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ON PRESERVATION OF BAIRE, WEAK BAIRE, AND μ-VOLTERRA SPACES

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Abstract. In this article, we concentrate on what kind of functions in generalized topology preserved the image and inverse image of Baire space, weak Baire space, and μ-Volterra space. We proved that the images of Baire space and μ-Volterra space are preserved under (μ, κ)-continuous and somewhat (μ, κ)-open surjective mapping. We further show that the inverse image of Baire space and μ-Volterra space are preserved by somewhat (μ, κ)-continuous and (μ, κ)-open bijective mapping. Finally, we established that weak Baire space is invariant under somewhat (μ, κ)-continuous and somewhat (μ, κ)-open injective mapping.

1. Introduction

Generalized topology is the most suitable generalization of the general topology first proposed by Császár [**1**]. He introduced the concept of continuity in generalized topology, termed (μ, κ)-continuous [**2**]. Following his pioneering work, researchers have explored various weaker forms of continuity, including weak (μ, κ)-continuous [**15**], almost (μ, κ)-continuous [**14**], and feebly (μ, κ)-continuous [**12**]. In recent years, authors have also investigated Baire spaces [**16**], weak Baire spaces [**12**], and μ-Volterra spaces [**9**] in generalized topology. Gentry and Hoyle [**9**] initially introduced the concepts of somewhat continuity and somewhat open functions. B. Roy later established that somewhat (μ, κ)-open functions and feebly (μ, κ)-open functions are equivalent [**6**]. Our research indicates that somewhat (μ, κ)-continuous functions are even weaker than feebly (μ, κ)-continuous functions.

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In this article, we concentrate on what kind of function in generalized topology preserved the image and the inverse image of Baire space, weak Baire space, and μ-Volterra space. The concept of Baire space in generalized topology was initially presented by Li and Lin [**16**], who demonstrated that (μ, κ)-continuous and (μ, κ)-open surjective mappings preserve Baire spaces. We were able to extend this result by replacing the (μ, κ)-open mapping with a somewhat (μ, κ)-open mapping. Additionally, Korczak-Kubiak et al. [**8**] defined μ-strongly nowhere dense sets and weak Baire spaces. Vadakasi and Renukadevi [**12**] later proved that weak Baire spaces are preserved under feebly (μ, κ)-continuous and feebly (μ, κ)-open injective mappings. Our work generalizes this result by proving that weak Baire spaces remain invariant under somewhat (μ, κ)-continuous and somewhat (μ, κ)-open injective mapping.

Jeyanthi and Geetha [**10**] introduced the concept of μ-Volterra spaces in generalized topology in 2020 and established several fundamental results. This raises the natural question of which types of mappings preserve the image and inverse image of μ-Volterra spaces. We address this question affirmatively in Theorems 6.1 and 6.2.

2. Preliminaries

Let $X \neq \emptyset$ and $exp(X)$ denote the power set of X; then $\mu \subseteq exp(X)$ is called a generalized topology [**2**] on X if $\emptyset \in \mu$ and arbitrary union of elements of μ belong to μ. If (X, μ) is a generalized topological space (briefly, GTS) and $S \subseteq X$, then S is μ-open if $S \in \mu$. The complement of the μ-open set is μ-closed. The interior of S is the largest μ-open subset contained in S, and the closure of S is the smallest μ-closed subset containing S; they are also denoted by $i_\mu S$ and $c_\mu S$, respectively. We denote $\mu_x = \{G \in \mu : x \in G\}$ and $\tilde{\mu} = \{G \in \mu : \mu \neq \emptyset\}$. In this paper, (X, μ) and (Y, κ) always mean GTS, and no separation axioms are assumed without mention. Some straightforward proofs are omitted for simplicity. The definitions and theorems we are using in this article are as follows.

Definition 2.1. *Let (X, μ) be a GTS. Then a subset S of X is called*

(a) *μ-dense [**13**] in X, if $c_\mu S = X$.*
(b) *μ-nowhere dense [**13**] in X, if $i_\mu c_\mu S = \emptyset$.*
(c) *μ-codense [**7**] in X, if $X - S$ is μ-dense in X.*
(d) *μ-first category [**16**] in X, if there exists a sequence $\{S_n\}$ consisting of μ-nowhere dense subsets of X such that $S = \bigcap_{n\in\mathbb{N}} S_n$.*
(e) *μ-second category [**16**] in X, if S is not μ-first category in X.*
(f) *μ-residual [**16**] in X, if $X - S$ is μ-first category in X.*

Theorem 2.1. *[**16**] Let (X, μ) be a GTS and $S \subseteq X$. Then S is μ-dense in X if and only if $G \cap S \neq \emptyset$ for any $G \in \mu - \{\emptyset\}$.*

Theorem 2.2. *[**16**] Let (X, μ) be a GTS and $S \subseteq X$. Then S is μ-nowhere dense in X if and only if $c_\mu S$ is μ-codense in X.*

Definition 2.2. *A function $\Phi : (X, \mu) \to (Z, \kappa)$ is called*

(a) *(μ,κ)-continuous [**2**] if $S \in \kappa$ implies that $\Phi^{-1}(S) \in \mu$.*
(b) *weakly (μ,κ)-continuous [**15**] if for each $x \in X$ and $T \in \kappa_{\Phi(x)}$, there exists $S \in \mu_x$ such that $\Phi(S) \subseteq c_\kappa T$.*
(c) *almost (μ,κ)-continuous [**14**] if for each $x \in X$ and $T \in \kappa_{\Phi(x)}$, there exists $S \in \mu_x$ such that $\Phi(S) \subseteq i_\kappa c_\kappa T$.*
(d) *feebly (μ,κ)-continuous [**12**] if $i_\mu(\Phi^{-1}(T)) \neq \emptyset$ for every $T \subseteq Y$ and $i_\kappa(T) \neq \emptyset$.*
(e) *feebly (μ,κ)-open [**16**] if $i_\kappa(\Phi(S)) \neq \emptyset$ for each $S \in \tilde{\mu}$.*
(f) *somewhat (μ,κ)-open [**6**] if $S \in \tilde{\mu}$, then there is a $T \in \tilde{\kappa}$ such that $T \subseteq \Phi(S)$.*
(g) *somewhat (μ,κ)-continuous [**6**] if for each $S \in \kappa$ and $\Phi^{-1}(S) \neq \emptyset$, then there is a $T \in \tilde{\mu}$ such that $T \subseteq \Phi^{-1}(S)$.*

THEOREM 2.3. *[**6**] If $\Phi : (X,\mu) \to (Z,\kappa)$ is a function, then the following are equivalent:*

(a) *Φ is somewhat (μ,κ)-open.*
(c) *If S is κ-dense in Z, then $\Phi^{-1}(S)$ is μ-dense in X.*

THEOREM 2.4. *[**6**] If $\Phi : (X,\mu) \to (Z,\kappa)$ is a function, then the following conditions are equivalent:*

(a) *Φ is somewhat (μ,κ)-continuous,*
(b) *If S is a κ-closed subset of Z such that $\Phi^{-1}(S) \neq X$, then there is a proper μ-closed subset T of X such that $\Phi^{-1}(S) \subseteq T$.*
(c) *If P is a μ-dense in X, then $\Phi(P)$ is a κ-dense in $\Phi(X)$.*

3. Comparisons with other functions

REMARK 3.1. For various forms of continuity in GTS, the implications listed below hold true, while the converse is not true, as shown by the example we provide.

$$\begin{array}{ccc}
(\mu,\kappa)-continuous & \Rightarrow & almost\ (\mu,\kappa)-continuous \\
\Downarrow & & \Downarrow \\
feebly\ (\mu,\kappa)-continuous & & weakly\ (\mu,\kappa)-continuous \\
\Downarrow & & \\
somewhat\ (\mu,\kappa)-continuous & &
\end{array}$$

EXAMPLE 3.1. *Let $X = \{\alpha,\beta,\gamma\}$ and consider two GTS as $\mu = \{\emptyset,\{\beta\}\}$ and $\kappa = \{\emptyset,\{\alpha,\beta\}\}$ on X. Then the identity function $\Phi : (X,\mu) \to (X,\kappa)$ is somewhat (μ,κ)-continuous but neither weakly (μ,κ)-continuous nor almost (μ,κ)-continuous.*

EXAMPLE 3.2. *Let $X = \{\alpha,\beta,\gamma,\delta\}$ and consider two GTS as $\mu = \{\emptyset,\{\alpha\},\{\alpha,\beta\},\{\beta,\gamma\},\{\alpha,\beta,\gamma\}\}$ and $\kappa = \{\emptyset,\{\alpha,\gamma\},\{\beta,\gamma\},\{\alpha,\beta,\gamma\}\}$ on X. Define a function $\Phi : (X,\mu) \to (X,\kappa)$ by $\Phi(\alpha) = \beta, \Phi(\beta) = \delta, \Phi(\gamma) = \gamma, \Phi(\delta) = \delta$. Then Φ is almost (μ,κ)-continuous and weakly (μ,κ)-continuous but Φ is not somewhat (μ,κ)-continuous.*

EXAMPLE 3.3. *Let* $X = \{\alpha, \beta, \gamma, \delta\}$ *and consider two GTS as* $\mu = \{\emptyset, \{\alpha, \beta, \gamma\}\}$, $\kappa = \{\emptyset, \{\beta\}, \{\alpha, \gamma\}, \{\alpha, \beta, \gamma\}\}$. *Define* $\Phi : (X, \mu) \to (X, \kappa)$ *by* $\Phi(\alpha) = \alpha$, $f(\beta) = \gamma$, $f(\gamma) = \gamma$, $f(\delta) = \gamma$. *Then* Φ *is somewhat* (μ, κ)*-continuous, but not feebly* (μ, κ)*-continuous as* $T = \{\alpha, \beta\}$*, then* $i_\kappa T = \{\beta\} \neq \emptyset$ *and* $i_\mu(\Phi^{-1}(T)) = \emptyset$.

PROPOSITION 3.1. *Let* $\Phi : (X, \mu) \to (Z, \kappa)$ *be a mapping. Then* Φ *is feebly* (μ, κ)*-open if and only if* Φ *is somewhat* (μ, κ)*-open.*

PROOF. It is clear from Theorem 3.3 in [**6**]. □

4. Preservation of Baire space

DEFINITION 4.1. *[16] A GTS* (X, μ) *is said to be Baire, if for any sequence* $\{S_n\}$ *consisting of* μ*-open and* μ*-dense subsets of* X*, then* $\bigcap_{n\in\mathbb{N}} S_n$ *is* μ*-dense in* X*.*

LEMMA 4.1. *[4] A mapping* $\Phi : (X, \mu) \to (Z, \kappa)$ *is* (μ, κ)*-continuous if and only if* $c_\mu(\Phi^{-1}(S)) \subseteq \Phi^{-1}(c_\kappa(S))$ *for any* $S \subseteq Z$*.*

LEMMA 4.2. *[16] A mapping* $\Phi : (X, \mu) \to (Z, \kappa)$ *is* (μ, κ)*-open if and only if* $\Phi^{-1}(c_\kappa(S)) \subseteq c_\mu(\Phi^{-1}(S))$ *for any* $S \subseteq Z$*.*

THEOREM 4.1. *Let* $\Phi : (X, \mu) \to (Z, \kappa)$ *is somewhat* (μ, κ)*-open and* (μ, κ)*-continuous surjection. If* X *is a Baire space, then* Z *is also a Baire space.*

PROOF. Let $\{S_n\}$ be a sequence of κ-open and κ-dense subsets of Z. Since Φ is somewhat (μ, κ)-open by Theorem 2.3 $\Phi^{-1}(S_n)$ is μ-dense in X for all $n \in \mathbb{N}$. Note that if Φ is (μ, κ)-continuous, then $\Phi^{-1}(S_n)$ is μ-open in X for all $n \in \mathbb{N}$. Since X is a Baire space, therefore $\bigcap_{n\in\mathbb{N}} \Phi^{-1}(S_n)$ is μ-dense in X.

Again, Φ is (μ, κ)-continuous, and by the Lemma 4.1 we have

$$X = c_\mu\Big(\bigcap_{n\in\mathbb{N}} \Phi^{-1}(S_n)\Big) = c_\mu\Big(\Phi^{-1}\Big(\bigcap_{n\in\mathbb{N}} S_n\Big)\Big) \subseteq \Phi^{-1}\Big(c_\mu\Big(\bigcap_{n\in\mathbb{N}} S_n\Big)\Big).$$

Since Φ is a surjection, it entails that

$$Z = \Phi(X) \subseteq \Phi\Big(\Phi^{-1}\Big(c_\mu\Big(\bigcap_{n\in\mathbb{N}} S_n\Big)\Big)\Big) = c_\mu\Big(\bigcap_{n\in\mathbb{N}} S_n\Big).$$

Hence $\bigcap_{n\in\mathbb{N}} S_n$ is κ-dense in Z. Thus Z is a Baire space. □

THEOREM 4.2. *Let* $\Phi : (X, \mu) \to (Z, \kappa)$ *is somewhat* (μ, κ)*-continuous and* (μ, κ)*-open bijection. If* Z *is a Baire space, then* X *is also a Baire space.*

PROOF. Let $\{S_n\}$ be a sequence of μ-open and μ-dense subsets of X. Since Φ is (μ, κ)-open, it implies that $\Phi(S_n)$ is κ-open for all $n \in \mathbb{N}$. Also, Φ is a somewhat (μ, κ)-continuous surjection, then by Theorem 2.4 we have that $\Phi(S_n)$ is κ-dense in Z for all $n \in \mathbb{N}$. By assumption, Z is a Baire space, then

$$c_\kappa\Big(\bigcap_{n\in\mathbb{N}} \Phi(S_n)\Big) = Z.$$

Using Lemma 4.2, we get

$$X=\Phi^{-1}(Z)=\Phi^{-1}\Big(c_\kappa\bigcap_{n\in\mathbb{N}}\Phi(S_n)\Big)\subseteq c_\mu\Big(\Phi^{-1}\Big(\bigcap_{n\in\mathbb{N}}\Phi(S_n)\Big)\Big).$$

Now Φ is injection implies that

$$X\subseteq c_\mu\Big(\Phi^{-1}\Big(\bigcap_{n\in\mathbb{N}}\Phi(S_n)\Big)\Big)=c_\mu\Big(\bigcap_{n\in\mathbb{N}}\Phi^{-1}\Big(\Phi(S_n)\Big)\Big)=c_\mu\Big(\bigcap_{n\in\mathbb{N}}S_n\Big).$$

Hence X is a Baire space. □

5. Preservation of weak Baire space

Definition 5.1. *Let (X,μ) be a GTS. $A\subseteq X$ is called*

(a) *μ-strongly nowhere dense [**8**] if for any $S\in\tilde{\mu}$, there exists $T\in\tilde{\mu}$ such that $T\subseteq S$ and $T\cap A=\emptyset$.*
(b) *μ-s-meager [**8**] if there exists a sequence $\{S_n\}$ consisting of μ-strongly nowhere dense subsets of X such that $A=\bigcap\limits_{n\in\mathbb{N}}S_n$.*
(c) *μ-s-second category [**8**] if A is not μ-s-meager set.*

Definition 5.2. *[**8**] A GTS (X,μ) is called weak Baire space if each set $S\in\tilde{\mu}$ is of the μ-s-second category in X.*

Theorem 5.1. *A mapping $\Phi:(X,\mu)\to(Z,\kappa)$ is somewhat (μ,κ)-open and somewhat (μ,κ)-continuous injection. Then S is a μ-strongly nowhere dense in X if and only if $\Phi(S)$ is a κ-strongly nowhere dense in Y.*

Proof. Let S be a μ-strongly nowhere dense in X and $U\in\tilde{\kappa}$. Since Φ is somewhat (μ,κ)-continuous and $\Phi^{-1}(U)\neq\emptyset$, there exists $T\in\tilde{\mu}$ such that $T\subseteq\Phi^{-1}(U)$. By assumption, there exists $Q\in\tilde{\mu}$ such that $Q\subseteq T$ and $Q\cap S=\emptyset$. Again, Φ is somewhat (μ,κ)-open and $Q\in\tilde{\mu}$, then there exists $P\in\tilde{\kappa}$ such that $P\subseteq\Phi(Q)$. This implies that $i_\kappa(\Phi(Q))\in\tilde{\kappa}$. Therefore

$$i_\kappa(\Phi(Q))\subseteq\Phi(Q)\subseteq\Phi(T)\subseteq\Phi(\Phi^{-1}(U))=U.$$

Also $\emptyset=\Phi(\emptyset)=\Phi(Q\cap S)=\Phi(Q)\cap\Phi(S)$ as Φ is injective. Then $i_\mu(\Phi(Q))\cap\Phi(S)=\emptyset$. Hence $\Phi(S)$ is a κ-strongly nowhere dense in Z.

Conversely let $\Phi(S)$ be a κ-strongly nowhere dense in Z and $R\in\tilde{\mu}$. Since Φ is somewhat (μ,κ)-open, so there exists $T\in\tilde{\kappa}$ such that $T\subseteq\Phi(R)$. By assumption there exists $Q\in\tilde{\kappa}$ such that $Q\subseteq T$ and $Q\cap\Phi(S)=\emptyset$. Again Φ is somewhat (μ,κ)-continuous and $Q\in\tilde{\kappa}$, then there exists $P\in\tilde{\mu}$ such that $P\subseteq\Phi^{-1}(Q)$. This implies that $i_\mu(\Phi^{-1}(Q))\in\tilde{\mu}$. Therefore

$$i_\mu(\Phi^{-1}(Q))\subseteq\Phi^{-1}(Q)\subseteq\Phi^{-1}(T)\subseteq\Phi^{-1}(\Phi(R))=R$$

as Φ is injective. Also $\emptyset=\Phi^{-1}(\emptyset)=\Phi^{-1}(Q\cap\Phi(S))=\Phi^{-1}(Q)\cap S$. Then $i_\mu(\Phi^{-1}(Q))\cap S=\emptyset$. Thus S is a μ-strongly nowhere dense in X. □

Using the fact that every feebly (μ, κ)-continuous function is somewhat (μ, κ)-continuous with the same Example 4.9 (a), (b), (e) and (f) in [**12**], we can conclude that the conditions somewhat (μ, κ)-open and injection cannot be dropped in Theorem 5.1.

EXAMPLE 5.1. *Let $X = \{p, q, r, s\}$ and consider two GTS as $\mu = \{\emptyset, \{p\}, \{q\}, \{p, q\}, \{p, r\}, \{q, r\}, \{p, q, r\}\}$ and $\kappa = \{\emptyset, \{p\}, \{r\}, \{s\}, \{p.r\}, \{p, s\}, \{r, s\}, \{p, r, s\}\}$ on X. Define a mapping $\Phi : (X, \mu) \to (X, \kappa)$ by $\Phi(p) = r, \Phi(q) = p, \Phi(r) = s, \Phi(s) = q$. Then Φ is somewhat (μ, κ)-open and injective but Φ is not somewhat (μ, κ)-continuous. Thus $\{r, s\}$ is μ-strongly nowhere dense set in (X, μ), then $\Phi(\{r, s\})$ is not κ-strongly nowhere dense set in (X, κ).*

EXAMPLE 5.2. *Let $X = \{p, q, r, s\}$ and consider two GTS as $\mu = \{\emptyset, \{p\}, \{p, q\}, \{q, r\}, \{p, q, r\}\}$ and $\kappa = \{\emptyset, \{r\}, \{s\}, \{p, s\}, \{r, s\}, \{p, r, s\}\}$ on Z. Define a mapping $\Phi : (X, \mu) \to (X, \kappa)$ by $\Phi(p) = r, \Phi(q) = s, \Phi(r) = p, \Phi(s) = q$. Then Φ is somewhat (μ, κ)-open and injective but Φ is not somewhat (μ, κ)-continuous. Thus $\{p\}$ is κ-strongly nowhere dense set in (X, κ), then $\Phi^{-1}(\{p\})$ is not μ-strongly nowhere dense set in (X, μ).*

These two examples ensure that the condition somewhat (μ, κ)-continuous cannot be dropped in Theorem 5.1.

LEMMA 5.1. *[**6**] Let (X, μ) be a generalized topology and $A \subseteq Y \subseteq X$. If Y is μ-s-meager set in X, then A is a μ-s-meager set in X.*

THEOREM 5.2. *Let $\Phi : (X, \mu) \to (Z, \kappa)$ is somewhat (μ, κ)-open and somewhat (μ, κ)-continuous injection. Then (X, μ) is a weak Baire space if and only if (Z, κ) is a weak Baire space.*

PROOF. Let (X, μ) be a weak Baire space. Suppose that (Z, κ) is not a weak Baire space, then there exists $S \in \tilde{\kappa}$ such that S is a κ-s-meager set. Therefore, $S = \bigcup_{n \in \mathbb{N}} S_n$, where each S_n is a κ-strongly nowhere dense subset of Z. The assumption and Theorem 5.1 imply that for each $n \in \mathbb{N}$, $\Phi^{-1}(S_n)$ is a μ-strongly nowhere dense subset of X. Thus, $\Phi^{-1}(S) = \bigcup_{n \in \mathbb{N}} \Phi^{-1}(S_n)$ is a μ-s-meager set. Then, by Lemma 5.1, $i_\mu(\Phi^{-1}(S))$ is also a μ-s-meager set. Since Φ is somewhat (μ, κ)-continuous, then $i_\mu(\Phi^{-1}(S)) \in \tilde{\mu}$, which is a contradiction. Hence, (Z, κ) is not a weak Baire space.

Conversely, let (Z, κ) be a weak Baire space. Suppose that (X, μ) is not a weak Baire space, then there exists a $S \in \tilde{\mu}$ such that S is a μ-s-meager set. Therefore, $S = \bigcup_{n \in \mathbb{N}} S_n$, where each S_n is a μ-strongly nowhere dense subset of X. The assumption, Theorem 5.1 imply that for each $n \in \mathbb{N}$, $\Phi(S_n)$ is a κ-strongly nowhere dense subset of Z. Thus, $\Phi(S) = \bigcup_{n \in \mathbb{N}} \Phi(S_n)$ is a κ-s-meager set. Therefore, by Lemma 5.1, $i_\kappa(\Phi(S))$ is also a κ-s-meager set. Since Φ is somewhat (μ, κ)-open, then $i_\kappa(\Phi(S)) \in \tilde{\kappa}$, which is a contradiction. Thus, (X, μ) is a weak Baire space. □

6. Preservation of μ-Volterra space

DEFINITION 6.1. *[**10**] A subset S of a GTS (X,μ) is said to be a μ-G_δ set if it can be expressed as a countable intersection of μ-open sets in X.*

DEFINITION 6.2. *[**10**] A GTS (X,μ) is called μ-Volterra space, if the intersection of any two μ-dense and μ-G_δ subsets of X is μ-dense in X.*

THEOREM 6.1. *Let $\Phi : (X,\mu) \to (Z,\kappa)$ be a somewhat (μ,κ)-open, (μ,κ) continuous and surjection. If X is a μ-Volterra space, then Z is also a κ-Volterra space.*

PROOF. Let S and T be κ-dense and κ-G_δ subsets of Z. Then there exists countable index subsets κ and Ω such that $S = \bigcap_{\alpha\in\kappa} S_\alpha$ and $T = \bigcap_{\beta\in\Omega} T_\beta$, where each S_α and T_β are κ-open in Z for $\alpha \in \kappa$ and $\beta \in \Omega$, respectively. Since Φ is (μ,κ)-continuous, then $\Phi^{-1}(S_\alpha)$ and $\Phi^{-1}(T_\beta)$ are μ-open in X for all $\alpha \in \kappa$ and $\beta \in \Omega$. Then

$$\Phi^{-1}(S) = \bigcap_{\alpha\in\kappa} \Phi^{-1}(S_\alpha) \quad \textit{and} \quad \Phi^{-1}(T) = \bigcap_{\beta\in\Omega} \Phi^{-1}(T_\beta).$$

which imply that both $\Phi^{-1}(S)$ and $\Phi^{-1}(T)$ are μ-G_δ sets. Since Φ is somewhat (μ,κ)-open and Theorem 2.3 both implies that $\Phi^{-1}(S)$ and $\Phi^{-1}(T)$ are μ-dense in X. Since X is μ-Volterra, thus $\Phi^{-1}(S\cap T) = \Phi^{-1}(S) \cap \Phi^{-1}(T)$ is μ-dense in X. Again, Φ is (μ,κ)-continuous, implying that Φ is somewhat (μ,κ)-continuous surjection. Then, by Theorem 2.4, we have $\Phi(\Phi^{-1}(S\cap T)) = S\cap T$ is κ-dense in Z. Hence, Z is a κ-Volterra space. □

THEOREM 6.2. *Let $\Phi : (X,\mu) \to (Z,\kappa)$ be a somewhat (μ,κ)-continuous and (μ,κ)-open bijection. If Z is κ-Volterra space, then X is μ-Volterra space.*

PROOF. Suppose that S and T are μ-dense and μ-G_δ subsets of X. Then there exists countable index subsets κ and Ω such that $S = \bigcap_{\alpha\in\kappa} S_\alpha$ and $T = \bigcap_{\beta\in\Omega} T_\beta$, where each S_α and T_β are μ-open in X for $\alpha \in \kappa$ and $\beta \in \Omega$, respectively. Since Φ is (μ,κ)-open, then $\Phi(S_\alpha)$ and $\Phi(T_\beta)$ are κ-open in Z for all $\alpha \in \kappa$ and $\beta \in \Omega$. Therefore

$$\Phi(S) = \bigcap_{\alpha\in\kappa} \Phi(S_\alpha) \quad \textit{and} \quad \Phi(T) = \bigcap_{\beta\in\Omega} \Phi(T_\beta)$$

as Φ is injection, which implies that both $\Phi(S)$ and $\Phi(T)$ are κ-G_δ set. Note that Φ is somewhat (μ,κ)-continuous surjection, by Theorem 2.4 $\Phi(S)$ and $\Phi(T)$ are κ-dense in Z. Since Z is κ-Volterra, then $\Phi(S)\cap\Phi(T) = \Phi(S\cap T)$ is κ-dense in Z as Φ is injective. Since Φ is (μ,κ)-open, it implies that Φ is somewhat (μ,κ)-open. Then, by Theorem 2.3 $\Phi^{-1}(\Phi(S\cap T)) = S\cap T$ is μ-dense in X. Hence, X is a μ-Volterra space. □

7. Some other preservation

Definition 7.1. *[**11**] A GTS (X, μ) is irreducible if the intersection of any two non empty μ-open sets is non empty.*

Theorem 7.1. *Let $\Phi : (X, \mu) \to (Z, \kappa)$ be somewhat (μ, κ)-continuous and surjection. If X is irreducible, then Z is also irreducible.*

Proof. Suppose that G and H are non empty κ-open sets in Z. Since Φ is surjective, $\Phi^{-1}(G)$ and $\Phi^{-1}(H)$ are both non-empty. Now Φ is somewhat (μ, κ)-continuous, then there exist two non empty μ-open sets S and T such that $S \subseteq \Phi^{-1}(G)$ and $T \subseteq \Phi^{-1}(H)$. Since X is irreducible, then $S \cap T \neq \emptyset$. Also, we have $S \cap T \subseteq \Phi^{-1}(G \cap H)$, which implies that $G \cap H \neq \emptyset$. Hence, Z is irreducible. □

Lemma 7.1. *Let $\Phi : (X, \mu) \to (Z, \kappa)$ be a (μ, κ)-open injection, then $c_\kappa(\Phi(S)) \subseteq \Phi(c_\mu(S))$ for any $S \subseteq X$.*

Proof. Suppose that $y \notin \Phi(c_\mu S)$, imply $\Phi^{-1}(y) \notin c_\mu S$, then there exists a μ-open set V containing $\Phi^{-1}(y)$ such that $Y \cap S = \phi$. By assumption, $\Phi(V)$ is κ-open in Z and $\Phi(V) \cap \Phi(S) = \emptyset$. This implies that $y \notin c_\kappa(\Phi(S))$. Thus, $c_\kappa \Phi(S) \subseteq \Phi(c_\mu S)$. □

Theorem 7.2. *Let $\Phi : (X, \mu) \to (Z, \kappa)$ be a somewhat (μ, κ)-continuous and (μ, κ)-open bijective mapping. Then, for any $S \subseteq X$ following holds*

(a) *whenever S is μ-nowhere dense in X, then $\Phi(S)$ is also κ-nowhere dense in Z.*
(b) *whenever S is μ-first category, then $\Phi(S)$ is also κ-first category in Z.*
(c) *whenever S is μ-residual in X, then $\Phi(S)$ is also κ-residual in Z.*
(d) *whenever S is κ-second category in Z, then $\Phi^{-1}(S)$ is also μ-second category in X.*

Proof. (a) Let S be a μ-nowhere dense in X. Then $X - c_\mu S$ is μ-dense in X by Theorem 2.2. Since Φ is somewhat (μ, κ)-continuous and bijection, then by Theorem 2.4 implies $\Phi(X - c_\mu S) = Y - \Phi(c_\mu(S))$ is κ-dense in Z. Note that Φ is a (μ, κ)-open injection from Lemma 7.1. Therefore $Y - \Phi(c_\mu(S)) \subseteq Y - c_\kappa(\Phi(S)) \subseteq Y$, implies that $Y - c_\kappa(\Phi(S))$ is κ-dense in Z. Hence $\Phi(S)$ is κ-nowhere dense in Z.

(b) Let S be the μ-first category in X. Then there exists a sequence $\{S_n\}$ of μ-nowhere dense subsets of X such that $S = \bigcup_{n \in \mathbb{N}} S_n$. By (a), each $\Phi(S_n)$ is κ-nowhere dense in Z. Therefore

$$\Phi(S) = \Phi\Big(\bigcup_{n \in \mathbb{N}} S_n\Big) = \bigcup_{n \in \mathbb{N}} \Phi(S_n).$$

Hence, $\Phi(S)$ is the κ-first category in Z.

The proofs of (c) and (d) follows the straight-forward consequences of (b), so we omit these. □

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