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SOME COMMON FIXED-POINT AND FIXED-CIRCLE RESULTS IN SYMMETRIC S-MULTIPLICATIVE METRIC SPACE

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Abstract. In this paper, we introduce a new metric space, called symmetric S-multiplicative metric space, and establish a common fixed point theorem for four self-mappings with a family of functions in this space. We also investigate some geometric interpretations of fixed point theorems on the circle and disc.

1. Introduction and preliminaries

The Banach contraction principle was first stated explicitly in 1922 [**23**]. The term fixed point theory referred on those fixed points theoretic results in which geometric conditions on the underlying spaces and for mappings play a crucial role. For the past several years metric fixed point theory has been flourishing area for many mathematicians. Since then, topology, functional analysis and nonlinear analysis had been reliant on metric space. This space's topological nature with applications in fixed-point theory has attracted the interest of numerous mathematicians(see [**1**], [**9**] - [**19**]).

In 1906, Frechet [**10**] introduced metric spaces. Numerous generalisations of the concept of metric space have been constructed and different fixed point theorems have been proved in recent years. In 2008, Bashirov et al. [**1**] developed a different type of metric spaces which is known as multiplicative metric spaces and they also established the corresponding Banach fixed point result in the same spaces. In 2012, Sedghi et al. [**21**] established a fixed point theorem for a self-mapping

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on a complete S-metric spaces. In 2019, Mustafa et al. [**25**] generalized S-metric spaces into S_p-metric spaces and investigated the existence of a fixed point for such mappings under various contractive conditions. Most recently in 2024, Adewale et al. [**17**] stated and proved some fixed-point theorems in S-multiplicative metric spaces, which shows that some fixed point theorems are equivalent to those of corresponding fixed-point results in S-metric spaces.
We first present some important definitions and notations which will be used in the main results as follows:

Definition 1.1. *[**10**] Let Y be a non-empty set. A mapping $d : Y \times Y \to [0, +\infty)$ is said to be metric on Y, if and only if for all $\mu, \tau, \rho \in Y$, it satisfies the following conditions:*

(1) $d(\mu, \tau) \geqslant 0$,
(2) $d(\mu, \tau) = 0$ *if and only if* $\mu = \tau$,
(3) $d(\mu, \tau) = d(\tau, \mu)$,
(4) $d(\mu, \tau) \leqslant d(\mu, \rho) + d(\rho, \tau)$.

The pair (Y, d) is called a (standard) metric space.

Definition 1.2. *[**21**] Let Y be a non-empty set. A mapping $S : Y \times Y \times Y \to [0, +\infty)$ is said to be S-metric on Y, if and only if for all $\mu, \tau, \rho, a \in Y$, it satisfies the following conditions:*

(1) $S(\mu, \tau, \rho) = 0$ *if and only if* $\mu = \tau = \rho$,
(2) $S(\mu, \tau, \rho) \leqslant S(\mu, \mu, a) + S(\tau, \tau, a) + S(\rho, \rho, a)$.

The pair (Y, S) is called a S-metric space.

example 1.1. *[**22**] Let $Y = \mathbb{R}$. The function $S : Y \times Y \times Y \to [0, +\infty)$ defined as*

$$S(\mu, \tau, \rho) = |\mu - \rho| + |\tau - \rho|$$

for each $\mu, \tau, \rho \in Y$ is an S-metric on Y. This S-metric is called a usual S-metric.

The relation between a metric d and S-metric defined on a set Y is given in the following lemma.

Lemma 1.1. *[**13**] Let (Y, d) be a metric space. The function $S_d : Y \times Y \times Y \to [0, +\infty)$ defined as $S_d(\mu, \tau, \rho) = d(\mu, \rho) + d(\tau, \rho)$ for each $\mu, \tau, \rho \in Y$ is an S-metric on the set Y. The metric S_d is called the S-metric generated by the metric d.*

Note that there exists an S-metric S satisfying $S \neq S_d$ for all metrics d.

example 1.2. *[**15**] Let $Y = \mathbb{R}$ and define the function*

$$S(\mu, \tau, \rho) = |\mu - \rho| + |\mu + \rho - 2\tau|$$

for all $\mu, \tau, \rho \in Y$. Then, (Y, S) is an S-metric space. There does not exist any metric d such that $S = S_d$.

Definition 1.3. *[**1**] Let Y be a non-empty set. A mapping $d^* : Y \times Y \to [0, +\infty)$ is said to be multiplicative metric on Y if and only if for all $\mu, \tau, \rho \in Y$, it satisfies the following conditions:*

(1) $d^*(\mu,\tau) \geqslant 1$,
(2) $d^*(\mu,\tau) = 1$ *if and only if* $\mu = \tau$,
(3) $d^*(\mu,\tau) = d^*(\tau,\mu)$,
(4) $d^*(\mu,\tau) \leqslant d^*(\mu,\rho).d^*(\rho,\tau)$.

The pair (Y, d^*) *is called a multiplicative metric space.*

Remark 1.1. [**18**] If the logarithm of condition 4. of Definition (1.3) is taken, the multiplicative metric space is equivalent to the (standard) metric space. Indeed, this situation is easy to see. Since the logarithm function is a non-decreasing, we obtain

$$\begin{aligned} d(\mu,\tau) &= \ln(d^*(\mu,\tau)) \\ &\leqslant \ln(d^*(\mu,\rho).d^*(\rho,\tau)) \\ &= \ln(d^*(\mu,\rho)) + \ln(d^*(\rho,\tau)) \\ &= d(\mu,\rho) + d(\rho,\tau). \end{aligned}$$

Definition 1.4. *[**17**] Let Y be a non-empty set. A mapping $S^* : Y \times Y \times Y \to [0,+\infty)$ is said to be S-multiplicative metric on Y, if and only if for all $\mu,\tau,\rho,a \in Y$, it satisfies the following conditions:*

(1) $S^*(\mu,\tau,\rho) = 1$ *if and only if* $\mu = \tau = \rho$,
(2) $S^*(\mu,\tau,\rho) \leqslant S^*(\mu,\mu,a) \times S^*(\tau,\tau,a) \times S^*(\rho,\rho,a)$.

The pair (Y, S^*) *is called an S-multiplicative metric space.*

Remark 1.2. If we take the logarithm function to condition 2. of the Definition (1.4), we obtain the S-metric space. It can be seen as follows:

$$\begin{aligned} S(\mu,\tau,\rho) &= \ln(S^*(\mu,\tau,\rho)) \\ &\leqslant \ln(S^*(\mu,\mu,a) \times S^*(\tau,\tau,a) \times S^*(\rho,\rho,a)) \\ &= \ln(S^*(\mu,\mu,a)) + \ln(S^*(\tau,\tau,a)) + \ln(S^*(\rho,\rho,a)) \\ &= S(\mu,\mu,a) + S(\tau,\tau,a) + S(\rho,\rho,a). \end{aligned}$$

Definition 1.5. *[**17**] Let (Y,S) be an S-multiplicative metric space. For $\mu \in Y$, $r > 0$, the S-sphere with center μ and radius r is $\underline{S}(\mu,r) = \{\tau \in Y : S(\mu,\tau,\tau) < r\}$.*

Definition 1.6. *[**17**] Let (Y,S) and (Y',S') be two S-multiplicative metric spaces, a function $T : Y \to Y'$ is S-continuous at a point $\mu \in Y$ if*

$$T^{-1}(S_{S'}(T(\mu),r)) \in \tau(S),$$

for all $r > 1$. T is S-continuous if it is S-continuous at all points of Y.

Definition 1.7. *[**17**] Let (Y,S) be an S-multiplicative metric space and $\{\mu_n\}$ a sequence in Y. Then $\{\mu_n\}$ converges to μ if and only if $S(\mu_n,\mu,\mu) \to 1$ as $n \to \infty$.*

Definition 1.8. *[**17**] Let (Y,S) be an S-multiplicative metric space and $\{\mu_n\}$ a sequence in Y. Then $\{\mu_n\}$ is said to be a Cauchy sequence if and only if $S(\mu_n,\mu_m,\mu_l) \to 1$ as $n,m,l \to \infty$.*

In this paper, we introduce symmetric S-multiplicative metric space and establish common fixed point theorem in this space and an example is also furnished to validate our results.

2. Main results

DEFINITION 2.1. *Let (Y,S) be an S-multiplicative metric space. An S-multiplicative metric space is called symmetric if $S(\mu,\mu,\tau) = S(\tau,\tau,\mu)$ for all $\mu,\tau \in Y$.*

EXAMPLE 2.1. *Let $Y = \mathbb{R}^+$ and define $S : Y \times Y \times Y \to [0,+\infty)$ by*

$$S(\mu,\tau,\rho) = |cos(\rho(\mu-\tau))|.$$

Here $S(\mu,\tau,\rho) \leqslant S(\mu,\mu,a) \times S(\tau,\tau,a) \times S(\rho,\rho,a)$ and $S(\mu,\mu,\tau) = S(\tau,\tau,\mu)$, $\forall \mu,\tau,\rho,a \in Y$. Clearly, the pair (Y,S) is a symmetric S-multiplicative metric space but it is not an S-metric space.

EXAMPLE 2.2. *Let $Y = \mathbb{R}^+$ and define $S : Y \times Y \times Y \to [0,+\infty)$ by*

$$S(\mu,\tau,\rho) = \begin{cases} 1, & \mu = \tau = \rho \\ e^{\cos(\rho(\mu-\tau))}, & \textit{otherwise.} \end{cases}$$

Then, (Y,S) is a symmetric S-multiplicative metric space but it is not an S-metric space.

To obtain a new common fixed-point theorem, we are inspired by the function family F_6 introduced in [**20**]. We modify these families as follows:
Let Φ be the family of all lower semi-continuous functions $\phi : {\mathbb{R}_+}^6 \to \mathbb{R}$ that satisfy the following condition:
(ϕ^*). For all $\mu,\tau,\rho \geqslant 0$, there exists a $r \in [0,1)$ such that $\mu \leqslant \phi(\mu,\tau,\tau,\mu,1,\rho)$ with $\rho \leqslant \mu^2\tau$, then $\mu \leqslant \tau^r$.

EXAMPLE 2.3. *Define the function $\phi : {\mathbb{R}_+}^6 \to \mathbb{R}$ such that*

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6) = [max\{t_1,t_2\}]^r,$$

with $r \in [0,1)$. Then obviously $\phi \in \Phi$.

Now, we present the following result.

THEOREM 2.1. *Let (Y,S) be a complete continuous symmetric S-multiplicative metric space. Let $f,g,F,G : Y \to Y$ are four continuous mappings satisfying the following conditions.*

1. *$f(Y) \subset G(Y)$ and $g(Y) \subset F(Y)$,*
2. *For all $\mu,\tau \in Y$ and $\phi \in \Phi$,*
$$S(f\mu,f\mu,g\tau) \leqslant \phi \begin{pmatrix} S(f\mu,f\mu,g\tau), S(\mu,\mu,\tau), S(\mu,\mu,f\mu), \\ S(\tau,\tau,g\tau), S(\tau,\tau,f\mu), S(\mu,\mu,g\tau) \end{pmatrix},$$
3. *For all $\mu,\tau \in Y$ and $\phi \in \Phi$,*
$$S(F\mu,F\mu,G\tau) \leqslant \phi \begin{pmatrix} S(F\mu,F\mu,G\tau), S(\mu,\mu,\tau), S(\mu,\mu,F\mu), \\ S(\tau,\tau,G\tau), S(\tau,\tau,F\mu), S(\mu,\mu,G\tau) \end{pmatrix},$$

then f, g, F and G have a common fixed point in Y.

PROOF. Let $\rho_0 \in Y,\ \rho_1 = f\rho_0$ and $\rho_2 = g\rho_1$. Using the condition 2, we get

$$\begin{aligned} S(f\rho_0, f\rho_0, g\rho_1) &= S(\rho_1, \rho_1, \rho_2) \\ &\leqslant \phi\left(\begin{array}{l} S(f\rho_0, f\rho_0, g\rho_1), S(\rho_0, \rho_0, \rho_1), S(\rho_0, \rho_0, f\rho_0), \\ S(\rho_1, \rho_1, g\rho_1), S(\rho_1, \rho_1, f\rho_0), S(\rho_0, \rho_0, g\rho_1) \end{array} \right) \\ &= \phi\left(\begin{array}{l} S(\rho_1, \rho_1, \rho_2), S(\rho_0, \rho_0, \rho_1), S(\rho_0, \rho_0, \rho_1), \\ S(\rho_1, \rho_1, \rho_2), S(\rho_1, \rho_1, \rho_1), S(\rho_0, \rho_0, \rho_2) \end{array} \right) \\ &= \phi\left(\begin{array}{l} S(\rho_1, \rho_1, \rho_2), S(\rho_0, \rho_0, \rho_1), S(\rho_0, \rho_0, \rho_1), \\ S(\rho_1, \rho_1, \rho_2), 1, S(\rho_0, \rho_0, \rho_2) \end{array} \right). \end{aligned} \tag{2.1}$$

By using condition 2 of Definition 1.4 and the symmetry, we have

$$\begin{aligned} S(\rho_0, \rho_0, \rho_2) &= S(\rho_2, \rho_2, \rho_0) \\ &\leqslant (S(\rho_2, \rho_2, \rho_1))^2 S(\rho_0, \rho_0, \rho_1) \\ &= (S(\rho_1, \rho_1, \rho_2))^2 S(\rho_0, \rho_0, \rho_1). \end{aligned} \tag{2.2}$$

Using (2.1), (2.2) and (ϕ^*), there exists a $r \in [0, 1)$ such that

$$S(\rho_1, \rho_1, \rho_2) \leqslant (S(\rho_0, \rho_0, \rho_1))^r.$$

Continuing this process with the condition 1, we can define the sequence ρ_n as follows:

$$\rho_{2n+1} = f\rho_{2n} = G\rho_{2n} \text{ and } \rho_{2n} = g\rho_{2n-1} = F\rho_{2n-1}.$$

Using condition 2, for $\mu = \rho_{2n}$ and $\tau = \rho_{2n+1}$, we have

$$\begin{aligned} S(f\rho_{2n}, f\rho_{2n}, g\rho_{2n+1}) &= S(\rho_{2n+1}, \rho_{2n+1}, \rho_{2n+2}) \\ &\leqslant \phi\left(\begin{array}{l} S(f\rho_{2n}, f\rho_{2n}, g\rho_{2n+1}), S(\rho_{2n}, \rho_{2n}, \rho_{2n+1}), \\ S(\rho_{2n}, \rho_{2n}, f\rho_{2n}), S(\rho_{2n+1}, \rho_{2n+1}, g\rho_{2n+1}), \\ S(\rho_{2n+1}, \rho_{2n+1}, f\rho_{2n}), S(\rho_{2n}, \rho_{2n}, g\rho_{2n+1}) \end{array} \right) \\ &= \phi\left(\begin{array}{l} S(\rho_{2n+1}, \rho_{2n+1}, \rho_{2n+2}), S(\rho_{2n}, \rho_{2n}, \rho_{2n+1}), \\ S(\rho_{2n}, \rho_{2n}, \rho_{2n+1}), S(\rho_{2n+1}, \rho_{2n+1}, \rho_{2n+2}), \\ S(\rho_{2n+1}, \rho_{2n+1}, \rho_{2n+1}), S(\rho_{2n}, \rho_{2n}, \rho_{2n+2}) \end{array} \right) \\ &= \phi\left(\begin{array}{l} S(\rho_{2n+1}, \rho_{2n+1}, \rho_{2n+2}), S(\rho_{2n}, \rho_{2n}, \rho_{2n+1}), \\ S(\rho_{2n}, \rho_{2n}, \rho_{2n+1}), S(\rho_{2n+1}, \rho_{2n+1}, \rho_{2n+2}), \\ 1, S(\rho_{2n}, \rho_{2n}, \rho_{2n+2}). \end{array} \right). \end{aligned} \tag{2.3}$$

By using condition 2 of Definition 1.4 and the symmetry, we have

$$\begin{aligned} S(\rho_{2n}, \rho_{2n}, \rho_{2n+2}) &= S(\rho_{2n+2}, \rho_{2n+2}, \rho_{2n}) \\ &\leqslant (S(\rho_{2n+2}, \rho_{2n+2}, \rho_{2n+1}))^2 S(\rho_{2n}, \rho_{2n}, \rho_{2n+1}) \\ &= (S(\rho_{2n+1}, \rho_{2n+1}, \rho_{2n+2}))^2 S(\rho_{2n}, \rho_{2n}, \rho_{2n+1}). \end{aligned} \tag{2.4}$$

Using (2.3), (2.4) and (ϕ^*), there exists a $r \in [0, 1)$ such that

$$S(\rho_{2n+1}, \rho_{2n+1}, \rho_{2n+2}) \leqslant (S(\rho_{2n}, \rho_{2n}, \rho_{2n+1}))^r. \tag{2.5}$$

Using condition 2, for $\mu = \rho_{2n-1}$ and $\tau = \rho_{2n}$, we have

$$\begin{aligned}
S(F\rho_{2n-1}, F\rho_{2n-1}, G\rho_{2n}) &= S(\rho_{2n}, \rho_{2n}, \rho_{2n+1}) \\
&\leqslant \phi\left(\begin{array}{l} S(F\rho_{2n-1}, F\rho_{2n-1}, G\rho_{2n}), S(\rho_{2n-1}, \rho_{2n-1}, \rho_{2n}), \\ S(\rho_{2n-1}, \rho_{2n-1}, F\rho_{2n-1}), S(\rho_{2n}, \rho_{2n}, G\rho_{2n}), \\ S(\rho_{2n}, \rho_{2n}, F\rho_{2n-1}), S(\rho_{2n-1}, \rho_{2n-1}, G\rho_{2n}) \end{array}\right) \\
&= \phi\left(\begin{array}{l} S(\rho_{2n}, \rho_{2n}, \rho_{2n+1}), S(\rho_{2n-1}, \rho_{2n-1}, \rho_{2n}), \\ S(\rho_{2n-1}, \rho_{2n-1}, \rho_{2n}), S(\rho_{2n}, \rho_{2n}, \rho_{2n+1}), \\ S(\rho_{2n}, \rho_{2n}, \rho_{2n}), S(\rho_{2n-1}, \rho_{2n-1}, \rho_{2n+1}) \end{array}\right) \\
&= \phi\left(\begin{array}{l} S(\rho_{2n}, \rho_{2n}, \rho_{2n+1}), S(\rho_{2n-1}, \rho_{2n-1}, \rho_{2n}), \\ S(\rho_{2n-1}, \rho_{2n-1}, \rho_{2n}), S(\rho_{2n}, \rho_{2n}, \rho_{2n+1}), \\ 1, S(\rho_{2n-1}, \rho_{2n-1}, \rho_{2n+1}) \end{array}\right). \quad (2.6)
\end{aligned}$$

By using condition 2 of Definition 1.4 and the symmetry, we have

$$\begin{aligned}
S(\rho_{2n-1}, \rho_{2n-1}, \rho_{2n+1}) &= S(\rho_{2n+1}, \rho_{2n+1}, \rho_{2n-1}) \\
&\leqslant (S(\rho_{2n+1}, \rho_{2n+1}, \rho_{2n}))^2 S(\rho_{2n-1}, \rho_{2n-1}, \rho_{2n}) \\
&= (S(\rho_{2n}, \rho_{2n}, \rho_{2n+1}))^2 S(\rho_{2n-1}, \rho_{2n-1}, \rho_{2n}). \quad (2.7)
\end{aligned}$$

Using (2.6), (2.7) and (ϕ^*), there exists a $r \in [0, 1)$ such that

$$S(\rho_{2n}, \rho_{2n}, \rho_{2n+1}) \leqslant (S(\rho_{2n-1}, \rho_{2n-1}, \rho_{2n}))^r. \quad (2.8)$$

Using (2.5) and (2.8), we get

$$S(\rho_{2n+1}, \rho_{2n+1}, \rho_{2n+2}) \leqslant (S(\rho_{2n}, \rho_{2n}, \rho_{2n+1}))^r \leqslant (S(\rho_{2n-1}, \rho_{2n-1}, \rho_{2n}))^{r^2}.$$

Continuing this process, we obtain

$$S(\rho_n, \rho_n, \rho_{n+1}) \leqslant (S(\rho_0, \rho_0, \rho_1))^{r^n}. \quad (2.9)$$

By using condition 2 of Definition 1.4, we have

$$S(\rho_n, \rho_m, \rho_m) \leqslant S(\rho_n, \rho_n, \rho_{n+1})(S(\rho_m, \rho_m, \rho_{n+1}))^2.$$

Continuing this process with $m > n$, we obtain

$$\begin{aligned}
S(\rho_n, \rho_m, \rho_m) \leqslant\ & S(\rho_n, \rho_n, \rho_{n+1}) \times (S(\rho_{n+1}, \rho_{n+1}, \rho_{n+2}))^2 \\
& \times (S(\rho_{n+2}, \rho_{n+2}, \rho_{n+3}))^4 \times ... \times (S(\rho_{m-1}, \rho_{m-1}, \rho_m))^{2n}]. \quad (2.10)
\end{aligned}$$

From (2.9) and (2.10), we have

$$\begin{aligned}
S(\rho_n, \rho_m, \rho_m) \leqslant\ & (S(\rho_0, \rho_0, \rho_1))^{r^n} \times (S(\rho_0, \rho_0, \rho_1))^{r^{2n}} \\
& \times (S(\rho_0, \rho_0, \rho_1))^{r^{4n}} \times ... \times (S(\rho_0, \rho_0, \rho_1))^{r^{2n^2}}].
\end{aligned}$$

Taking the limit of $S(\rho_n, \rho_m, \rho_m)$ as $n, m \to \infty$, we have

$$\lim_{n\to\infty} S(\rho_n, \rho_m, \rho_m) = 1.$$

For $n, m, l \in \mathbb{N}$ with $n > m > l$,

$$S(\rho_n, \rho_m, \rho_l) \leqslant S(\rho_n, \rho_n, \rho_{n-1}) \times S(\rho_m, \rho_m, \rho_{n-1}) \times S(\rho_l, \rho_l, \rho_{n-1})].$$

Taking the limit of $S(\rho_n, \rho_m, \rho_l)$ as $n, m, l \to \infty$, we have

$$\lim_{n,m,l\to\infty} S(\rho_n, \rho_m, \rho_l) = 1.$$

Hence $\{\rho_n\}$ is a Cauchy sequence. By the completeness of (Y, S), there exists $\rho \in Y$ such that $\{\rho_n\}$ is S-convergent to ρ, that is, $\lim_{n\to\infty} S(\rho_n, \rho_n, \rho) = 1$.
Next, we establish that ρ is a common fixed point of f, g, F and G. Using the condition (2) of Definition 1.4, for $\mu = \rho_{2n}$ and $\tau = \rho$, we have

$$\begin{aligned} S(f\rho_{2n}, f\rho_{2n}, g\rho) &= S(\rho_{2n+1}, \rho_{2n+1}, g\rho) \\ &\leqslant \phi\begin{pmatrix} S(f\rho_{2n}, f\rho_{2n}, g\rho), S(\rho_{2n}, \rho_{2n}, \rho), S(\rho_{2n}, \rho_{2n}, f\rho_{2n}), \\ S(\rho, \rho, g\rho), S(\rho, \rho, f\rho_{2n}), S(\rho_{2n}, \rho_{2n}, g\rho) \end{pmatrix} \\ &= \phi\begin{pmatrix} S(\rho_{2n+1}, \rho_{2n+1}, g\rho), S(\rho_{2n}, \rho_{2n}, \rho), S(\rho_{2n}, \rho_{2n}, \rho_{2n+1}), \\ S(\rho, \rho, g\rho), S(\rho, \rho, \rho_{2n+1}), S(\rho_{2n}, \rho_{2n}, g\rho) \end{pmatrix} \end{aligned}$$

and taking $n \to \infty$, we get

$$\begin{aligned} S(\rho, \rho, g\rho) &\leqslant \phi\begin{pmatrix} S(\rho, \rho, g\rho), S(\rho, \rho, \rho), S(\rho, \rho, \rho), \\ S(\rho, \rho, g\rho), S(\rho, \rho, \rho), S(\rho, \rho, g\rho) \end{pmatrix} \\ &= \phi\left(S(\rho, \rho, g\rho), 1, 1, S(\rho, \rho, g\rho), 1, S(\rho, \rho, g\rho) \right). \end{aligned} \tag{2.11}$$

Therefore

$$S(\rho, \rho, g\rho) \leqslant (S(\rho, \rho, g\rho))^2.1 = (S(\rho, \rho, g\rho))^2. \tag{2.12}$$

Using (2.11), (2.12) and (ϕ^*), there exists a $r \in [0, 1)$ such that

$$S(\rho, \rho, g\rho) \leqslant 1^r = 1,$$

that is, $g\rho = \rho$. Using the continuity hypothesis of f, we have

$$\begin{aligned} &\lim_{n\to\infty} S(\rho_{2n}, \rho_{2n}, \rho) = 1 \\ &\Rightarrow \lim_{n\to\infty} S(f\rho_{2n}, f\rho_{2n}, f\rho) = 1 \\ &\Rightarrow \lim_{n\to\infty} S(\rho_{2n+1}, \rho_{2n+1}, f\rho) = 1 \\ &\Rightarrow S(\rho, \rho, f\rho) = 1 \\ &\Rightarrow f\rho = \rho. \end{aligned}$$

Hence ρ is a common fixed point f and g. Using condition 3, for $\mu = \rho_{2n}$ and $\tau = \rho$, we have

$$\begin{aligned} S(F\rho_{2n}, F\rho_{2n}, G\rho) &= S(\rho_{2n+1}, \rho_{2n+1}, \rho) \\ &\leqslant \phi\begin{pmatrix} S(F\rho_{2n}, F\rho_{2n}, G\rho), S(\rho_{2n}, \rho_{2n}, \rho), S(\rho_{2n}, \rho_{2n}, F\rho_{2n}), \\ S(\rho, \rho, G\rho), S(\rho, \rho, F\rho_{2n}), S(\rho_{2n}, \rho_{2n}, G\rho) \end{pmatrix} \\ &= \phi\begin{pmatrix} S(\rho_{2n+1}, \rho_{2n+1}, G\rho), S(\rho_{2n}, \rho_{2n}, \rho), S(\rho_{2n}, \rho_{2n}, \rho_{2n+1}), \\ S(\rho, \rho, G\rho), S(\rho, \rho, \rho_{2n+1}), S(\rho_{2n}, \rho_{2n}, G\rho) \end{pmatrix}, \end{aligned}$$

and taking $n \to \infty$, we get

$$S(\rho,\rho,G\rho) \leqslant \phi \left(\begin{array}{c} S(\rho,\rho,G\rho), S(\rho,\rho,\rho), S(\rho,\rho,\rho), \\ S(\rho,\rho,G\rho), S(\rho,\rho,\rho), S(\rho,\rho,G\rho) \end{array} \right)$$

$$= \phi \left(\; S(\rho,\rho,G\rho), 1, 1, S(\rho,\rho,G\rho), 1, S(\rho,\rho,G\rho) \; \right). \tag{2.13}$$

Therefore

$$S(\rho,\rho,G\rho) \leqslant (S(\rho,\rho,G\rho))^2.1 = (S(\rho,\rho,G\rho))^2. \tag{2.14}$$

Using (2.13), (2.14) and (ϕ^*), there exists a $r \in [0,1)$ such that

$$S(\rho,\rho,G\rho) \leqslant 1^r = 1,$$

that is, $G\rho = \rho$. Using the continuity hypothesis of F, we have

$$\begin{aligned} & \lim_{n\to\infty} S(\rho_{2n},\rho_{2n},\rho) = 1 \\ & \Rightarrow \lim_{n\to\infty} S(F\rho_{2n},F\rho_{2n},F\rho) = 1 \\ & \Rightarrow \lim_{n\to\infty} S(\rho_{2n+1},\rho_{2n+1},F\rho) = 1 \\ & \Rightarrow S(\rho,\rho,F\rho) = 1 \\ & \Rightarrow F\rho = \rho. \end{aligned}$$

Consequently, we get

$$\rho = f\rho = g\rho = F\rho = G\rho.$$

Thus ρ is a common fixed point of self-mappings f, g, F and G.

□

3. Some fixed-circle and fixed-disc results

Ozgur and Tas [**16**] introduced a study of the theory of fixed circles in metric spaces, establishing the existence and uniqueness criteria of fixed circles for self-mappings. Later, the theory of fixed circles has been extended to some generalized metric spaces with different geometric approximations ([**6**], [**14**], [**7**], [**11**], [**12**], [**4**]). Motivated by the application of fixed circle theorems and the ongoing work on various generalised metric spaces, we think it is very interesting to introduce the notion of 'fixed circle' and 'fixed disc' in symmetric S-multiplicative metric spaces.

We will first introduce the idea of a fixed circle and a fixed disc in symmetric S-multiplicative metric spaces and then we will study the fixed circle theorems.

Definition 3.1. *Let (Y,S) be a S-multiplicative metric space and $\rho_0, \rho_1, \rho_2 \in Y, r_1 \in [1,\infty)$.*

- *The circle is defined by*

$$C^S_{\rho_1,r_1} = \{\mu \in Y : S(\mu,\mu,\rho_1) = r_1\}.$$

- *The disc is defined by*

$$D^S_{\rho_1,r_1} = \{\mu \in Y : S(\mu,\mu,\rho_1) \leqslant r_1\}.$$

DEFINITION 3.2. *Let $f : Y \to Y$ be a self-mapping where (Y, S) is a symmetric S-multiplicative metric space and let $Fix(f)$ be set of all fixed points of f, then a geometric figure $\mathcal{F}$ (circle or disc) is said to be a fixed figure of f if $\mathcal{F}$ is contained in $Fix(f)$.*

THEOREM 3.1. *Let (Y, S) be a symmetric S-multiplicative metric space, $f : Y \to Y$ be a mapping and $r_1 = \inf\{S(\mu, \mu, f\mu) : \mu \notin Fix(f)\}$. If there exist $\rho_1 \in Y$ and $\phi \in \Phi$ for all $\mu \in Y - \{\rho_1\}$ such that $\mu \notin Fix(f)$ implies*

$$S(f\mu, f\mu, \mu) \leqslant \phi \begin{pmatrix} S(f\mu, f\mu, \mu), S(\mu, \mu, \rho_1), \\ S(\mu, \mu, f\rho_1), S(f\mu, f\mu, \mu), \\ S(f\rho_1, f\rho_1, \rho_1), S(f\mu, f\mu, \rho_1) \end{pmatrix}, \tag{3.1}$$

and $f\rho_1 = \rho_1$ with $f\mu \in D^S_{\rho_1,r_1}$ then $D^S_{\rho_1,r_1} \subset Fix(f)$. Especially, we have $C^S_{\rho_1,r_1} \subset Fix(f)$.

PROOF. Let $r_1 = 1$. Then we have $D^S_{\rho_1,r_1} = \{\rho_1\}$. By the hypothesis $f\rho_1 = \rho_1$, we obtain

$$D^S_{\rho_1,r_1} \subset Fix(f).$$

Let $r_1 > 1$ and $\mu \in D^S_{\rho_1,r_1}$ such that $\mu \notin Fix(f)$. Using (3.1), we have

$$\begin{aligned} S(f\mu, f\mu, \mu) &\leqslant \phi \begin{pmatrix} S(f\mu, f\mu, \mu), S(\mu, \mu, \rho_1), \\ S(\mu, \mu, f\rho_1), S(f\mu, f\mu, \mu), \\ S(f\rho_1, f\rho_1, \rho_1), S(f\mu, f\mu, \rho_1) \end{pmatrix} \\ &= \phi \begin{pmatrix} S(f\mu, f\mu, \mu), S(\mu, \mu, \rho_1), \\ S(\mu, \mu, \rho_1), S(f\mu, f\mu, \mu), \\ S(\rho_1, \rho_1, \rho_1), S(f\mu, f\mu, \rho_1) \end{pmatrix} \\ &= \phi \begin{pmatrix} S(f\mu, f\mu, \mu), S(\mu, \mu, \rho_1), \\ S(\mu, \mu, \rho_1), S(f\mu, f\mu, \mu), \\ 1, S(f\mu, f\mu, \rho_1) \end{pmatrix} \\ &= \phi(S(f\mu, f\mu, \mu), r_1, r_1, S(f\mu, f\mu, \mu), 1, r_1), \end{aligned}$$

since $r_1 \leqslant (S(f\mu, f\mu, \mu))^2 r_1$. Using (ϕ^*), there exists a $r \in [0, 1)$ such that

$$S(f\mu, f\mu, \mu) \leqslant (r_1)^r \leqslant (S(f\mu, f\mu, \mu))^r \leqslant S(f\mu, f\mu, \mu),$$

which is a contradiction. Hence $\mu \in Fix(f)$. Consequently, we get $D^S_{\rho_1,r_1} \subset Fix(f)$. Using the similar arguments, you can easily see that

$$C^S_{\rho_1,r_1} \subset Fix(f).$$

□

Now, we present the following illustrative example of the geometric results established above.

EXAMPLE 3.1. *Let $Y = [-1, 1] \cup \{-7, -2, 2, 7, 8, 11\}$ and define symmetric S-multiplicative metric space as*

$$S(\mu, \tau, \rho) = |cos(\mu - \tau)\rho|, \ \forall \mu, \tau, \rho \in Y.$$

Also define the function $f: Y \to Y$ *by*

$$f(\mu) = \begin{cases} 1, & for\ \mu = 7 \\ \mu, & otherwise, \end{cases}$$

for all $\mu \in Y$ *and the function* $\phi : {\mathbb{R}^+}^6 \to \mathbb{R}$ *as*

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = (t_2)^r,\ with\ r \in [0, 1).$$

Under these hypothesis, we get

$$r_1 = \inf\{S(\mu, \mu, f\mu) : \mu \notin Fix(f)\} = \inf\{S(\mu, \mu, f\mu) : \mu = 7\} = 1.$$

(1) *If we take* $\rho_1 = -1$ *and* $r = \frac{1}{2}$ *then* f *satisfies the conditions of Theorem 3.1. Therefore, we get*

$$C^S_{\rho_1, r_1} \subset Fix(f) = Y - \{7\}.$$

(2) *If we take* $\rho_1 = \frac{1}{2}$ *and* $r = \frac{1}{2}$ *then* f *satisfies the conditions of Theorem 3.1. Therefore, we get*

$$D^S_{\rho_1, r_1} \subset Fix(f) = Y - \{7\}.$$

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