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QUOTIENT NEARNESS RINGS

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Abstract. In 2019, Öztürk et al. defined gamma semigroups, the first algebraic structure on weak nearness approximation spaces (see [**8**]). This new view, which completely changes the nearness of algebraic structures, was first expressed in an article called nearness *d*-algebras (see [**9**]). Afterwards, the normal nearness subgroups, nearness cosets of the nearness groups and quotient nearness groups were defined (see [**13**], [**14**]). In the light of the aforementioned studies, we present this study in which quotient nearness rings are defined and their properties are examined.

1. Introduction

Set theory is a very important tool for mathematicians and engineers use when conducting scientific research and studies. Because the real world is uncertain, imprecise, and absolute, researchers have defined new approaches where ordinary set theory fails. One of these approaches is rough set theory, on which they base set theory. The uncertainty that rough set theory deals with involves indistinguishable elements that have different values in the decision features ([**3**], [**15**]).

Near set theory, which is based on the determination of universal sets according to the available information of the objects and is a generalization of rough set theory, was put forward by Peters in 2007. One can see [**6**], [**16**], [**17**], [**18**], [**19**] and [**20**] for more information on the near set theory.

We may use near set theory to turn elements in algebraic structures into concrete elements. Algebraic structures as we know them are not useful for difficult problems in daily life, because these structures consist of non-empty abstract points. In near set theory, we use perceptual objects (non-abstract points) that have some

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properties, such as the color of an apple, its degree of ripeness, etc. In algebraic structures defined on weak nearness approximation spaces, the main tool is upper approximations of subsets of perceptual objects. Nearness approach is studied with non-abstract points in algebraic structures and upper approximation of perceptual objects are taken into account for the nearness of binary operations. This is the important difference between nearness algebraic structures and classical algebraic structures. The basic property of classical algebraic structures is as follows: Let G be a non-empty set. If elements of G has one and only one property then upper and lower approximation and itself of this set are equal to each other for $r = n$ $(n \in \mathbb{Z})$. That is, $N_r(B)^*G = G = N_r(B)_* G$. In real life, many perceptual objects have more than one property. For this reason, we think that the nearness algebraic structure with the property $G \underset{\neq}{\subset} N_r(B)^* G$ should be examined (see [**9**]).

In 2012, İnan and Öztürk analyzed the concept of nearness groups and investigated its basic properties (see [**1**]). Other algebraic structure studies on nearness approximation spaces are [**2**], [**4**], [**5**], [**7**], [**10**], [**11**], [**12**], [**14**] and [**21**]. Then, in 2019, Öztürk et. al. defined the first algebraic structure which is gamma semigroup on weak nearness approximation spaces (see [**8**]). After this paper, the view of the nearness of algebraic structures has completely changed. This view was first expressed a paper called "nearness d-algebras" in 2021 (see [**9**]). Afterwards, in 2023, Öztürk defined the normal nearness groups and quotient nearness groups (see [**13**]).

In classical ring theory, over the years, many methods have been developed to study rings by dividing rings into smaller parts such as subrings, ideals, and quotient rings. In classical ring theory, the concept of cosets of a subring or ideal of a ring is a crucial notion in the study of quotient rings. In this paper, we have expanded this concept to nearness rings. More precisely, we define quotient nearness rings using the concept of nearness, which brings a different approach to algebraic structures, and examine some of their properties.

2. Preliminaries

In this section, topics that will help the main topic of the paper will be given. Let $\mathcal{O}$ be a set of perceptual objects which are points definable by their characteristic. An objects description is defined by means of a tuple of function values $\Phi(x)$ associated with an object $x \in \mathcal{X} \subsetneqq \mathcal{O}$. The important thing to notice is the choice of functions $\varphi_i \in B$ used to describe any object of interest. Assume that $B \subseteq \mathcal{F}$ is a given set of functions representing features of sample object $\mathcal{X} \subset \mathcal{O}$. In combination, the function representing object features provide a basis for an object description $\Phi : \mathcal{O} \to \mathbb{R}^L$, $\Phi(x) = (\varphi_1(x), \varphi_2(x), ..., \varphi_L(x))$ a vector containing measurements (returned values) associated with each function values $\varphi_i(x)$, where the description length $|\Phi| = L$ (see [**16**]).

Definition 2.1. ([***16***]) *Let $\mathcal{O}$ be a set of perceptual objects, $\mathcal{F}$ be a set of the probe functions, $x, x' \in \mathcal{O}$, and $B \subseteq \mathcal{F}$.*

$$\sim_{B_r} = \{(x, x') \in \mathcal{O} \times \mathcal{O} \mid \triangle_{\varphi_i} = \mid \varphi_i(x) - \varphi_i(x') \mid = 0 \text{ for all } \varphi_i \in B\}$$

is called the indiscernibility relation on $\mathcal{O}$, where description length $1 \leqslant i \leqslant \mid \Phi \mid$.

Definition 2.2. ([**16**]) *Let $\mathcal{O}$ be a set of perceptual objects, $\mathcal{F}$ be a set of the probe functions, $X, X' \subseteq \mathcal{O}$, and $B \subseteq \mathcal{F}$. Then, X is called near X' if there exists $x \in X, x' \in X', \varphi_i \in B$ such that $x \sim_{\varphi_i} x'$.*

Definition 2.3. ([**6**]) *Let $\mathcal{O}$ be a set of perceptual objects, Φ be an object description and $A \subseteq \mathcal{O}$. Then, the set description of A is defined as*

$$Q(A) = \{\Phi(a) \mid a \in A\}.$$

Definition 2.4. ([**6**]) *Let $\mathcal{O}$ be a set of perceptual objects, A and B be any two subsets of $\mathcal{O}$. If $Q(A) \cap Q(B) \neq \emptyset$, then A is called descriptively near B and denoted by $A\delta_\Phi B$.*

Definition 2.5. ([**20**]) *Let $\mathcal{O}$ be a set of perceptual objects and A be a subset of $\mathcal{O}$. Then, the descriptive nearness collection $\xi_\Phi(A)$ is defined by*

$$\xi_\Phi(A) = \{B \in \mathcal{P}(\mathcal{O}) \mid A\delta_\Phi B\}$$

where $\mathcal{P}(\mathcal{O})$ is power set of $\mathcal{O}$.

Definition 2.6. ([**10**]) *Let $\mathcal{O}$ be a set of perceptual objects, $\mathcal{F}$ be a set of probe functions, $\sim_{B_r}$ be indiscernibility relation, and $N_r(B)$ be a collection of partitions. Then, $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B))$ is called a weak nearness approximation space.*

Theorem 2.1. ([**8**]) *Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B))$ be a weak nearness approximation space and $X, Y \subset \mathcal{O}$. Then, the following statements hold;*
i) $N_r(B)_* X \subseteq X \subseteq N_r(B)^* X$,
ii) $N_r(B)^* (X \cup Y) = (N_r(B)^* X) \cup (N_r(B)^* Y)$,
iii) $N_r(B)^* (X \cap Y) \subseteq (N_r(B)^* X) \cap (N_r(B)^* Y$,
iv) $X \subseteq Y$ implies $N_r(B)^* X \subseteq N_r(B)^* Y$.

Definition 2.7. ([**21**]) *Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B))$ be a weak nearness approximation space and $G \subseteq \mathcal{O}$, and "$\cdot$" be an operation by $\cdot : G \times G \to N_r(B)^* G$. G is called a group on $\mathcal{O}$, or shortly, nearness group if the following properties are satisfied:*
NG_1) $x \cdot y \in N_r(B)^* G$ for all $x, y \in G$,
NG_2) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ property holds in $N_r(B)^* G$ for all $x, y, z \in G$,
NG_3) There exists $e \in N_r(B)^* G$ such that $x \cdot e = x = e \cdot x$ for all $x \in G$,
NG_4) There exists $y \in G$ such that $x \cdot y = e = y \cdot x$ for all $x \in G$.

Definition 2.8. ([**21**]) *Let G be a group on weak nearness approximation space $\mathcal{O}$ and H be a non-empty subset of G. H is called a nearness subgroup of G if H is a nearness group relative to the operation in G and it is denoted by $H \preccurlyeq G$.*

Let G be a nearness group and H be a nearness subgroup of G. Let e_H denote the identity of H and e_G denote the identity of G. The identity elements of G and H are equal to each other, i.e., $e_H = e_G$.

Definition 2.9. ([**21**]) *Let $(G, \cdot)$ be a nearness group and $\sim_{B_r}$ be an indiscernibility relation on G. Then, $\sim_{B_r}$ is called a congruence indiscernibility relation on nearness group G, if $x \sim_{B_r} y$, where $x, y \in G$ implies $(x + a) \sim_{B_r} (y + a)$, $(a + x) \sim_{B_r} (a + y)$, and $x \cdot a \sim_{B_r} y \cdot a$, and $a \cdot x \sim_{B_r} a \cdot y$ for all $a \in G$.*

THEOREM 2.2. ([**21**]) *Let* G *be a group on weak nearness approximation space* $\mathcal{O}$ *and* H *be a non-empty subset of* G. *Then,* H *is a nearness subgroup of* G *if and only if*

i) $x \cdot y \in N_r(B)^* H$ *for all* $x, y \in H$,

ii) $x^{-1} \in H$ *for all* $x \in H$.

DEFINITION 2.10. ([**21**]) *Let* G *be a nearness group such that* $N_r(B)^*(N_r(B)^* G) = N_r(B)^* G$ *and* H *be a non-empty subset of* $N_r(B)^* G$. *If the following properties are hold, then* H *is called upper nearness subgroup of* G *and it is denoted by* $H \preccurlyeq G$.

i) $x \cdot y \in N_r(B)^* H$ *for all* $x, y \in H$,

ii) $x^{-1} \in H$ *for all* $x \in H$.

THEOREM 2.3. ([**21**]) *Let* G *be a nearness group and* H *be a nearness subgroup of* G *such that* $N_r(B)^*(N_r(B)^* H) = N_r(B)^* H$. *Then*

i) $N_r(B)^*(HH^{-1}) = N_r(B)^* H$.

ii) $N_r(B)^*(HH) = N_r(B)^* H$.

DEFINITION 2.11. ([**14**]) *Let* G *be a nearness group,* H *be a nearness subgroup of* G, *and* $a, b \in G$. *We say that* a *is right (resp. left) congruent to* b *nearness modulo* H, *denoted by* $a \cong_R b$ *(near-mod*H*) (resp.* $a \cong_L b$ *(near-mod*H*)) if* $a \cdot b^{-1} \in N_r(B)^* H$ *(resp.* $a^{-1} \cdot b \in N_r(B)^* H$*).*

THEOREM 2.4. ([**14**]) *Let* G *be a nearness group,* H *be a nearness subgroup of* G *such that* $N_r(B)^*(N_r(B)^* H) = N_r(B)^* H$, *and* $a, b \in G$. *If* $\sim_{B_r}$ *is an indiscernibility relation on* G, *then the relation* $a \cong_R b$ *(near-mod*H*) is an equivalence relation.*

DEFINITION 2.12. ([**14**]) *Let* G *be a nearness group and* $a \in G$. *If* H *is a nearness subgroup of* G *such that* $N_r(B)^*(N_r(B)^* H) = N_r(B)^* H$, *then* $[a]_R = \{x \in N_r(B)^* G \mid a \cong_R x \text{ (near-mod}H)\}$. *The set* $[a]_R$ *is called a right nearness equivalence class with respect to (near-mod* H*) (or* $\cong_R$*) determined by* a.

Similarly, the left nearness equivalence class is defined.

DEFINITION 2.13. ([**14**]) *Let* G *be a nearness group and* $a \in G$. *If* H *is a nearness subgroup of* G, *then* $(N_r(B)^* H)a = \{ha \in N_r(B)^* G \mid h \in N_r(B)^* H\}$. $(N_r(B)^* H)a$ *is called a right near-coset of* H *in* G.

Similarly, the left near-coset is defined.

THEOREM 2.5. ([**14**]) *Let* H *be a nearness subgroup of a nearness group* G *such that* $N_r(B)^*(N_r(B)^* H) = N_r(B)^* H$, $\sim_{B_r}$ *be a congruence indiscernibility relation on* G, *and* $a, b \in G$. *The following properties hold:*

i) If $(N_r(B)^* H)a = (N_r(B)^* H)b$, *then* $ab^{-1} \in N_r(B)^* H$,

ii) If $a(N_r(B)^* H) = b(N_r(B)^* H)$, *then* $a^{-1}b \in N_r(B)^* H$.

THEOREM 2.6. ([**14**]) *Let* G *be a nearness group such that* $N_r(B)^*(N_r(B)^* G) = N_r(B)^* G$, H *be a nearness subgroup of* G *such that* $N_r(B)^*(N_r(B)^* H) = N_r(B)^* H$. *If* $\sim_{B_r}$ *is a congruence indiscernibility relation on* G, *and* $a \in G$, *then* $(N_r(B)^* H)a = \{x \in N_r(B)^* G \mid a \cong_R x \text{ (near-mod}H)\}$.

DEFINITION 2.14. (*[**13**]*) *Let G be a nearness group. A nearness subgroup H of G is called a normal nearness subgroup of G if $((N_r(B)^* H)a = a(N_r(B)^* H)$ for all $a \in G$ and it is denoted by $H \trianglelefteq G$. If $H \neq G$, then it is denoted by $H \triangleleft G$.*

THEOREM 2.7. (*[**13**]*) *Let G be a nearness group such that $N_r(B)^* (N_r(B)^* G) = N_r(B)^* G$, H be a normal nearness subgroup of G such that $N_r(B)^* (N_r(B)^* H) = N_r(B)^* H$, and $\sim_{B_r}$be a congruence indiscernibility relation on G. If G/H is the set of all (left) near-cosets of H in G, then G/H is a nearness group under the operation given by $a(N_r(B)^* H)b(N_r(B)^* H) = ab(N_r(B)^* H)$ for all $a, b \in G$.*

DEFINITION 2.15. (*[**13**]*) *Let G be a nearness group such that $N_r(B)^* (N_r(B)^* G) = N_r(B)^* G$, H be a nearness subgroup of G such that $N_r(B)^* (N_r(B)^* H) = N_r(B)^* H$. The nearness group G/H is called the quotient nearness group or factor nearness group of G by H.*

DEFINITION 2.16. (*[**10**]*) *Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B))$ be a weak nearness approximation space and $R \subseteq \mathcal{O}$, and "$+$" and "$\cdot$" be operations by $+ : R \times R \to N_r(B)^* R$ and $\cdot : R \times R \to N_r(B)^* R$, respectively. R is called a ring on $\mathcal{O}$, or shortly, nearness ring if the following properties are satisfied:*

NR_1) $(R, +)$ is an abelian group on $\mathcal{O}$ with identity element 0_R,

NR_2) $(R, \cdot)$ is a semigroup on $\mathcal{O}$,

NR_3) For all $x, y, z \in R$,

$$x \cdot (y + z) = x \cdot y + x \cdot z \quad \textit{and} \ (x + y) \cdot z = x \cdot z + y \cdot z$$

properties hold in $N_r(B)^* R$.

If in addition:

NR_4) If $x \cdot y = y \cdot x$ for all $x, y \in R$, then the R is said to be a commutative nearness ring.

NR_5) If $N_r(B)^* R$ contains an element 1_R such that $x \cdot 1_R = x = 1_R \cdot x$ for all $x \in R$, then R is said to be a nearness ring with identity.

DEFINITION 2.17. (*[**10**]*) *Let $(R, +, \cdot)$ be a ring on $\mathcal{O}$, where $\mathcal{O}$ is a weak nearness approximation space, and S be a non-empty subset of R. S is called subnearness ring of R if S is a nearness ring with binary operations "$+$" and "$\cdot$"on nearness ring R.*

DEFINITION 2.18. (*[**10**]*) *Let $(R, +, \cdot)$ be a nearness ring and I be a non-empty subset of R. I is called a left (right) nearness ideal of R provided $x - y \in N_r(B)^* I$, $r \cdot x \in N_r(B)^* I$ ($x - y \in N_r(B)^* I$, $x \cdot r \in N_r(B)^* I$) for all $x, y \in I$ and for all $r \in R$, respectively. A non-empty set I of a nearness ring R is called a nearness ideal of R if I is both a left and a right nearness ideal of R.*

DEFINITION 2.19. (*[**10**]*) *Let $(R, +, \cdot)$ be a nearness ring and $\sim_{B_r}$be an indiscernibility relation on R. Then, $\sim_{B_r}$ is called a congruence indiscernibility relation on nearness ring R, if $x \sim_{B_r} y$, where $x, y \in R$ implies $(x + r) \sim_{B_r} (y + r)$, $(r + x) \sim_{B_r} (r + y)$, and $x \cdot r \sim_{B_r} y \cdot r$, and $r \cdot x \sim_{B_r} r \cdot y$ for all $r \in R$.*

Lemma 2.1. ([**10**]) *Let $(R,+,\cdot)$ be a nearness ring. If $\sim_{B_r}$is a congruence indiscernibility relation on R, then $[x]_{B_r}+[y]_{B_r}\subseteq[x+y]_{B_r}$, and $[x]_{B_r}\cdot[y]_{B_r}\subseteq[x\cdot y]_{B_r}$ for all $x,y\in R$.*

Let $(R,+,\cdot)$ be a nearness ring. Let $X+Y=\{x+y\mid x\in X,\text{ and }y\in Y\}$ and $X\cdot Y=\{\sum\limits_{finite}x_iy_i\mid x_i\in X\ ,\text{ and }y_i\in Y\}$, where X and Y are subsets of R.

Lemma 2.2. ([**10**]) *Let $(R,+,\cdot)$ be a nearness ring and $\sim_{B_r}$be a congruence indiscernibility relation on R. The following properties hold:*

i) If $X,Y\subseteq R$, then $(N_r(B)^*X)+(N_r(B)^*Y)\subseteq N_r(B)^*(X+Y)$,

ii) If $X,Y\subseteq R$, then $(N_r(B)^*X)\cdot(N_r(B)^*Y)\subseteq N_r(B)^*(X\cdot Y)$.

Definition 2.20. ([**10**]) *Let R be a nearness ring and A,B and P be nearness ideals of R. P is called a prime nearness ideal of R if $AB\subseteq N_r(B)^*P$ implies that either $A\subseteq N_r(B)^*P$ or $B\subseteq N_r(B)^*P$.*

In the other words, let R be nearness ring and P be nearness ideal of R. P is called a prime nearness ideal of R if $ab\in N_r(B)^*P$ implies that either $a\in N_r(B)^*P$ or $b\in N_r(B)^*P$ for any $a,b\in R$.

3. Quotient nearness rings

The role played by normal nearness subgroups in the nearness group theory is played by the nearness ideals in the theory of the nearness rings. Let R be a nearness ring such that $N_r(B)^*\left(N_r(B)^*R\right)=N_r(B)^*R$, I be a nearness ideal of R such that $N_r(B)^*\left(N_r(B)^*I\right)=N_r(B)^*I$, and $\sim_{B_r}$ be a congruence indiscernibility relation on R. Then, $(R,+)$ is the commutative nearness group. Since every nearness subgroup of a commutative nearness group is a normal nearness group, $(I,+)$ is a normal nearness subgroup of $(R,+)$. Thus the quotient (factor) nearness group $(R/I,\oplus)$ is defined by Theorem 2.7.

Now, we can give the following theorem.

Theorem 3.1. *Let R be a nearness ring such that $N_r(B)^*\left(N_r(B)^*R\right)=N_r(B)^*R$, I be a nearness ideal of R such that $N_r(B)^*\left(N_r(B)^*I\right)=N_r(B)^*I$, $\sim_{B_r}$ be a congruence indiscernibility relation on R and $a,b\in R$. Then,*

i) $\left(N_r(B)^*I\right)+a=\left(N_r(B)^*I\right)+b$ if and only if $a-b\in N_r(B)^*I$,

ii) $a+\left(N_r(B)^*I\right)=b+\left(N_r(B)^*I\right)$ if and only if $b-a\in N_r(B)^*I$,

iii) "$a\cong b\ (near\text{-}\operatorname{mod}I)\Leftrightarrow a-b\in N_r(B)^*I$" is a nearness equivalence relation.

Proof. *i)* $(\Rightarrow)$ The proof is similar to Theorem 2.5.
$(\Leftarrow)$ Let $a-b\in N_r(B)^*I$. Then $[a-b]_{B_r}\cap I\neq\emptyset$. In this case, there exists x such that $x\in[a-b]_{B_r}$ and $x\in I$. Therefore, $x\sim_{B_r}a-b$, $x\in I$. Since $\sim_{B_r}$ is a congruence indiscernibility relation on R and I is a commutative nearness subgroup of R, we have $a\sim_{B_r}x+b$, $x\in I$. On the other hand, $y\in\left(N_r(B)^*I\right)+a$ implies

$y = z + a$, $z \in N_r(B)^* I$. Thus, there exists w such that $w \in [z]_{B_r}$ and $w \in I$. Hence, we have $z \sim_{B_r} w$, $w \in I$. Since $\sim_{B_r}$ is a congruence indiscernibility relation, we get that $z + a \sim_{B_r} w + a$, $w \in I$. Thus, $y \sim_{B_r} w + a$, $w \in I \Longrightarrow y - w \sim_{B_r} a$, $-w \in I \Longrightarrow y - w \sim_{B_r} x + b$; $-w, x \in I$. Therefore, since $\sim_{B_r}$ is a conruence indiscernibility relation and I is a commutative nearness subgroup of R, we obtain $y - b \sim_{B_r} x + w$, $x + w \in N_r(B)^* I$, and so $x + w \in [y - b]_{B_r}$ and $x + w \in N_r(B)^* I$, e.i., $[y - b]_{B_r} \cap N_r(B)^* I \neq \emptyset$. Hence, we have $y - b \in N_r(B)^* I$, and so there exists $x \in N_r(B)^* I$ such that $y - b = x$, i.e., $y = x + b \in \left(N_r(B)^* I\right) + b$. In this case, we obtain $\left(N_r(B)^* I\right) + a \subseteq \left(N_r(B)^* I\right) + b$. Similarly, $\left(N_r(B)^* I\right) + b \subseteq \left(N_r(B)^* I\right) + a$ is obtained.
ii) The proof is similar to (*i*).
iii) The proof is similar to Theorem 2.4. □

LEMMA 3.1. *Let R be a nearness ring such that $N_r(B)^* \left(N_r(B)^* R\right) = N_r(B)^* R$, I be a nearness ideal of R such that $N_r(B)^* \left(N_r(B)^* I\right) = N_r(B)^* I$, $\sim_{B_r}$ be a congruence indiscernibility relation on R and $a, b, c, d \in R$. If $a \cong b$ (near-$\bmod I$) and $c \cong d$ (near-$\bmod I$), then $a + c \cong b + d$ (near-$\bmod I$) and $ac \cong bd$ (near-$\bmod I$).*

PROOF. Let $a \cong b$ (*near*-$\bmod I$) and $c \cong d$ (*near*-$\bmod I$). Thus we get $a - b \in N_r(B)^* I$ and $c - d \in N_r(B)^* I$ by Theorem 3.1 (*iii*). If $a - b \in N_r(B)^* I$, then $[a - b]_{B_r} \cap I \neq \emptyset$, and so there exists n such that $n \in [a - b]_{B_r}$ and $n \in I$. Then, since $n \sim_{B_r} a - b$, $n \in I$, $a \sim n + b$, $n \in I$. On the other hand, if $c - d \in N_r(B)^* I$, then $[c - d]_{B_r} \cap I \neq \emptyset$. In this case, there exists m such that $m \in [c - d]_{B_r}$, $m \in I$. Hence $m \sim_{B_r} c - d$, $m \in I$ and since $\sim_{B_r}$is a congruence indiscernibility relation and I is a commutative nearness subgroup of R, we obtain $c \sim_{B_r} m + d$, $m \in I$. Therefore, we get $(a + c) - (b + d) \sim_{B_r} n + m$, $n + m \in N_r(B)^* I$. Since $n + m \in [(a + c) - (b + d)]_{B_r}$, we get $[(a + c) - (b + d)]_{B_r} \cap N_r(B)^* I \neq \emptyset$. That is, $a + c - (b + d) \in N_r(B)^* I$. From Theorem 3.1 (*iii*), we obtain $a + c \cong b + d$ (*near*-$\bmod I$).

From $ac \sim_{B_r} (n + b)(m + d)$, $n, m \in I$, we have $ac \sim_{B_r} bd + nd + bm + nm$. Since I is a nearness ideal of R such that $N_r(B)^* \left(N_r(B)^* I\right) = N_r(B)^* I$, $nd + bm + nm \in N_r(B)^* I$. Thus, $ac - bd \sim_{B_r} nd + bm + nm$, $nd + bm + nm \in N_r(B)^* I$. Hence, we get $[ac - bd] \cap N_r(B)^* I \neq \emptyset$. That is, $ac - bd \in N_r(B)^* I$. From Theorem 3.1 (*iii*), we get $ac \cong bd$ (*near*-$\bmod I$). □

REMARK 3.1. Assume that $m \in N_r(B)^* I$. We have $[m]_{B_r} \cap I \neq \emptyset$. Let us take $x \in [m]_{B_r}$, $x \in I$. Thus, $m \sim_{B_r} x$. Since $\sim_{B_r}$ is a congruence indiscernibility relation on R, we have $-m \sim_{B_r} -x$, $-x \in I$. Therefore, we have $[-m] \cap I \neq \emptyset$. That is, $-m \in N_r(B)^* I$.

Let I be a nearness ideal of the nearness ring R such that $N_r(B)^*\left(N_r(B)^* I\right) = N_r(B)^* I$ and $a \in N_r(B)^* R$. In this case,

$$\begin{aligned}
[a]_I &= \left\{b \in N_r(B)^* R \mid\ a \cong b\,(near\text{-}\operatorname{mod} I)\right\} \\
&= \left\{b \in N_r(B)^* R \mid\ a - b \in N_r(B)^* I\right\} \\
&= \left\{b \in N_r(B)^* R \mid\ a - b = m,\, m \in N_r(B)^* I\right\} \\
&= \left\{b \in N_r(B)^* R \mid\ b = a + n,\ -m = n,\, n \in N_r(B)^* I\right\} \\
&= a + N_r(B)^* I.
\end{aligned}$$

Thus, $a + N_r(B)^* I = \left\{a + n \mid\ n \in N_r(B)^* I\right\}$ is called the nearness equivalence class of element a with respect to $(near\text{-}\operatorname{mod} I)$. Thus, we have

$$R/I = \left\{a + N_r(B)^* I \mid\ a \in R\right\}.$$

Moreover, from Theorem 3.1 and Lemma 3.1, we have the following operations "$\oplus$" and "$\odot$" are well-defined.

$$\left(a + N_r(B)^* I\right) \oplus \left(b + N_r(B)^* I\right) = (a+b) + N_r(B)^* I$$
$$\left(a + N_r(B)^* I\right) \odot \left(b + N_r(B)^* I\right) = (ab) + N_r(B)^* I$$

for all $a, b \in R$. It is similar to the proof of Theorem 2.7, $(R/I, \oplus, \odot)$ is a nearness ring. Therefore, we can give the following theorem without proof.

Theorem 3.2. *Let R be a nearness ring such that $N_r(B)^*\left(N_r(B)^* R\right) = N_r(B)^* R$, I be a nearness ideal of R such that $N_r(B)^*\left(N_r(B)^* I\right) = N_r(B)^* I$, $\sim_{B_r}$ be a congruence indiscernibility relation on R. If R/I is the set of all nearness equivalence classes respect to near-*$\operatorname{mod} I$*, then R/I is a nearness ring under the operations given by*

$$\left(a + N_r(B)^* I\right) \oplus \left(b + N_r(B)^* I\right) = (a+b) + N_r(B)^* I$$
$$\left(a + N_r(B)^* I\right) \odot \left(b + N_r(B)^* I\right) = (ab) + N_r(B)^* I$$

for all $a, b \in R$.

Definition 3.1. *Let R be a nearness ring such that $N_r(B)^*\left(N_r(B)^* R\right) = N_r(B)^* R$, I be a nearness ideal of R such that $N_r(B)^*\left(N_r(B)^* I\right) = N_r(B)^* I$, $\sim_{B_r}$ be a congruence indiscernibility relation on R. The nearness ring R/I is called the quotient (factor) nearness ring of R by I.*

If R is a nearness ring with identity such that $N_r(B)^*\left(N_r(B)^* R\right) = N_r(B)^* R$, then the identity element of the quotient nearness ring R/I is $[1_R] = 1_R + N_r(B)^* I$ and the zero element is $[0_R] = 0_R + N_r(B)^* I = N_r(B)^* I$. Furthermore, if R is commutative nearness ring, so is R/I.

Definition 3.2. *Let $\mathcal{O}$ be a set of perceptual objects, I be a nearness ideal of a nearness ring R, R/I be a set of all nearness equivalence classes respect to near-*$\operatorname{mod} I$*, $\xi_\Phi(A)$ be a descriptive nearness collection, and $A \in P(\mathcal{O})$, where $P(\mathcal{O})$ is the power set of $\mathcal{O}$. Then,*

$$N_r(B)^*(R/I) = \bigcup_{\xi_\Phi(A) \cap R/I \neq \emptyset} \xi_{\Phi(A)}$$

is called upper approximation of R/I.

EXAMPLE 3.1. *Let $\mathcal{O} = \{0, 1, a, b, c, d, e, f, g, h, i, j, k, l, m, n\}$ be set of perceptual objects, where*

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
$$c = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$
$$g = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad j = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$
$$k = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad l = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad m = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad n = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

for $U = \{(a_{ij}) : a_{ij} \in \mathbb{Z}_2\}$, $B = \{\Psi_1, \Psi_2, \Psi_3\} \subseteq \mathcal{F}$ be a set of probe functions, $R = \{d, k, n\}$ be a subset of $\mathcal{O}$ and $r = 1$. Values of the probe functions

$$\begin{aligned} &\Psi_1 : \mathcal{O} \to V_1 = \{\beta_1, \beta_2, \beta_3, \beta_5, \beta_7\}, \\ &\Psi_2 : \mathcal{O} \to V_2 = \{\beta_1, \beta_2, \beta_4, \beta_5, \beta_6\}, \\ &\Psi_3 : \mathcal{O} \to V_3 = \{\beta_1, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7\} \end{aligned}$$

are given in the table:

	0	1	a	b	c	d	e	f	g	h	i	j	k	l	m	n
Ψ_1	β_3	β_3	β_5	β_3	β_7	β_3	β_2	β_2	β_1	β_1	β_7	β_1	β_3	β_2	β_7	β_1
Ψ_2	β_2	β_1	β_4	β_1	β_5	β_2	β_4	β_6	β_1	β_2	β_6	β_2	β_1	β_6	β_5	β_2
Ψ_3	β_6	β_6	β_3	β_1	β_4	β_1	β_5	β_5	β_6	β_6	β_7	β_1	β_6	β_3	β_4	β_1

In this case, let us find the nearness equivalence classes of $\mathcal{O}$ according to the relationship $\sim_{B_r}$.

$$\begin{aligned} [0]_{\Psi_1} &= \{x \in \mathcal{O} \mid \Psi_1(x) = \Psi_1(0) = \beta_3\} = \{0, 1, b, d, k\} \\ &= [1]_{\Psi_1} = [b]_{\Psi_1} = [d]_{\Psi_1} = [k]_{\Psi_1}, \\ [a]_{\Psi_1} &= \{x \in \mathcal{O} \mid \Psi_1(x) = \Psi_1(a) = \beta_5\} = \{a\}, \\ [c]_{\Psi_1} &= \{x \in \mathcal{O} \mid \Psi_1(x) = \Psi_1(c) = \beta_7\} = \{c, i, m\} \\ &= [i]_{\Psi_1} = [m]_{\Psi_1}, \\ [e]_{\Psi_1} &= \{x \in \mathcal{O} \mid \Psi_1(x) = \Psi_1(e) = \beta_2\} = \{e, f, l\} \\ &= [f]_{\Psi_1} = [l]_{\Psi_1}, \\ [g]_{\Psi_1} &= \{x \in \mathcal{O} \mid \Psi_1(x) = \Psi_1(g) = \beta_1\} = \{g, h, j, n\} \\ &= [h]_{\Psi_1} = [j]_{\Psi_1} = [n]_{\Psi_1}. \end{aligned}$$

Then we obtain $\xi_{\Psi_1} = \left\{[0]_{\Psi_1}, [a]_{\Psi_1}, [c]_{\Psi_1}, [e]_{\Psi_1}, [g]_{\Psi_1}\right\}$.

$$\begin{aligned}
[0]_{\Psi_2} &= \{x \in \mathcal{O} \mid \Psi_2(x) = \Psi_2(0) = \beta_2\} = \{0, d, h, j, n\} \\
&= [d]_{\Psi_2} = [h]_{\Psi_2} = [j]_{\Psi_2} = [n]_{\Psi_2}, \\
[1]_{\Psi_2} &= \{x \in \mathcal{O} \mid \Psi_2(x) = \Psi_2(1) = \beta_1\} = \{1, b, g, k\} \\
&= [b]_{\Psi_2} = [g]_{\Psi_2} = [k]_{\Psi_2}, \\
[a]_{\Psi_2} &= \{x \in \mathcal{O} \mid \Psi_2(x) = \Psi_2(a) = \beta_4\} = \{a, e\} \\
&= [e]_{\Psi_2}, \\
[c]_{\Psi_2} &= \{x \in \mathcal{O} \mid \Psi_2(x) = \Psi_2(c) = \beta_5\} = \{c, m\} \\
&= [m]_{\Psi_2}, \\
[f]_{\Psi_2} &= \{x \in \mathcal{O} \mid \Psi_2(x) = \Psi_2(f) = \beta_6\} = \{f, i, l\} \\
&= [i]_{\Psi_2} = [l]_{\Psi_2},
\end{aligned}$$

Thus we get $\xi_{\Psi_2} = \left\{[0]_{\Psi_2}, [1]_{\Psi_2}, [a]_{\Psi_2}, [c]_{\Psi_2}, [f]_{\Psi_2}\right\}$. *Also, we can write*

$$\begin{aligned}
[0]_{\Psi_3} &= \{x \in \mathcal{O} \mid \Psi_3(x) = \Psi_3(0) = \beta_6\} = \{0, 1, g, h, k\} \\
&= [1]_{\Psi_3} = [g]_{\Psi_3} = [h]_{\Psi_3} = [k]_{\Psi_3}, \\
[a]_{\Psi_3} &= \{x \in \mathcal{O} \mid \Psi_3(x) = \Psi_3(a) = \beta_3\} = \{a, l\} \\
&= [l]_{\Psi_3}, \\
[b]_{\Psi_3} &= \{x \in \mathcal{O} \mid \Psi_3(x) = \Psi_3(b) = \beta_1\} = \{b, d, j, n\} \\
&= [d]_{\Psi_3} = [j]_{\Psi_3} = [n]_{\Psi_3}, \\
[c]_{\Psi_3} &= \{x \in \mathcal{O} \mid \Psi_3(x) = \Psi_3(c) = \beta_4\} = \{c, m\} \\
&= [m]_{\Psi_3}, \\
[e]_{\Psi_3} &= \{x \in \mathcal{O} \mid \Psi_3(x) = \Psi_3(e) = \beta_5\} = \{e, f\} \\
&= [f]_{\Psi_3}, \\
[i]_{\Psi_3} &= \{x \in \mathcal{O} \mid \Psi_3(x) = \Psi_3(i) = \beta_7\} = \{i\}.
\end{aligned}$$

Thus, we get $\xi_{\Psi_3} = \left\{[0]_{\Psi_3}, [a]_{\Psi_3}, [b]_{\Psi_3}, [c]_{\Psi_3}, [e]_{\Psi_3}, [i]_{\Psi_3}\right\}$. *Hence, for* $r = 1,$ *a set of partitions of* $\mathcal{O}$ *is* $N_1(B) = \{\xi_{\Psi_1}, \xi_{\Psi_2}, \xi_{\Psi_3}\}$.

$$\begin{aligned}
N_1(B)^* R &= \bigcup_{[x]_{\Psi_i} \cap R \neq \varnothing} [x]_{\Psi_i} \\
&= \{0, 1, b, d, g, h, j, k, n\}.
\end{aligned}$$

Considering the following table of operation:

$+$	d	k	n
d	0	n	k
k	n	0	d
n	k	d	0

Thus, $(R,+)$ is a commutative nearness group. Now, let the following table of operation

$\cdot$	d	k	n
d	0	h	h
k	g	1	h
n	g	g	0

In this case, $(R,+,\cdot)$ is a nearness ring such that $N_1(B)^*\left(N_1(B)^*R\right)=N_1(B)^*R$.

Let $I=\{k,n\}$ be a subset of R. Thus, I is a nearness ideal of R such that $N_1(B)^*\left(N_1(B)^*I\right)=N_1(B)^*I=\{0,1,b,d,k,j,g,h,n\}$. From Definition 2.13, we have $[x]_I=\left(N_1(B)^*I\right)+x=\{y+x\in N_1(B)^*R\mid y\in N_1(B)^*I\}$. We can compute the all nearness equivalence classes (respect to near-mod I) of R by I, and so we get $[d]_I=[n]_I=\{0,d,k,n\}$ and $[k]_I=\{0,1,b,d,g,j,k,n\}$. Thus, we have $R/I=\{[d]_I,[k]_I\}$. Moreover, although $0_R\notin R$, $[0]_I=N_1(B)^*I$. Now, let's get the set of $N_1(B)^*(R/I)$.

$$\begin{aligned}\mathcal{Q}(R/I)&=\{\Phi(A)\mid A\in R/I\}\\&=\{\Phi([d]_I),\Phi([k]_I)\}\\&=\{\{\Phi(0),\Phi(d),\Phi(k),\Phi(n)\},\\&\qquad\{\Phi(0),\Phi(1),\Phi(b),\Phi(d),\Phi(g),\Phi(j),\Phi(k),\Phi(n)\}\},\\&=\{\{(\beta_3,\beta_2,\beta_6),(\beta_3,\beta_2,\beta_1),(\beta_3,\beta_1,\beta_6),(\beta_1,\beta_2,\beta_1)\},\\&\qquad\{(\beta_3,\beta_2,\beta_6),(\beta_3,\beta_1,\beta_6),(\beta_3,\beta_1,\beta_1),(\beta_3,\beta_2,\beta_1),(\beta_1,\beta_1,\beta_6),\\&\qquad(\beta_1,\beta_2,\beta_1),(\beta_3,\beta_1,\beta_6),(\beta_1,\beta_2,\beta_1)\}\}.\end{aligned}$$

For $[k]_I\in R/I$, we get that

$$\begin{aligned}\mathcal{Q}([k]_I)&=\{\Phi(0),\Phi(1),\Phi(b),\Phi(d),\Phi(g),\Phi(j),\Phi(k),\Phi(n)\}\\&=(\beta_3,\beta_2,\beta_6),(\beta_3,\beta_1,\beta_6),(\beta_3,\beta_1,\beta_1),(\beta_3,\beta_2,\beta_1),(\beta_1,\beta_1,\beta_6),\\&\qquad(\beta_1,\beta_2,\beta_1),(\beta_3,\beta_1,\beta_6),(\beta_1,\beta_2,\beta_1)\}.\end{aligned}$$

Since $\mathcal{Q}([k]_I)\cap\mathcal{Q}([k]_I)\neq\emptyset$, it follows that $[k]_I\in\xi_\Phi([k]_I)$. Therefore, $\mathcal{Q}([k]_I)\cap R/I\neq\emptyset$ and $[k]_I\in N_1(B)^*(R/I)$ by Definition 3.2.
For $[d]_I,[n]_I\in R/I$, we get that $[d]_I=[n]_I=\{k,d,n,0\}$. Thus, we have

$$\begin{aligned}\mathcal{Q}([d]_I)&=\{\Phi(0),\Phi(d),\Phi(k),\Phi(n)\}\\&=\{(\beta_3,\beta_2,\beta_6),(\beta_3,\beta_2,\beta_1),(\beta_3,\beta_1,\beta_6),(\beta_1,\beta_2,\beta_1)\}\end{aligned}$$

Since $\mathcal{Q}([d]_I) \cap \mathcal{Q}([d]_I) \neq \emptyset$, *it follows that* $[d]_I \in \xi_\Phi([d]_I)$. *Therefore,* $\mathcal{Q}([d]_I) \cap R/I \neq \emptyset$ *and* $[d]_I \in N_1(B)^*(R/I)$ *by Definition 3.2. If we calculate the other nearness equivalence classes (respect to near-*mod I*) of* R *by* I*, then we get*
$N_1(B)^*(R/I) = \{[0]_I, [1]_I, [b]_I, [d]_I, [g]_I, [h]_I, [j]_I, [k]_I\}$ *by Definition 3.2.*

Considering the following the tables of operations

$\oplus$	$[d]_I$	$[k]_I$
$[d]_I$	$[0]_I$	$[d]_I$
$[k]_I$	$[d]_I$	$[0]_I$

$\odot$	$[d]_I$	$[k]_I$
$[d]_I$	$[0]_I$	$[h]_I$
$[k]_I$	$[g]_I$	$[1]_I$

Then, $(R/I, \oplus, \odot)$ *is a nearness ring, i.e.,* R/I *is a quotient (factor) nearness ring of* R *by* I*.*

LEMMA 3.2. *Let* R *be a nearness ring with identity such that*
$N_r(B)^*\left(N_r(B)^* R\right) = N_r(B)^* R$, I *be a nearness ideal of* R *such that*
$N_r(B)^*\left(N_r(B)^* I\right) = N_r(B)^* I$, $\sim_{B_r}$ *be a congruence indiscernibility relation on* R*. If* $1_R \in N_r(B)^* I$*, then* $N_r(B)^* I = N_r(B)^* R$*.*

PROOF. Since I is a nearness ideal of R, we have $I \subseteq R$, and so $N_r(B)^* I \subseteq N_r(B)^* R$ by Theorem 2.1.

Now, let $x \in N_r(B)^* R$. Then, $[x]_I \cap R \neq \emptyset$. Let us take $r \in R$ and $r \in [x]_I$. On the other words, if $1_R \in N_r(B)^* I$, then $[1_R]_I \cap I \neq \emptyset$. Assume that $a \in [1_R]$ and $a \in I$. Since I is a nearness ideal of R, we get $ar \in N_r(B)^* I$ and also, from hypothesis, $ar \in [1_R]_I [x]_I \subseteq [1_R x]_I = [x]_I$ by Lemma 2.1. Hence we obtain $[x]_I \cap N_r(B)^* I \neq \emptyset$. Thus, $x \in N_r(B)^*\left(N_r(B)^* I\right) = N_r(B)^* I$. Therefore, $N_r(B)^* R \subseteq N_r(B)^* I$. □

DEFINITION 3.3. *Let* $R \neq \{0_R\}$ *be a nearness ring.* R *is called a nearness integral domain if the commutative nearness ring with identity, where* $x \neq 0_R$ *and* $y \neq 0_R$ *imply* $xy \neq 0_R$ *for all* $x, y \in R$*.*

On the other words, the nearness integral domain may also be defined as follows: A zero divisor of a commutative nearness ring R is an element $x \neq 0_R$ of R such that $xy = 0_R$ for some $0_R \neq y \in R$. A commutative nearness ring $R \neq \{0_R\}$ is integral domain if and only if R has no zero divisor.

THEOREM 3.3. *Let* $R \neq \{0_R\}$ *be a commutative nearness ring such that*
$N_r(B)^*\left(N_r(B)^* R\right) = N_r(B)^* R$, P *be a nearness ideal of* R *such that*
$N_r(B)^*\left(N_r(B)^* P\right) = N_r(B)^* P$, $\sim_{B_r}$ *be a congruence indiscernibility relation on* R*. Then,* P *is a prime nearness ideal if and only if* R/I *is a nearness integral domain.*

PROOF. Suppose that P is prime nearness ideal of R. Since P is nearness ideal, R/P is a nearness ring by Theorem 3.2. Also, since R is a commutative, R/P is commutative. Therefore, we must show that $[x]_P = [0]_P$ or $[y]_P = [0]_P$ when $[x]_P [y]_P = [0]_P$ for some $[x]_P, [y]_P \in R/P$. Let $[x]_P = x + N_r(B)^* P$ and $[y]_P = y+$

$N_r(B)^* P$, $x, y \in R$. If $[x]_P [y]_P = [0]_P$, then $\left(x + N_r(B)^* P\right)\left(y + N_r(B)^* P\right) = N_r(B)^* P$. From here, since $xy + N_r(B)^* P = N_r(B)^* P$, we have $xy \in N_r(B)^* P$. By hypothesis, $x \in N_r(B)^* P$ or $y \in N_r(B)^* P$. That is, $[x]_P = [0]_P$ or $[y]_P = [0]_P$.

Conversely, let R/P be a nearness integral domain. Then, $P \neq R$ is a nearness ideal of R. Now, let $xy \in N_r(B)^* P$ for $x, y \in R$. Hence, from Theorem 3.1, we have $xy + N_r(B)^* P = N_r(B)^* P$. From this, we get that $[x]_P [y]_P = [0]_P$ for some $[x]_P, [y]_P \in R/P$. Since R/P is commutative integral domain, $[x]_P = [0]_P$ or $[y]_P = [0]_P$. That is, $x + N_r(B)^* P = N_r(B)^* P$ or $y + N_r(B)^* P = N_r(B)^* P$. Thus, we get $x \in N_r(B)^* P$ or $y \in N_r(B)^* P$ by Theorem 3.1. Thus, P is a prime nearness ideal. □

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