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IRREDUCIBILITY AND RIGIDITY IN DIGITAL IMAGES

Laurence Boxer

Abstract. We study how the properties of irreducibility and rigidity in digital images interact with Cartesian products, wedges, and cold and freezing sets.

1. Introduction

The properties of irreducibility and rigidity in digital images were introduced in [**13**] and have been studied in subsequent papers, including [**5**, **7**, **8**, **10**]. In the current work, we study implications of these properties for Cartesian products, wedges, and cold and freezing sets.

2. Preliminaries

We use $\mathbb{N}$ for the set of natural numbers, $\mathbb{N}^* = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ for the set of integers, and $\#X$ for the number of distinct members of X.

We typically denote a (binary) digital image as (X, κ), where $X \subset \mathbb{Z}^n$ for some $n \in \mathbb{N}$ and κ represents an adjacency relation of pairs of points in X. Thus, (X, κ) is a graph, in which members of X may be thought of as black points, and members of $\mathbb{Z}^n \setminus X$ as white points, of a picture of some "real world" object or scene.

2.1. Adjacencies. This section is largely quoted or paraphrased from [**6**].

Let $u, n \in \mathbb{N}$, $1 \leqslant u \leqslant n$. For $X \subset \mathbb{Z}^n$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in X$ are c_u-adjacent if and only if

- $x \neq y$, and
- for at most u indices i, $\mid x_i - y_i \mid = 1$, and
- for all indices j such that $\mid x_j - y_j \mid \neq 1$, we have $x_j = y_j$.

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The c_u adjacencies are the adjacencies most used in digital topology, especially c_1 and c_n.

In low dimensions, it is also common to denote a c_u adjacency by the number of points that can have this adjacency with a given point in $\mathbb{Z}^n$. E.g.,

- in $\mathbb{Z}$, c_1-adjacency is 2-adjacency;
- in $\mathbb{Z}^2$, c_1-adjacency is 4-adjacency and c_2-adjacency is 8-adjacency;
- in $\mathbb{Z}^3$, c_1-adjacency is 6-adjacency, c_2-adjacency is 18-adjacency, and c_3-adjacency is 26-adjacency.

We use the notations $y \leftrightarrow_\kappa x$, or, when the adjacency κ can be assumed, $y \leftrightarrow x$, to mean x and y are κ-adjacent. The notations $y \leftrightarroweq_\kappa x$, or, when κ can be assumed, $y \leftrightarroweq x$, mean either $y = x$ or $y \leftrightarrow_\kappa x$.

A sequence $P = \{y_i\}_{i=0}^m$ in a digital image (X, κ) is a *κ-path from $a \in X$ to $b \in X$* if $a = y_0$, $b = y_m$, and $y_i \leftrightarroweq_\kappa y_{i+1}$ for $0 \leqslant i < m$.

X is *κ-connected* [**16**], or *connected* when κ is understood, if for every pair of points $a, b \in X$ there exists a κ-path in X from a to b.

A *(digital) κ-closed curve* is a path $S = \{s_i\}_{i=0}^{m-1}$ such that $s_0 \leftrightarrow_\kappa s_{m-1}$, and $i \neq j$ implies $s_i \neq s_j$. If also $0 \leqslant i < m$ implies the only κ-adjacent members of S to x_i are $x_{(i-1) \bmod m}$ and $x_{(i+1) \bmod m}$, then S is a *(digital) κ-simple closed curve.*

2.2. Digitally continuous functions. This section is largely quoted or paraphrased from [**6**].

Digital continuity is defined to preserve connectedness, as at Definition 2.1 below. By using adjacency as our standard of "closeness," we get Theorem 2.1 below.

DEFINITION 2.1. [**2**] (generalizing a definition of [**16**]) *Let (X, κ) and (Y, λ) be digital images. A function $f : X \to Y$ is* (κ, λ)-continuous *if for every κ-connected $A \subset X$ we have that $f(A)$ is a λ-connected subset of Y.*

When $X \cup Y \subset (Z^n, \kappa)$, we use the abbreviation *κ-continuous* for *(κ, κ)-continuous.*

When the adjacency relations are understood, we will simply say that f is *continuous.* Continuity can be expressed in terms of adjacency of points:

THEOREM 2.1. [**2, 16**] *A function $f : X \to Y$ is continuous if and only if $x \leftrightarrow x'$ in X implies $f(x) \leftrightarroweq f(x')$.*

See also [**11**,**12**], where similar notions are referred to as *immersions*, *gradually varied operators*, and *gradually varied mappings*.

A digital *isomorphism* (called *homeomorphism* in [**1**]) is a (κ, λ)-continuous surjection $f : X \to Y$ such that $f^{-1} : Y \to X$ is (λ, κ)-continuous.

A *digital interval* is a set denoted $[a, b]_{\mathbb{Z}}$ where $a, b \in \mathbb{Z}$, $a \leqslant b$, and

$$[a, b]_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid a \leqslant z \leqslant b\}$$

with the c_1 adjacency in $\mathbb{Z}$.

Let $X \subset \mathbb{Z}^n$. The *boundary of X* [**15**] is

$$Bd(X) = \{x \in X \mid \text{there exists } y \in \mathbb{Z}^n \setminus X \text{ such that } y \leftrightarrow_{c_1} x\}.$$

A homotopy between continuous functions may be thought of as a continuous deformation of one of the functions into the other over a finite time period.

DEFINITION 2.2. ([**2**]; see also [**14**]) *Let X and Y be digital images. Let $f, g : X \to Y$ be (κ, κ')-continuous functions. Suppose there is a positive integer m and a function $F : X \times [0, m]_{\mathbb{Z}} \to Y$ such that*

- *for all $x \in X$, $F(x, 0) = f(x)$ and $F(x, m) = g(x)$;*
- *for all $x \in X$, the induced function $F_x : [0, m]_{\mathbb{Z}} \to Y$ defined by*

$$F_x(t) \;=\; F(x, t) \text{ for all } t \in [0, m]_{\mathbb{Z}}$$

  is $(2, \kappa')-$continuous. Thus, $\{F_x(t)\}_{t=0}^{m}$ is a path in Y.
- *for all $t \in [0, m]_{\mathbb{Z}}$, the induced function $F_t : X \to Y$ defined by*

$$F_t(x) \;=\; F(x, t) \text{ for all } x \in X$$

  is $(\kappa, \kappa')-$continuous.

Then F is a digital $(\kappa, \kappa')-$homotopy between *f and g, and f and g are* digitally $(\kappa, \kappa')-$homotopic in *Y.* □

THEOREM 2.2. [**3**] *Let S be a simple closed κ-curve and let $H : S \times [0, m]_{\mathbb{Z}} \to S$ be a (κ, κ)-homotopy between an isomorphism H_0 and $H_m = f$, where $f(S) \neq S$. Then $\#S = 4$.*

The literature uses *path* polymorphically: a (c_1, κ)-continuous function $f : [0, m]_{\mathbb{Z}} \to X$ is a κ-path if $f([0, m]_{\mathbb{Z}})$ is a κ-path from $f(0)$ to $f(m)$ as described above.

We use id_X to denote the *identity function*, $id_X(x) = x$ for all $x \in X$.

Given a digital image (X, κ), we denote by $C(X, \kappa)$ the set of κ-continuous functions $f : X \to X$.

Given $f \in C(X, \kappa)$, a *fixed point* of f is a point $x \in X$ such that $f(x) = x$. $Fix(f)$ will denote the set of fixed points of f. We say f is a *retraction*, and the set $Y = f(X)$ is a *retract of* X, if $f|_Y = id_Y$; thus, $Y = Fix(f)$.

DEFINITION 2.3. [**5**] *Let (X, κ) be a digital image. We say $A \subset X$ is a* freezing set *for X if given $g \in C(X, \kappa)$, $A \subset Fix(g)$ implies $g = id_X$. A freezing set A is* minimal *if no proper subset of A is a freezing set for (X, κ).*

EXAMPLE 2.1. *We have the following examples from* [**5**].

- *$\{a, b\}$ is a minimal freezing set for $[a, b]_{\mathbb{Z}}$.*
- *Given $X \subset \mathbb{Z}^n$ such that X is finite and $1 \leqslant u \leqslant n$, $Bd(X)$ is a freezing set for (X, c_u) (not necessarily minimal).*
- *$\Pi_{i=1}^{n}\{a_i, b_i\}$ is a freezing set for (X, c_1), where $X = \Pi_{i=1}^{n}[a_i, b_i]_{\mathbb{Z}}$ (minimal for $n \in \{1, 2\}$; not necessarily minimal for $n > 2$).*

The following elementary assertion was noted in [**5**].

LEMMA 2.1. *Let (X, κ) be a connected digital image for which A is a freezing set. If $A \subset A' \subset X$, then A' is a freezing set for (X, κ).*

Definition 2.4. [**5**] *Given* $s \in \mathbb{N}^*$*, we say* $A \subset X$ *is an* s-cold set *for the connected digital image* (X, κ) *if given* $g \in C(X, \kappa)$ *such that* $g|_A = id_A$*, then for all* $x \in X$*, there is a* κ*-path in* X *of length at most* s *from* x *to* $g(x)$*. A* cold set *is a 1-cold set.*

Example 2.2. [**5**] $\{0\}$ *is a cold set, but not a freezing set, for* $[0,1]_{\mathbb{Z}}$.

Note a 0-cold set is a freezing set [**5**].

Let $X \subset \mathbb{Z}^n$, $x = (x_1, \ldots, x_n) \in Z^n$, where each $x_i \in \mathbb{Z}$. For each index i, the *projection map* (onto the i^{th} coordinate) $p_i : X \to \mathbb{Z}$ is given by $p_i(x) = x_i$.

2.3. Tools for determining fixed point sets.

Theorem 2.3. [**5**] *Let* A *be a freezing set for the digital image* (X, κ) *and let* $F : (X, \kappa) \to (Y, \lambda)$ *be an isomorphism. Then* $F(A)$ *is a freezing set for* (Y, λ).

Proposition 2.1. [**10**] *Let* (X, κ) *be a digital image and* $f \in C(X, \kappa)$*. Suppose* $x, x' \in Fix(f)$ *are such that there is a unique shortest* κ*-path* P *in* X *from* x *to* x'*. Then* $P \subset Fix(f)$.

The following lemma may be understood as saying that if q and q' are adjacent with q in a given direction from q', and if f pulls q further in that direction, then f also pulls q' in that direction.

Lemma 2.2. [**5**] *Let* $(X, c_u) \subset \mathbb{Z}^n$ *be a digital image,* $1 \leqslant u \leqslant n$*. Let* $q, q' \in X$ *be such that* $q \leftrightarrow_{c_u} q'$*. Let* $f \in C(X, c_u)$.

(1) *If* $p_i(f(q)) < p_i(q) < p_i(q')$ *then* $p_i(f(q')) < p_i(q')$.
(2) *If* $p_i(f(q)) > p_i(q) > p_i(q')$ *then* $p_i(f(q')) > p_i(q')$.

2.4. Irreducible and rigid images.

Definition 2.5. [**13**] *A finite image* X *is* reducible *when it is homotopy equivalent to an image of fewer points. Otherwise, we say* X *is* irreducible.

Lemma 2.3. [**13**] *A finite image* X *is reducible if and only if* id_X *is homotopic to a nonsurjective map.*

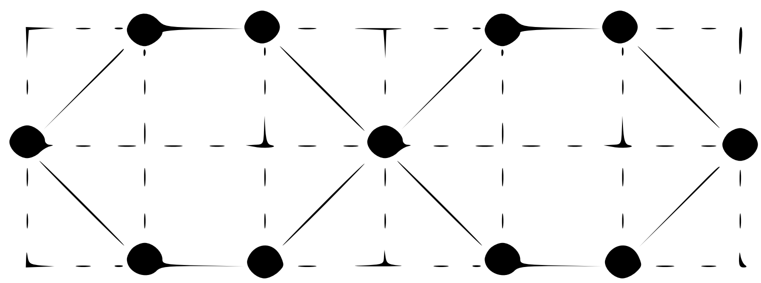

Figure 1. [**9**] Example of a rigid digital image - a wedge of digital simple closed curves

LEMMA 2.4. [**13**] *A finite image X is reducible if and only if id_X is homotopic in one step to a nonsurjective map.*

DEFINITION 2.6. [**13**] *We say an image X is* rigid *if the only map homotopic to id_X is id_X.*

Figure 1 shows an example of a rigid digital image.

PROPOSITION 2.2. [**13**] *A finite rigid digital image is irreducible.*

That the converse of Proposition 2.2 is not generally valid, is shown by the following example.

EXAMPLE 2.3. [**13**] *A digital simple closed curve is irreducible but not rigid.*

3. Products

For Cartesian products of digital images $X = \Pi_{i=1}^{v}(X_i, \kappa_i)$ and $1 \leqslant u \leqslant v$, we often use the generalized normal product adjacency [**4**] $NP_u(\kappa_1, \ldots, \kappa_v)$: given distinct $x, x' \in X$, $x = (x_1, \ldots, x_v)$, $x' = (x'_1, \ldots, x'_v)$, where $x_i, x'_i \in X_i$, we have $x \leftrightarrow_{NP_u(\kappa_1,\ldots,\kappa_v)} x'$ if and only if

- for at least 1 and at most u indices i, $x_i \leftrightarrow_{\kappa_i} x'_i$, and
- for all other indices j, $x_j = x'_j$.

THEOREM 3.1. [**10**] *Let (X_i, κ_i) be a digital image, $1 \leqslant i \leqslant v$. Let $X = \Pi_{i=1}^{v} X_i$. If $(X, NP_v(\kappa_1, \ldots, \kappa_v))$ is rigid, then each (X_i, κ_i) is rigid.*

At Corollary 3.1 below, we obtain an analogous result for irreducible digital images.

THEOREM 3.2. *Let (X_i, κ_i) be a finite digital image, $1 \leqslant i \leqslant v$. Let $X = \Pi_{i=1}^{v} X_i$. If for some j, (X_j, κ_j) is reducible, then $(X, NP_v(\kappa_1, \ldots, \kappa_v))$ is reducible.*

PROOF. By Lemma 2.3, there is a κ_j-homotopy $H_j : X_j \times [0, m]_{\mathbb{Z}} \to X_j$ from id_{X_j} to a nonsurjective map $f_j : X_j \to X_j$. For $i \neq j$, let $H_i : X_i \times [0, m]_{\mathbb{Z}} \to X_i$ be the trivial homotopy $H_i(x_i, t) = x_i$. Then $H : X \times [0, m]_{\mathbb{Z}} \to X$, given by

$$H(x_1, \ldots, x_v, t) = (H_1(x_1, t), \ldots, H_v(x_v, t))$$

is an $NP_v(\kappa_1, \ldots, \kappa_v)$-homotopy from id_X to a nonsurjective map. The assertion follows from Lemma 2.3. □

EXAMPLE 3.1. *Let (X_1, c_2) be the rigid digital image of Figure 1. By Proposition 2.2, (X_1, c_2) is irreducible. Let $(X_2, c_1) = [0, 1]_{\mathbb{Z}}$. Clearly, (X_2, c_1) is reducible. By Theorem 3.2, $(X_1 \times X_2, NP_2(c_2, c_1))$ is reducible.*

As an immediate consequence of Theorem 3.2, we have the following.

COROLLARY 3.1. *Let (X_i, κ_i) be a finite digital image, $1 \leqslant i \leqslant v$. Let $X = \Pi_{i=1}^{v} X_i$. If $(X, NP_v(\kappa_1, \ldots, \kappa_v))$ is irreducible, then each (X_i, κ_i) is irreducible.*

4. Wedges

Let $X \cup Y \subset (\mathbb{Z}^n, \kappa)$ such that there is a point $x_0 \in \mathbb{Z}^n$ with $X \cap Y = \{x_0\}$. Suppose $x \in X$, $y \in Y$, and $x \leftrightarrows_\kappa y$ imply $x_0 \in \{x, y\}$. Then $X' = X \cup Y$ is the ($\kappa-$)*wedge* of X and Y, denoted $X' = X \vee Y$. We call x_0 the *wedge point* of X'.

In this section, we explore the preservation of irreducibility and of rigidity by the wedge construction.

LEMMA 4.1. *Let* $(X, \kappa) = (X_0, \kappa) \vee (X_1, \kappa)$ *where* x_0 *is the wedge point. The function* $r : X \to X_0$ *given by*

$$r(x) = \begin{cases} x & \text{if } x \in X_0; \\ x_0 & \text{if } x \notin X_0, \end{cases}$$

is κ*-continuous and is a* κ*-retraction.*

PROOF. Elementary and left to the reader. □

We have the following.

THEOREM 4.1. [**10**] *Let* $(X, \kappa) = (X_0, \kappa) \vee (X_1, \kappa)$ *where* x_0 *is the wedge point. Suppose* $\#X_0 > 1$ *and* $\#X_1 > 1$*. Suppose* (X_0, κ) *and* (X_1, κ) *are both connected. If* X_0 *and* X_1 *are both rigid, then* X *is rigid.*

We obtain a similar result for the property of irreducibility in the following.

THEOREM 4.2. *Let* $(X, \kappa) = (X_0, \kappa) \vee (X_1, \kappa)$ *where* x_0 *is the wedge point, i.e.,* $\{x_0\} = X_0 \cap X_1$*. Suppose* $\#X_0 > 1$ *and* $\#X_1 > 1$*. If* X_0 *and* X_1 *are both irreducible, then* X *is irreducible.*

PROOF. Suppose otherwise. Then there is a digital homotopy

$$H : X \times [0, m]_{\mathbb{Z}} \to X$$

between id_X and a continuous function $f : X \to X$ such that f is not a surjection. Without loss of generality, there exists $y \in X_0$ such that $y \notin f(X)$.

Let R be the retraction of Lemma 4.1. Then $R \circ H : X_0 \times [0, m]_{\mathbb{Z}} \to X_0$ is a κ-homotopy from id_{X_0} to $R \circ f|_{X_0}$, and $y \notin R \circ f(X_0)$. By Lemma 2.3, this is contrary to the assumption that X_0 is irreducible. The assertion follows. □

The converse of Theorem 4.1 is not generally valid, as shown by Example 3.11 of [**10**].

PROPOSITION 4.1. *(Corollary 3.13 of* [**13**]*) A digital simple closed curve of at least 5 points is irreducible but not rigid.*

For the following Example 4.1 and Theorem 4.3, we have

- $(X, \kappa) = (Y, \kappa) \vee (S, \kappa)$, where $\#Y > 1$, (Y, κ) is irreducible or rigid, and (S, κ) is a digital simple closed curve of at least 5 points.
- $S = \{s_i\}_{i=0}^n$ is a circular listing of the members of S, where $s_0 = x_0$, which is the wedge point.

- Functions $R,\ R_1 : X \to X$ are given by
$$R(x) = \left\{ \begin{array}{ll} x_0 & \text{if } x \in Y; \\ x & \text{if } x \in S, \end{array} \right\},\ R_1(x) = \left\{ \begin{array}{ll} x_0 & \text{if } x \in S; \\ x & \text{if } x \in Y \end{array} \right\}.$$
- Given a homotopy $H : X \times [0,m]_{\mathbb{Z}} \to X$ from id_X to $f \in C(X,\kappa)$, let $G : S \times [0,1]_{\mathbb{Z}} \to S$ be given by
$$G(s,t) = R(H(s,t))$$
and let $G_1 : Y \times [0,1]_{\mathbb{Z}} \to Y$ be given by
$$G_1(x,t) = R_1(H(x,t)).$$

EXAMPLE 4.1. *Let $(X,\kappa) = (Y,\kappa) \vee (S,\kappa)$, where $\#Y > 1$, (Y,κ) is irreducible, and (S,κ) is a digital simple closed curve of at least 5 points. Then (X,κ) is irreducible.*

PROOF. The assertion follows from Theorem 4.2 and Proposition 4.1. □

THEOREM 4.3. *Let $(X,\kappa) = (Y,\kappa) \vee (S,\kappa)$, where Y is finite and $\#Y > 1$, (Y,κ) is rigid, and (S,κ) is a digital simple closed curve of at least 5 points. Then (X,κ) is rigid.*

PROOF. We argue by contradiction. Suppose $f \in C(X,\kappa)$ such that $f \neq id_X$ and there is a homotopy $H : X \times [0,m]_{\mathbb{Z}} \to X$ from id_X to f. By Definition 2.6, we may assume $m = 1$.

Let x_0 be the wedge point, i.e., $\{x_0\} = Y \cap S$, where $\{x_i\}_{i=0}^{n-1}$ is a circular ordering of the distinct members of S. Consider the following cases.

- $f(x_0) = H(x_0,1) \in Y \setminus \{x_0\}$. Then we must have $H(x_1,1) = x_0$ and $H(x_{n-1},1) = x_0$.

  By Lemma 4.1, R is a retraction of X to S. We have
$$R(f(x_0)) = x_0 = R(f(x_1)) \tag{4.1}$$
  Then G is a homotopy from id_S to a map that is non-injective, hence non-surjective; this is impossible by Proposition 4.1 and Lemma 2.3.
- $f(x_0) = H(x_0,1) \in S \setminus \{x_0\}$.

  By Lemma 4.1, R_1 is a retraction. Since Y is connected and has more than 1 point, there exists $y \in Y$ such that $y \leftrightarrow x_0$. However, y is not adjacent to any member of S other than x_0. Therefore, $H(y,1) \in Y$. Hence
$$x_0 \leftrightarrow H(x_0,1) \leftrightarrow H(y,1) = x_0$$
  and
$$G_1(x_0,1) = R_1(H(x_0,1)) = x_0 = R_1(H(y,1)) \tag{4.2}$$
  Then G_1 is a homotopy from id_Y to a map that, by (4.2), is not id_Y. This is impossible, since Y is rigid.
- $f(s) = H(s,1) \in Y \setminus \{x_0\}$ for some $s \in S \setminus \{x_0\}$. This is impossible, as the only member of S that is within 1 step of $Y \setminus \{x_0\}$ is x_0.

- $f(y) = H(y,1) \in S \setminus \{x_0\}$ for some $y \in Y \setminus \{x_0\}$. This is impossible, as the only member of Y that is within 1 step of $S \setminus \{x_0\}$ is x_0.
- $f(x_i) = H(x_i, 1) = x_j$ for some indices satisfying $i \neq j$. The continuity of f implies f "pulls" x_0 into S, i.e., $f(x_0) \in S \setminus \{x_0\}$, which, we saw above, is impossible.
- $f(y) = H(y,1) \in Y \setminus \{y\}$ for some $y \in Y \setminus \{x_0\}$. Then G_1 is a homotopy from id_Y to a nonidentity function on Y; this is impossible, since Y is rigid.

The hypotheses of the cases listed above exhaust all possibilities. Since each case yields a contradiction, we must have $f = id_X$. Thus (X, κ) is rigid. □

5. Cold and freezing sets

Let (X, κ) be a digital image. Let $n \in \mathbb{N}^*$. We say $f \in C(X, \kappa)$ is an *n-map* [**8**] if $x \in X$ implies there is a κ-path in X of length at most n from x to $f(x)$.

The following was observed in the proof of Proposition 2.20 of [**8**].

LEMMA 5.1. *Let (X, κ) be a digital image. Let $f \in C(X.\kappa)$ be a 1-map. Then f is κ-homotopic to id_X.*

PROPOSITION 5.1. [**8**] *Let (X, κ) be a connected rigid digital image. Then the only 1-map in $C(X, \kappa)$ is id_X.*

THEOREM 5.1. [**5**] *Let (X, κ) be a connected rigid digital image. Then $A \subset X$ is a freezing set for (X, κ) if and only if A is a cold set for (X, κ).*

The converse of Theorem 5.1 is not generally valid, as the following shows.

EXAMPLE 5.1. *Let $X = [0, 2]_{\mathbb{Z}}$. Then (X, c_1) is not rigid. However, each cold set for (X, c_1) is freezing.*

PROOF. It is easily seen that (X, c_1) is not rigid. It is easily seen that $A_1 = \{0, 2\}$ and X are cold sets that are freezing. We show there are no other cold sets by showing A_1 is contained in any cold set A for (X, c_1).

Suppose $0 \notin A$. Then the function

$$f(x) = \begin{cases} 2 & \text{if } x = 0; \\ x & \text{if } x \neq 0, \end{cases}$$

satisfies $f \in C(X, c_1)$, $f|_A = id_A$, and $0 \not\simeq_{c_1} f(0)$. Thus A is not cold.

Similarly, if $2 \notin A$ then A is not cold. Thus $A_1 \subset A$. □

THEOREM 5.2. *Let (X, κ) be a digital image. Then X is rigid if and only if the only 1-map in $C(X, \kappa)$ is id_X.*

PROOF. If X is rigid, it follows from Lemma 5.1 that the only 1-map in $C(X, \kappa)$ is id_X.

Suppose the only 1-map in $C(X, \kappa)$ is id_X. Let $H : X \times [0, m]_{\mathbb{Z}} \to X$ be a homotopy from id_X to $g \in C(X, \kappa)$. We argue by induction to show each induced map $H_t(x) = H(x, t)$ is id_X.

Clearly $H_0 = id_X$. Suppose $H_k = id_X$ for some k, $0 \leqslant k < m$. Then the continuity properties of the homotopy H imply H_{k+1} is a 1-map. By Proposition 5.1, $H_{k+1} = id_X$. This completes the induction.

Hence $g = H_m = id_X$. This shows X is rigid. □

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Laurence Boxer, Department of Computer and Information Sciences, Niagara University, NY, USA; and
Department of Computer Science and Engineering, State University of New York at Buffalo, Buffalo, NY, USA

Email address: boxer@niagara.edu