambda S^{\sigma}(t,\lambda) &= 0 \\ C^{\Delta\Delta}(t,\lambda) + \lambda C^{\sigma}(t,\lambda) &= q(t), \end{aligned}$$

respectively, and so these functions and their Δ-derivatives are entire on λ for each fixed t (see [**25**]).

It is clear that the zeros of the function

$$\Delta(\lambda) := \det \begin{pmatrix} U(C) & V(C) \\ U(S) & V(S) \end{pmatrix}, \tag{2.3}$$

which is also entire, coincide with the eigenvalues of the problem (1)-(3).

We aim to examine the problem (1)-(3) on two different version of $\mathbb{T}$.

First, we asume $\mathbb{T}$ is a finite time scale such that

$$\mathbb{T} = \left\{\rho^r(a), \rho^{r-1}(a), ..., \rho^2(a), \rho(a), a, \sigma(a), \sigma^2(a), ..., \sigma^{m-1}(a), \sigma^m(a)\right\},$$

where $\sigma^j = \sigma^{j-1} \circ \sigma$, $\rho^j = \rho^{j-1} \circ \rho$ for $j \geqslant 2$, $\rho^r(a) = \alpha$, $\sigma^{m-1}(\alpha) = \beta$, $m \geqslant 2$, and $r \geqslant 2$.

THEOREM 2.1. *If $a_{1d_a}\mu(\alpha) - a_{2d_a} \neq 0$, eigenvalues-number of (1)-(3) is as follows with multiplications*

$$s := \begin{cases} d_a + d_b + n - 2, & \deg(b_2) \geqslant \deg(b_1) \\ d_a + d_b + n - 3, & \deg(b_1) > \deg(b_2) + 1 \end{cases}, \tag{2.4}$$

where n is the number of elements of $\mathbb{T}$ and equals clearly to $m + r + 1$.

PROOF. It can be calculated that

$$\begin{aligned} \Delta(\lambda) &= \det \begin{pmatrix} U(C) & V(C) \\ U(S) & V(S) \end{pmatrix} \\ &= \det \begin{pmatrix} a_1(\lambda) C(\alpha) + a_2(\lambda) C^{\Delta}(\alpha) & b_1(\lambda) C(\beta) + b_2(\lambda) C^{\Delta}(\beta) \\ a_1(\lambda) S(\alpha) + a_2(\lambda) S^{\Delta}(\alpha) & b_1(\lambda) S(\beta) + b_2(\lambda) S^{\Delta}(\beta) \end{pmatrix} \\ &= a_1(\lambda) b_1(\lambda) [C(\alpha) S(\beta) - S(\alpha) C(\beta)] \\ &\quad + a_1(\lambda) b_2(\lambda) \left[C(\alpha) S^{\Delta}(\beta) - S(\alpha) C^{\Delta}(\beta)\right] \\ &\quad + a_2(\lambda) b_1(\lambda) \left[C^{\Delta}(\alpha) S(\beta) - S^{\Delta}(\alpha) C(\beta)\right] \\ &\quad + a_2(\lambda) b_2(\lambda) \left[C^{\Delta}(\alpha) S^{\Delta}(\beta) - S^{\Delta}(\alpha) C^{\Delta}(\beta)\right]. \end{aligned}$$

In [**25**], it is given the following equalities

$$S(\alpha,\lambda) = (-1)^r \mu^{\rho}(a) \left[\mu^{\rho^2}(a) \mu^{\rho^3}(a) ... \mu^{\rho^r}(a)\right]^2 \lambda^{r-1} + O\left(\lambda^{r-2}\right),$$

$$S^{\sigma}(\alpha,\lambda) = (-1)^{r-1} \mu^{\rho}(a) \left[\mu^{\rho^2}(a) \mu^{\rho^3}(a) ... \mu^{\rho^{r-1}}(a)\right]^2 \lambda^{r-2} + O\left(\lambda^{r-3}\right),$$

$$S(\beta,\lambda) = (-1)^m \left[\mu(a) \mu^{\sigma}(a) ... \mu^{\sigma^{m-3}}(a)\right]^2 \lambda^{m-2} \mu^{\sigma^{m-2}}(a) + O\left(\lambda^{m-3}\right),$$

$$S^{\sigma}(\beta,\lambda) = (-1)^{m+1} \left[\mu(a) \mu^{\sigma}(a) ... \mu^{\sigma^{m-2}}(a)\right]^2 \lambda^{m-1} \mu^{\sigma^{m-1}}(a) + O\left(\lambda^{m-2}\right),$$

$$C(\alpha,\lambda) = (-1)^r \left[\mu^{\rho}(a) \mu^{\rho^2}(a) ... \mu^{\rho^r}(a)\right]^2 \lambda^r + O\left(\lambda^{r-1}\right),$$

$$C^{\sigma}(\alpha,\lambda) = (-1)^{r-1} \left[\mu^{\rho}(a) \mu^{\rho^2}(a) ... \mu^{\rho^{r-1}}(a)\right]^2 \lambda^{r-1} + O\left(\lambda^{r-2}\right),$$

$$C(\beta,\lambda) = (-1)^m \mu(a) \left[\mu^{\sigma}(a) \mu^{\sigma^2}(a) ... \mu^{\sigma^{m-3}}(a)\right]^2 \mu^{\sigma^{m-2}}(a) \lambda^{m-2} + O\left(\lambda^{m-3}\right),$$

$$C^{\sigma}(\beta,\lambda) = (-1)^{m+1} \mu(a) \left[\mu^{\sigma}(a) \mu^{\sigma^2}(a) ... \mu^{\sigma^{m-2}}(a)\right]^2 \mu^{\sigma^{m-1}}(a) \lambda^{m-1} + O\left(\lambda^{m-2}\right),$$

where $O(\lambda^l)$ denotes a polynomial whose degree is l. Taking into account asumptions on degrees of polynomials $a_i(\lambda)$ and $b_i(\lambda)$ it can be obtained that for $\deg(b_2) \geqslant \deg(b_1)$,

$$\Delta(\lambda) = \frac{b_{2d_b}}{\mu(\alpha)\mu(\beta)} \lambda^{d_a+d_b} \left[a_{1d_a}\mu(\alpha) - a_{2d_a}\right] \lambda^{d_a+d_b+n-2} + O(\lambda^{d_a+d_b+n-3}),$$

and for $\deg(b_1) > \deg(b_2) + 1$,

$$\Delta(\lambda) = \frac{b_{1d_b}}{\mu(\alpha)} \lambda^{d_a+d_b} \left[a_{1d_a}\mu(\alpha) - a_{2d_a}\right] \lambda^{d_a+d_b+n-3} + O(\lambda^{d_a+d_b+n-4}).$$

Hence, the proof is completed. $\square$

Corollary 2.1. *The eigenvalues-number of (1)-(3) does not depend on $q(t)$ or a but elements-number of $\mathbb{T}$ and the polynomials in the boundary conditions (2) and (3).*

Remark 2.1. If we take the time scale simply as $\mathbb{T} = \{1, 2, ..., n\}$, we can find all eigenvalue of the problem solving the equation $\det \begin{pmatrix} P \\ Q \end{pmatrix} = 0$, where $P = P_1 + P_2$,

$$P_1 = \begin{pmatrix} 1 & \lambda-2 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \lambda-2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \lambda-2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & \lambda-2 & 1 \end{pmatrix}_{(n-2)\times n},$$

$$P_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 & \overset{a.column}{-q(1)} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -q(2) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -q(3) & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & -q(n-2) & 0 & \cdots & 0 \end{pmatrix}_{(n-2)\times n}, \text{ and}$$

$$Q = \begin{pmatrix} a_1(\lambda) - a_2(\lambda) & a_2(\lambda) & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & b_1(\lambda) - b_2(\lambda) & b_2(\lambda) \end{pmatrix}_{2\times n}.$$

Example 2.1. *Consider the following problem on $\mathbb{T} = \{0, 1, 2, 3, 4, 5, 6\}$*

$$L : \begin{cases} -y^{\Delta\Delta}(t) + y(3) = \lambda y^{\sigma}(t), \ t \in \{0, 1, 2, 3, 4\} \\ (3\lambda^2 + 1)y(1) + \left(2\lambda^2 - 3\lambda - 3\right) y^{\Delta}(1) = 0, \\ \lambda^4 y(5) + \left(\lambda^2 - 1\right) y^{\Delta}(5) = 0. \end{cases}$$

Eigenvalues of L are the zeros of the polynomial

$$p(\lambda) = -\lambda^9 + 7\lambda^8 - 19\lambda^7 + 32\lambda^6 - 49\lambda^5 + 57\lambda^4 - 29\lambda^3 - 24\lambda^2 + 54\lambda - 20.$$

Now, we move our study to another special time scale: $\mathbb{T} = [\alpha, \delta_1] \cup [\delta_2, \beta]$, where $\alpha < a < \delta_1 < \delta_2 < \beta$. Suppose $a \in (\alpha, \delta_1)$ and $\beta - \delta_2 = \delta_1 - \alpha$. The similar analysis can be done for $a \in (\delta_2, \beta)$.

The following asymptotic relations for the solutions $S(t, \lambda)$ and $C(t, \lambda)$ can be proved by using a method similar to that in [**26**].

$$S(t,\lambda) = \begin{cases} \dfrac{\sin\sqrt{\lambda}\,(t-a)}{\sqrt{\lambda}}, \ t \in [\alpha, \delta_1], \\ \begin{array}{c} \delta^2\sqrt{\lambda}\cos\sqrt{\lambda}\,(\delta_1 - a)\sin\sqrt{\lambda}(\delta_2 - t) \\ + O\left(\exp|\tau|\,(t-a-\delta)\right) \end{array} ,t \in [\delta_2, \beta], \end{cases} \tag{2.5}$$

$$S^{\Delta}(t,\lambda) = \begin{cases} \cos\sqrt{\lambda}\,(t-a)\,, \ t \in [\alpha, \delta_1), \\ \begin{array}{c} -\delta^2\lambda\cos\sqrt{\lambda}\,(\delta_1 - a)\cos\sqrt{\lambda}(\delta_2 - t) \\ + O\left(\sqrt{\lambda}\exp|\tau|\,(t-a-\delta)\right) \end{array} , \ t \in [\delta_2, \beta], \end{cases} \tag{2.6}$$

$$C(t,\lambda) = \begin{cases} \cos\sqrt{\lambda}\,(t-a) + O\left(\dfrac{1}{\sqrt{\lambda}}\exp|\tau|\,|t-a|\right), \ t \in [\alpha, \delta_1], \\ \begin{array}{c} -\delta^2\lambda\sin\sqrt{\lambda}\,(\delta_1 - a)\sin\sqrt{\lambda}(\delta_2 - t) \\ + O\left(\sqrt{\lambda}\exp|\tau|\,(t-a-\delta)\right) \end{array} , \ t \in [\delta_2, \beta], \end{cases} \tag{2.7}$$

$$C^{\Delta}(t,\lambda) = \begin{cases} -\sqrt{\lambda}\sin\sqrt{\lambda}\,(t-a) + O\left(\exp|\tau|\,|t-a|\right), \quad t \in [\alpha, \delta_1), \\ \begin{array}{c} \delta^2\lambda^{3/2}\sin\sqrt{\lambda}\,(\delta_1 - a)\cos\sqrt{\lambda}(\delta_2 - t) \\ + O\left(\lambda\exp|\tau|\,(t-a-\delta)\right) \end{array} , \ t \in [\delta_2, \beta], \end{cases} \tag{2.8}$$

where $\delta = \delta_2 - \delta_1$, $\tau = \mathrm{Im}\sqrt{\lambda}$ and O denotes Landau's symbol.

Since $\beta - \delta_2 = \delta_1 - \alpha$, by calculating directly, it can be obtained from (8)-(11) that the equality

$$\Delta(\lambda) = \begin{cases} A(\lambda)\left[\sin 2\sqrt{\lambda}\,(\beta - \delta_2) + O\left(\frac{\exp|\tau|(\beta-\alpha-\delta)}{\sqrt{\lambda}}\right)\right], \ \deg b_2 \geqslant \deg b_1 \\ B(\lambda)\left[\cos 2\sqrt{\lambda}\,(\beta - \delta_2) + O\left(\frac{\exp|\tau|(\beta-\alpha-\delta)}{\sqrt{\lambda}}\right)\right], \ \deg b_2 < \deg b_1 \end{cases} \tag{2.9}$$

is valid for $|\lambda| \to \infty$, where $A(\lambda) = \frac{-\delta^2}{2} a_{2d_a} b_{2d_b} \lambda^{d_a + d_b + \frac{3}{2}}$ and $B(\lambda) = \frac{-\delta^2}{2} a_{2d_a} b_{1d_b} \lambda^{d_a + d_b + 1}$.

Consider the region

$$G_\varepsilon := \{\lambda \in \mathbb{C} : \lambda = \rho^2, \left|\rho - \rho_n^0\right| > \varepsilon, \ n = 1, 2, 3, ...\}$$

where ε is sufficiently small number, and $\rho_n^0 = \begin{cases} \frac{n\pi}{2(\beta-\delta_2)}, & \deg b_2 \geqslant \deg b_1 \\ \frac{\left(n+\frac{1}{2}\right)\pi}{2(\beta-\delta_2)}, & \deg b_2 < \deg b_1 \end{cases}$. There exist some positive constants C_ε for each ε, such that, the inequality

$$|\Delta(\lambda)| \geqslant C_\varepsilon |\lambda|^\gamma \exp 2 |\tau| (\beta - \delta_2)$$

holds for sufficiently large $\lambda \in G_n$, where $\gamma = \begin{cases} d_a + d_b + \frac{3}{2}, & \deg b_2 \geqslant \deg b_1 \\ d_a + d_b + 1, & \deg b_2 < \deg b_1 \end{cases}$.

Consequently, applying Rouche's theorem to $\Delta(\lambda)$ on $G_n := \{\lambda \in \mathbb{C} : \lambda = \rho^2, |\rho| < \rho_n^0 + \delta\}$ for sufficiently small δ and sufficiently large n, we obtain the following theorem.

THEOREM 2.2. *The problem (1)-(3) on* $\mathbb{T} = [\alpha, \delta_1] \cup [\delta_2, \beta]$ *has countable many eigenvalues, namely* λ_n *which are real for sufficiently large* n. *Moreover, the following asymptotic formula holds for* $n \to \infty$.

$$\sqrt{\lambda_n} = \begin{cases} \dfrac{n\pi}{2(\beta-\delta_2)} + O\left(\dfrac{1}{n}\right), & \deg b_2 \geqslant \deg b_1 \\ \dfrac{\left(n+\frac{1}{2}\right)\pi}{2(\beta-\delta_2)} + O\left(\dfrac{1}{n}\right), & \deg b_2 < \deg b_1 \end{cases} . \tag{2.10}$$

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