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NOTE ON WEAKLY 1-ABSORBING PRIME ELEMENTS

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Abstract. Let $\mathcal{L}$ be a *bounded distributive lattice*. Following the concept of 1-absorbing prime filters (resp. weakly 1-absorbing prime filters) of $\mathcal{L}$ [8], we define 1-absorbing prime elements (resp. weakly 1-absorbing prime elements). A proper element p of $\mathcal{L}$ is called 1-absorbing prime element (resp. weakly 1-absorbing prime element) of $\mathcal{L}$ if whenever non-zero elements $a, b, c \in \mathcal{L}$ and $p \leqslant a \vee b \vee c$ (resp. $p \leqslant a \vee b \vee c \neq 1$), then either $p \leqslant a \vee b$ or $p \leqslant c$. We will make an intensive investigate the basic properties and possible structures of these elements.

1. Introduction

All lattices considered in this paper are assumed to have a least element denoted by 0 and a greatest element denoted by 1, in other words they are bounded. In abstract algebra, structures composed of certain subjects tend to have their distinct qualities by way of lattice theoretic properties. This is the case (to mention only the presumably most prominent example) for the ideals of a commutative ring (see for instance [4, 6, 8, 9, 10, 11, 12]). The main aim of this article is that of extending some absorbing results obtained for rings theory to the theory of lattices.

Various generalizations of prime ideals of commutative rings have been studied. Badawi generalized the concept of prime ideals in [2]. We recall from [2] that a nonzero proper ideal I of R is said to be a 2-absorbing ideal of R if whenever $a, b, c \in R$ and $abc \in I$, then either $ab \in I$ or $ac \in I$ or $bc \in I$ (also see [6]). In 2003, Anderson and Smith in [1] defined weakly prime ideals which is a generalization of prime ideals (also see [7]). A proper ideal P of a ring R is said to be a weakly prime if $0 \neq xy \in P$ for each $x, y \in R$ implies either $x \in P$ or $y \in P$. Recently, Yassine et. al. defined a new class of ideals, which is an intermediate class of ideals

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between prime ideals and 2-absorbing ideals. Recall from [15] that a proper ideal P of R is said to be a 1-absorbing prime ideal if for each nonunits $x, y, z \in R$ with $xyz \in P$, then either $xy \in P$ or $z \in P$ (also see [5]). Koc et. al. in [13] investigated weakly 1-absorbing prime ideals. A proper ideal I of R is said to be a weakly 1-absorbing prime ideal if whenever $0 \neq abc \in I$ for some nonunits $a, b, c \in R$, then $ab \in I$ or $c \in I$ (also see [3, 8, 14]). Let $\mathcal{L}$ be a bounded distributive lattice. Our objective in this paper is to extend the notion of weakly 1-absorbing property in commutative rings to weakly 1-absorbing property in the lattices, and to investigate the relations between weakly 1-absorbing prime elements, 1-absorbing prime elements and weakly prime elements. Among many results in this paper, the first, introduction section contains elementary observations needed later on.

In Section 2, we give basic properties of 1-absorbing prime elements. We recall from [11] that a proper element p of a lattice $\mathcal{L}$ is called prime (resp. weakly prime) if $p \leqslant x \vee y$ (resp. $p \leqslant x \vee y \neq 1$), then either $p \leqslant x$ or $p \leqslant y$. At first, we give definitions of a 1-*absorbing prime element* (Definition 2.1) and we give an example (Example 2.1) of a 1-absorbing prime element of $\mathcal{L}$ that is not a prime element. It is shown (Theorem 2.1) that if $\mathcal{L}$ admits a 1-absorbing prime element that is not a prime element, then $\mathcal{L}$ is a A-*lattice* (i.e. it has at most one atom element). An element $1 \neq a \in \mathcal{L}$ is said to be a *direct meet* of 1 if $1 = a \vee b$ and $a \wedge b = 0$ for some element b of $\mathcal{L}$. In this case we write $1 = a \oplus b$. It is shown (Theorem 2.2) that if $\mathcal{L}$ is a non-A-lattice, then every nontrivial element of $\mathcal{L}$ is a 1-absorbing prime element if and only if $1 = c_1 \oplus c_2$, where c_1 and c_2 are coatom elements of $\mathcal{L}$. It is proved (Theorem 2.4) that p is a 1-absorbing prime element of $\mathcal{L}$ if and only if for any proper elements a, b, c of $\mathcal{L}$ such that $p \leqslant a \vee b \vee c$ implies that either $p \leqslant a \vee b$ or $p \leqslant c$. In the rest of this section, we provide an example of lattices for which their 1-absorbing prime elements and prime elements are the same. It is proved (Theorem 2.5) that if $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ is a decomposable lattice, then p is a 1-absorbing prime element of $\mathcal{L}$ if and only if p is a prime element of $\mathcal{L}$.

Section 3 is dedicated to the investigate the basic properties of weakly 1-absorbing prime elements. At first, we define definition of *weakly* 1-*absorbing prime element* (Definition 3.1) and we give an example (Example 3.1) of a weakly 1-absorbing prime element of $\mathcal{L}$ that is not a 1-absorbing prime element (so it is not a prime element of $\mathcal{L}$). It is proved (Theorem 3.1) that p is a weakly 1-absorbing prime element of $\mathcal{L}$ if and only if for each proper elements a, b, c of $\mathcal{L}$ such that $p \leqslant a \vee b \vee c \neq 1$, either $p \leqslant a \vee b$ or $p \leqslant c$. It is shown that (Theorem 3.2) that if p is a weakly 1-absorbing prime element of a uniform lattice $\mathcal{L}$ that is not 1-absorbing prime, then $p = 1$. In the Theorem 3.3, we give a condition under which a weakly 1-absorbing prime element of $\mathcal{L}$ is not a 1-absorbing prime element. Theorem 3.4 determines the class of lattices for which their weakly 1-absorbing prime elements and 1-absorbing prime elements are the same. In the Theorem 3.5, we give a characterization of weakly 1-absorbing prime elements of decomposable lattices. Also, we characterize lattices with the property that all proper elements are weakly 1-absorbing prime (Theorem 3.7). In particular, we prove that if every proper element of a lattice $\mathcal{L}$ is a weakly 1-absorbing prime, then $|\mathcal{A}(\mathcal{L})| \leqslant 2$, where $\mathcal{A}(\mathcal{L})$ is the set of all atom elements of $\mathcal{L}$ (Theorem 3.8).

A *poset* $(\mathcal{L}, \leqslant)$ is a *lattice* if $\sup\{a,b\} = a \vee b$ and $\inf\{a,b\} = a \wedge b$ exist for all $a, b \in \mathcal{L}$ (and call $\wedge$ the *meet* and $\vee$ the *join*). A lattice $\mathcal{L}$ is *complete* when each of its subsets X has a least upper bound and a greatest lower bound in $\mathcal{L}$. Setting $X = \mathcal{L}$, we see that any nonvoid complete lattice contains a least element 0 and greatest element 1 (in this case, we say that $\mathcal{L}$ is a lattice with 0 and 1). A lattice $\mathcal{L}$ is called a *distributive* lattice if $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all a, b, c in $\mathcal{L}$ (equivalently, $\mathcal{L}$ is distributive if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all a, b, c in $\mathcal{L}$).

An element x of a lattice $\mathcal{L}$ is *nontrivial* (resp. *proper*) if $x \neq 0, 1$ (resp. $x \neq 1$). An element x of a lattice $\mathcal{L}$ is called *essential*, if there is no nonzero $y \in \mathcal{L}$ such that $x \wedge y = 0$. An element $u \in \mathcal{L}$ is called *uniform* if for every $x, y \in \mathcal{L}$ the following implication holds: if $0 < x \leqslant u$ and $0 < y \leqslant u$, then $x \wedge y \neq 0$ (i.e. all nonzero elements from $[0, u]$ are essential in $[0, u]$). A lattice $\mathcal{L}$ is called uniform if 1 is *uniform* in $\mathcal{L}$. It is clear that a lattice $\mathcal{L}$ is uniform if and only if every non-zero element of $\mathcal{L}$ is essential in $\mathcal{L}$ [4]. We say that an element x in a lattice $\mathcal{L}$ is an *atom* (resp. *coatom*) if there is no $y \in \mathcal{L}$ such that $0 < y < x$ (resp. $x < y < 1$). We will use $\mathcal{L}^*$ to denote the set of all non-zero elements of $\mathcal{L}$. For terminology and notation not defined here, the reader is referred to [4].

2. Some basic properties of 1-absorbing prime elements

In this section, we collect some basic properties concerning 1-absorbing prime elements. We remind the reader with the following definition.

Definition 2.1. *A proper element p of a lattice $\mathcal{L}$ is called* 1-absorbing prime *if for all $a, b, c \in \mathcal{L}^*$ such that $p \leqslant a \vee b \vee c$, then either $p \leqslant a \vee b$ or $p \leqslant c$.*

Clearly, every prime element of $\mathcal{L}$ is 1-absorbing prime. But generally these two classes are different, as the following example shows.

example 2.1. *Let $D = \{1, 2, \cdots, n\}$. Then $\mathcal{L} = \{X : X \subseteq D\}$ forms a distributive lattice under set inclusion with greatest element D and least element $\emptyset$ (note that if $x, y \in \mathcal{L}$, then $x \vee y = x \cup y$ and $x \wedge y = x \cap y$). Set $p = \{1, 2\}$. Then p is clearly a 1-absorbing prime element of $\mathcal{L}$. Since $p \leqslant \{1\} \vee \{2\}$, $p \nleqslant \{1\}$ and $p \nleqslant \{2\}$, we conclude that p is not a prime element of $\mathcal{L}$. Thus a 1-absorbing prime element need not be a prime element.*

Theorem 2.1. *If p is a 1-absorbing prime element of a lattice $\mathcal{L}$ that is not a prime element for some element p of $\mathcal{L}$, then $\mathcal{L}$ is a A-lattice.*

Proof. By the hypothesis, there are non-zero elements $b, c \in \mathcal{L}$ such that $p \leqslant b \vee c$, $p \nleqslant b$ and $p \nleqslant c$. On the contrary, assume that a_1 and a_2 be two distinct atom elements of $\mathcal{L}$. Then $a_1 \wedge a_2 \leqslant a_1$ gives $a_1 \wedge a_2 = 0$, as a_1 is an atom. Therefore $p \leqslant b \vee c \vee a_1$ and $p \nleqslant b$ implies that $p \leqslant c \vee a_1$, as p is a absorbing prime element. Similarly, $p \leqslant c \vee a_2$. It follows that $p \leqslant (c \vee a_1) \wedge (c \vee a_2) = c \vee (a_1 \wedge a_2) = c$ which is impossible, as required. □

Corollary 2.1. *Suppose that a lattice $\mathcal{L}$ is not a A-lattice. Then a proper element p of $\mathcal{L}$ is a 1-absorbing prime element of $\mathcal{L}$ if and only if p is a prime element of $\mathcal{L}$.*

Proof. This is a direct consequence of Theorem 2.1. □

Proposition 2.1. *For a lattice $\mathcal{L}$ the following hold:*
(1) Every atom element of $\mathcal{L}$ is a prime element;
(2) Every atom element of $\mathcal{L}$ is a 1*-absorbing prime element.*

Proof. (1) Let a be an atom element of $\mathcal{L}$. On the contrary, assume that a is not prime. Then there are elements $b, c \in \mathcal{L}$ such that $a \leqslant b \vee c$, $a \nleqslant b$ and $a \nleqslant c$. Since a is an atom element, we conclude that $a \wedge b = 0 = a \wedge c$ which implies that $a = a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) = 0$ which is impossible. So a is prime.

(2) This is a direct consequence of (1). □

In the following theorem, we characterize lattices with the property that all nontrivial elements are 1-absorbing prime.

Theorem 2.2. *Suppose that a lattice $\mathcal{L}$ is not a A-lattice. The following statements are equivalent:*
(1) Every nontrivial element of $\mathcal{L}$ is a 1*-absorbing prime element;*
(2) $1 = c_1 \oplus c_2$, where c_1 and c_2 are coatom elements of $\mathcal{L}$.

Proof. (1) $\Rightarrow$ (2) By assumption, suppose that c_1 and c_2 are two distinct atom elements of $\mathcal{L}$. Then $c_1 \wedge c_2 \leqslant c_1$ gives $c_1 \wedge c_2 = 0$. We claim that $c_1 \vee c_2 = 1$. On the contrary, assume that $c_1 \vee c_2 \neq 1$. Then by using Corollary 2.1 and our hypothesis, we conclude that $c_1 \vee c_2$ is a prime element of $\mathcal{L}$. Then $c_1 \vee c_2 \leqslant c_1 \vee c_2$ gives $c_1 = c_2$ which is impossible. So $c_1 \vee c_2 = 1$ and then $1 = c_1 \oplus c_2$. It remains to show that c_1 and c_2 are coatom elements. If c_1 is not coatom, then there exists a nontrivial element c of $\mathcal{L}$ such that $c_1 < c$ with $c_2 \nleqslant c$ (so $c_2 \wedge c = 0$) and $c \vee c_2 = 1$. Hence, $c = c \wedge 1 = c \wedge (c_1 \vee c_2) = (c \wedge c_1) \vee (c \wedge c_2) = c_1$ which is a contradiction. Similarly, c_2 is a coatom element, i.e. (2) holds.

(2) $\Rightarrow$ (1) Let $1 = c_1 \oplus c_2$, where c_1 and c_2 are coatom elements of $\mathcal{L}$. At first, we show that c_1, c_2 are atom elements. If c_1 is not atom, there is a nontrivial element s of $\mathcal{L}$ such that $s < c_1$ with $s \nleqslant c_2$ (so $s \vee c_2 = 1$) and $s \wedge c_2 = 0$ which gives $c_1 = c_1 \wedge (s \vee c_2) = (c_1 \wedge s) \vee (c_1 \wedge c_2) = s$, a contradiction Thus c_1 is an atom. Likewise, c_2 is an atom. Let x be a nontrivial element of $\mathcal{L}$. Then $x = x \wedge (c_1 \vee c_2) = (c_1 \wedge x) \vee (c_2 \wedge x)$. Since $x \neq 0, 1$, we conclude that either $c_1 \leqslant x$ and $c_2 \wedge x = 0$ or $c_2 \leqslant x$ and $c_1 \wedge x = 0$; hence either $x = c_1$ or $x = c_2$. Therefore, x is a 1-absorbing prime element by Proposition 2.1. □

We say that a subset $S \subseteq \mathcal{L}$ is *join* if $0 \in S$ and $s_1 \vee s_2 \in S$ for all $s_1, s_2 \in S$. Clearly, if p is a non-zero prime element of $\mathcal{L}$, then $\mathcal{L} \smallsetminus \{x \in \mathcal{L} : p \leqslant x\}$ is a join subset of $\mathcal{L}$. Let $c \in \mathcal{L}$. We say that $S \wedge c = 0$ if $s \wedge c = 0$ for every $s \in S$.

Theorem 2.3. *For each nontrivial element p of a complete lattice $\mathcal{L}$ and a join subset S with $S \wedge p = 0$, there exists a* 1*-absorbing prime element $m = m(p, S)$ of $\mathcal{L}$ that satisfies $m \leqslant p$ is constructible.*

Proof. If $\Delta = \{x \in \mathcal{L} : x \leqslant p \text{ and } S \wedge x = 0\}$, then $p \in \Delta$, and so $\Delta \neq \emptyset$. Moreover, $(\Delta, \geqslant)$ is a partial order and Δ is inductive. Indeed, if $\{x_i\}_{i \in I}$ is a chain of elements of Δ, then $p' = \bigwedge_{i \in I} x_i \in \Delta$ is an upper bound for the chain. Then by Zorn's Lemma, Δ has a maximal element for $\geqslant$ and so there exists a minimal element (so an atom element) m such that $m \leqslant p$. Now the assertion follows from Proposition 2.1. $\square$

LEMMA 2.1. *Let p be a 1-absorbing prime element of $\mathcal{L}$. If $p \leqslant a \vee b \vee c$ for some $a, b \in \mathcal{L}^*$ and a proper element c of $\mathcal{L}$, then $p \leqslant a \vee b$ or $p \leqslant c$.*

PROOF. Suppose on the contrary that $p \leqslant a \vee b \vee c$, but $p \nleqslant a \vee b$ and $p \nleqslant c$. Then $p \leqslant a \vee b \vee c$ and $p \nleqslant a \vee b$ gives $c \neq 0$. Since p is a 1-absorbing prime element and $a, b, c \in \mathcal{L}^*$, we conclude that $p \leqslant a \vee b$ or $p \leqslant c$ which is impossible. $\square$

THEOREM 2.4. *Let p be a proper element of $\mathcal{L}$. The following statements are equivalent:*

(1) p is a 1-absorbing prime element of $\mathcal{L}$;

(2) For any proper elements a, b, c of $\mathcal{L}$ such that $p \leqslant a \vee b \vee c$ implies that either $p \leqslant a \vee b$ or $p \leqslant c$.

PROOF. (1) $\Rightarrow$ (2) Suppose that $p \leqslant a \vee b \vee c$ for some proper elements a, b, c of $\mathcal{L}$ and $p \nleqslant a \vee b$; so $c \neq 0$. If $a = 0 = b$, we are done. So either $a \neq 0$ or $b \neq 0$. We may assume that $a \neq 0$. Since $a, c \in \mathcal{L}^*$ and $p \leqslant a \vee c \vee b$, we conclude that either $p \leqslant a \vee c$ or $p \leqslant b \leqslant a \vee b$ by Lemma 2.1; hence $p \leqslant a \vee a \vee c$. It then follows from (1) that either $p \leqslant a \leqslant a \vee b$ or $p \leqslant c$ which gives $p \leqslant c$. The implication (2) $\Rightarrow$ (1) is clear. $\square$

Assume that $(\mathcal{L})_1, \leqslant_1), (\mathcal{L})_2, \leqslant_2)$ are lattices and let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$. We set up a partial order $\leqslant_c$ on $\mathcal{L}$ as follows: for each $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{L}$, we write $x \leqslant_c y$ if and only if $x_i \leqslant_i y_i$ for each $i \in \{1, 2\}$. The following notation below will be used in this paper: It is straightforward to check that $(\mathcal{L}, \leqslant_c)$ is a lattice with $x \vee_c y = (x_1 \vee y_1, x_2 \vee y_2)$ and $x \wedge_c y = (x_1 \wedge y_1, x_2 \wedge y_2)$. In this case, we say that $\mathcal{L}$ is a *decomposable lattice*.

PROPOSITION 2.2. *Suppose that $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ is a decomposable lattice and let $p = (p_1, p_2)$ be a proper element of $\mathcal{L}$. Then p is a prime element of $\mathcal{L}$ if and only if $p = (p_1, 0)$ for some prime element p_1 of $\mathcal{L}_1$ or $p = (0, p_2)$ for some prime element p_2 of $\mathcal{L}_2$.*

PROOF. Suppose that p is a prime element of $\mathcal{L}$. Then $p \leqslant (p_1, 0) \vee_c (0, p_2) = (p_1, p_2)$ gives either $p \leqslant (p_1, 0)$ or $p \leqslant (0, p_2)$; hence $p_2 = 0$ or $p_1 = 0$. Thus either $p = (p_1, 0)$ or $p = (0, p_2)$. Without loss of generality, we can assume that $p = (p_1, 0)$. Let $p_1 \leqslant a \vee b$ for some $a, b \in \mathcal{L}_1$. Then $p \leqslant (a \vee b, 0) = (a, 0) \vee_c (b, 0)$ gives either $p \leqslant (a, 0)$ or $p \leqslant (b, 0)$ which implies that either $p_1 \leqslant a$ or $p_1 \leqslant b$. Thus p_1 is a prime element. Conversely, suppose that either $p = (p_1, 0)$ or $p = (0, p_2)$, where p_1 is prime in $\mathcal{L}_1$ and p_2 is prime in $\mathcal{L}_2$. Let $p \leqslant (x, y) \vee_c (x', y') = (x \vee x', y \vee y')$ for some $(x, y), (x', y') \in \mathcal{L}$. We can assume that $p = (p_1, 0)$. Then $p_1 \leqslant x \vee x'$ gives either $p_1 \leqslant x$ or $p_1 \leqslant x'$ which implies that either $p \leqslant (x, y)$ or $p \leqslant (x', y')$; so p is prime. $\square$

PROPOSITION 2.3. *Suppose that $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ is a decomposable lattice and let $a = (a_1, a_2)$ be a proper element of $\mathcal{L}$. Then a is an atom element of $\mathcal{L}$ if and only* *if $a = (a_1, 0)$ for some atom element a_1 of $\mathcal{L}_1$ or $a = (0, a_2)$ for some atom element a_2 of $\mathcal{L}_2$.*

Proof. If a is an atom element of $\mathcal{L}$, then by Proposition 2.1, a is a prime element of $\mathcal{L}$ and so either $a = (a_1, 0)$ or $a = (0, a_2)$ by Proposition 2.2. Without loss of generality, we can assume that $a = (a_1, 0)$. Let $0 < s \leqslant a_1$. Then a is an atom element of $\mathcal{L}$ and $(0, 0) \leqslant_c (s, 0) \leqslant_c (a_1, 0)$ gives $a_1 = s$. Thus a_1 is an atom element of $\mathcal{L}_1$. The converse is similar. □

By our previous result, it can be easily seen that every decomposable lattice is not a A-lattice.

Corollary 2.2. *If $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ for some lattices $\mathcal{L}_1$ and $\mathcal{L}_2$, then a proper element p of $\mathcal{L}$ is a 1-absorbing prime element of $\mathcal{L}$ if and only if p is a prime element of $\mathcal{L}$.*

Proof. This is a direct consequence of Corollary 2.1. □

In view of Corollary 2.2 and Proposition 2.2, we have the following result.

Theorem 2.5. *Suppose that $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ is a decomposable lattice and let p be a proper element of $\mathcal{L}$. The following statements are equivalent:*
(1) p is a 1-absorbing prime element of $\mathcal{L}$;
(2) p is a prime element of $\mathcal{L}$;
(3) $p = (p_1, 0)$ for some prime element p_1 of $\mathcal{L}_1$ or $p = (0, p_2)$ for some prime element p_2 of $\mathcal{L}_2$.

3. Characterization of weakly 1-absorbing prime elements

In this section, the concept of weakly 1-absorbing prime element is introduced and investigated. We remind the reader with the following definition.

Definition 3.1. *A proper element p of $\mathcal{L}$ is called* weakly 1-absorbing prime *if for all $x, y, z \in \mathcal{L}^*$ such that $p \leqslant x \vee y \vee z \neq 1$, then either $p \leqslant x \vee y$ or $p \leqslant z$.*

example 3.1. *(1) It is easy to see that every 1-absorbing prime element is a weakly 1-absorbing prime element.*
(2) Let $D = \{a, b, c\}$. Then $\mathcal{L} = \{X : X \subseteq D\}$ forms a distributive lattice under set inclusion with greatest element D and least element $\emptyset$. Then 1 is clearly a weakly 1-absorbing prime element of $\mathcal{L}$. Since $1 \leqslant \{a\} \vee \{b\} \vee \{c\}$, $1 \nleqslant \{a\} \vee \{b\}$ and $1 \nleqslant \{c\}$, it follows that 1 is not a 1-absorbing prime element of $\mathcal{L}$. Thus a weakly 1-absorbing prime element need not be a 1-absorbing prime element.

Proposition 3.1. *Let p be a weakly 1-absorbing prime element of $\mathcal{L}$. If 1 is a 1-absorbing prime element, then p is a 1-absorbing prime element.*

Proof. Let $p \leqslant a \vee b \vee c$ for some $a, b, c \in \mathcal{L}^*$. If $a \vee b \vee c \neq 1$, then we have either $p \leqslant a \vee b$ or $p \leqslant c$. So assume that $1 \leqslant a \vee b \vee c$. Since 1 is a 1-absorbing prime element, we conclude that either $p \leqslant 1 \leqslant a \vee b$ or $p \leqslant 1 \leqslant c$, as needed. □

In the following theorem we give four other characterizations of weakly 1-absorbing prime elements.

THEOREM 3.1. *Let p be a proper element of a lattice $\mathcal{L}$. The following statements are equivalent:*

(1) p is a weakly 1-absorbing prime element of $\mathcal{L}$;

(2) For each $x, y \in \mathcal{L}^*$ with $p \nleqslant x \vee y$, $p \leqslant x \vee y \vee z$ if and only if $p \leqslant z$ or $x \vee y \vee z = 1$ for every proper element $z \in \mathcal{L}$;

(3) For each $x, y \in \mathcal{L}^*$ and a proper element z of $\mathcal{L}$ such that $p \leqslant x \vee y \vee z \neq 1$, either $p \leqslant x \vee y$ or $p \leqslant z$;

(4) For each $x \in \mathcal{L}^*$ and proper elements y, z of $\mathcal{L}$ such that $p \leqslant x \vee y \vee z \neq 1$, either $p \leqslant x \vee y$ or $p \leqslant z$;

(5) For each proper elements x, y, z of $\mathcal{L}$ such that $p \leqslant x \vee y \vee z$, either $p \leqslant x \vee y$ or $p \leqslant z$.

PROOF. (1) $\Rightarrow$ (2) One side is clear. To see the other side, assume that $p \leqslant x \vee y \vee z$, where $z \in \mathcal{L}$. Since $p \nleqslant x \vee y$ and $p \leqslant x \vee y \vee z$, we conclude that $z \neq 0$. If $x \vee y \vee z = 1$, then we are done. So suppose that $x \vee y \vee z \neq 1$. Since $p \nleqslant x \vee y$, $p \leqslant x \vee y \vee z$ and p is a weakly 1-absorbing prime element, we have $p \leqslant z$, and so (2) holds.

(2) $\Rightarrow$ (3) Let $p \leqslant x \vee y \vee z \neq 1$ for some $x, y \in \mathcal{L}^*$ and a proper element z of $\mathcal{L}$. If $p \leqslant x \vee y$, then we are done. So suppose that $p \nleqslant x \vee y$. Then by (2), we have either $p \leqslant z$ or $x \vee y \vee z = 1$ and this shows that $p \leqslant z$, i.e. (3) holds.

(3) $\Rightarrow$ (4) Let $p \leqslant x \vee y \vee z \neq 1$ for some $x \in \mathcal{L}^*$ and proper elements y, z of $\mathcal{L}$. On the contrary, assume that $p \nleqslant x \vee y$ and $p \nleqslant z$. Then $p \leqslant x \vee y \vee z \neq 1$ implies that $z \neq 0$ and $x \vee z \neq 1$. We claim that $p \nleqslant x \vee z$. Otherwise, $p \leqslant x \vee x \vee z \neq 1$ gives either $p \leqslant x$ or $p \leqslant z$ by (3) which is a contradiction. Thus $p \nleqslant x \vee z$. Now, since $p \leqslant x \vee y \vee z \neq 1$ and $x, z \in \mathcal{L}^*$, we conclude that either $p \leqslant x \vee z$ or $p \leqslant y \leqslant x \vee y$ which is impossible, i.e. (4) holds.

(4) $\Rightarrow$ (5) Let $p \leqslant x \vee y \vee z \neq 1$ for some proper elements x, y, z of $\mathcal{L}$. On the contrary, assume that $p \nleqslant x \vee y$ and $p \nleqslant z$. This shows that $z \neq 0$. Since $p \leqslant z \vee x \vee y \neq 1$ and $z \neq 0$, we conclude that either $p \leqslant x \vee z$ or $p \leqslant y \leqslant x \vee y$ by (4); hence $p \leqslant x \vee z$. Clearly, $x \vee z \neq 1$ and $x \neq 0$. Then $p \leqslant x \vee x \vee z \neq 1$ gives either $p \leqslant x \leqslant x \vee y$ or $p \leqslant z$ by (4), a contradiction. Thus either $p \leqslant x \vee y$ or $p \leqslant z$. The implication (5) $\Rightarrow$ (1) is clear. $\square$

PROPOSITION 3.2. *Assume that p is a weakly 1-absorbing prime element of an uniform lattice $\mathcal{L}$ and let there exist $a, b, c \in \mathcal{L}^*$ such that $a \vee b \vee c = 1$, $p \nleqslant a \vee b$ and $p \nleqslant c$. The following assertions hold:*

(1) $a \vee b \vee p = a \vee c \vee p = b \vee c \vee p = 1$.

(2) $a \vee p = c \vee p = b \vee p = 1$.

PROOF. (1) On the contrary, assume that $a \vee b \vee p \neq 1$. Then $p \leqslant (a \vee b) \vee (c \wedge p) = a \vee b \vee p \neq 1$. Since $a, b, c \wedge p \in \mathcal{L}^*$, $p \nleqslant a \vee b$ and p is a weakly 1-absorbing prime element, we conclude that $p \leqslant p \wedge c \leqslant c$, a contradiction. Thus $a \vee b \vee p = 1$. Now suppose that $a \vee c \vee p \neq 1$. Therefore $p \leqslant a \vee (b \wedge p) \vee c = a \vee c \vee p \neq 1$. Since $a, c, b \wedge p \in \mathcal{L}^*$ and $p \nleqslant c$, we have $p \leqslant a \vee (b \wedge p) = (a \vee b) \wedge (a \vee p) \leqslant a \vee b$, a contradiction. Thus $a \vee c \vee p = 1$. Likewise, $b \vee c \vee p = 1$.

(2) Suppose that $a \vee p \neq 1$. Since $a \vee (b \wedge p) \vee (c \wedge p) = (a \vee p) \wedge (a \vee b \vee c) = a \vee p \neq 1$, $a, b \wedge p, c \wedge p \in \mathcal{L}^*$ and p is a weakly 1-absorbing prime element, we conclude that

either $p \leqslant a \vee (b \wedge p) = (a \vee b) \wedge (a \vee p)$ or $p \leqslant c \wedge p$. Hence either $p \leqslant a \vee b$ or $p \leqslant c$ which is impossible. Thus $a \vee p = 1$. Similarly, $b \vee p = c \vee p = 1$. □

THEOREM 3.2. *If p is a weakly 1-absorbing prime element of a uniform lattice $\mathcal{L}$ that is not 1-absorbing prime, then $p = 1$.*

PROOF. Since p is a weakly 1-absorbing prime element of $\mathcal{L}$ that is not 1-absorbing prime, there exist $a, b, c \in \mathcal{L}^*$ such that $a \vee b \vee c = 1$, $p \nleqslant a \vee b$ and $p \nleqslant c$. On the contrary, assume that $p \neq 1$. Since $(a \wedge p) \vee (b \wedge p) \vee (c \wedge p) = p \wedge (a \vee b \vee c) = p \neq 1$, $p \wedge a, p \wedge b, p \wedge c \in \mathcal{L}^*$ and p is a weakly 1-absorbing prime filter, we conclude that either either $p \leqslant (a \wedge p) \vee (b \wedge p) = p \wedge (a \vee b)$ or $p \leqslant c \wedge p$, and so either $p \leqslant a \vee b$ or $p \leqslant c$, a contradiction. Therefore $p = 1$. □

A lattice $\mathcal{L}$ with 1 is called a *$\mathcal{L}$-domain* if $a \vee b = 1$ $(a, b \in \mathcal{L})$, then either $a = 1$ or $b = 1$. Clearly, a lattice $\mathcal{L}$ is a $\mathcal{L}$-domain if and only if 1 is prime.

PROPOSITION 3.3. *If $p \neq 1$ is an element of a $\mathcal{L}$-domain $\mathcal{L}$, then p is a weakly 1-absorbing prime element if and only if p is a 1-absorbing prime element.*

PROOF. One side is clear. To see the other side, assume that p is a weakly 1-absorbing prime element of $\mathcal{L}$ and $p \leqslant a \vee b \vee c$ for some $a, b, c \in \mathcal{L}^*$. If $a \vee b \vee c \neq 1$, then either $p \leqslant a \vee b$ or $p \leqslant c$. So we may assume that $a \vee b \vee c = 1$ and $p \nleqslant c$. Since $a \vee b \vee c = 1$ which is a prime element, we conclude that $p \leqslant 1 = a \vee b$. □

In the following theorem, we give a condition under which a weakly 1-absorbing prime element of $\mathcal{L}$ is not a 1-absorbing prime element.

THEOREM 3.3. *Let p be a weakly 1-absorbing prime element of an uniform lattice $\mathcal{L}$ and there exist $a, b, c \in \mathcal{L}^*$ such that $a \vee b \vee c = 1$, $a \vee b \neq 1$ and $c \neq 1$. Then p is not a 1-absorbing prime element if and only if $p = 1$.*

PROOF. If p is not a 1-absorbing prime element, then $p = 1$ by Theorem 3.2. Conversely, assume that $p = 1$. By the hypothesis, $p \leqslant a \vee b \vee c$, $p \nleqslant a \vee b$ and $p \nleqslant c$ gives p is not a 1-absorbing prime element. □

In the following results we show that weakly 1-absorbing prime elements are really of interest in indecomposable lattices.

PROPOSITION 3.4. *Suppose that $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ is a decomposable lattice and p_1 is a proper element of $\mathcal{L}_1$. Then the following statements are equivalent.*

(1) $(p_1, 0)$ is a weakly 1-absorbing prime element of $\mathcal{L}$;

(2) p_1 is a 1-absorbing prime element of $\mathcal{L}_1$;

(3) $(p_1, 0)$ is a 1-absorbing prime element of $\mathcal{L}$.

PROOF. (1) $\Rightarrow$ (2) Suppose that $p_1 \leqslant x \vee y \vee z$ for some $x, y, z \in \mathcal{L}_1^*$. If $1 \neq t \in \mathcal{L}_2$, then $(p_1, t) \leqslant (x, 0) \vee_c (y, 0) \vee_c (z, t) = (x \vee y \vee z, t) \neq (1, 1)$, and so either $(p_1, t) \leqslant (x, 0) \vee_c (y, 0) = (x \vee y, 0)$ or $(p_1, t) \leqslant (z, s)$. Hence, either $p_1 \leqslant x \vee y$ or $p_1 \leqslant z$. The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are clear. □

In the next theorem, we provide an example of lattices for which their 1-absorbing prime elements and weakly 1-absorbing prime elements are the same.

THEOREM 3.4. *Suppose that $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \mathcal{L}_3$ is a decomposable lattice and p is a nontrivial element of $\mathcal{L}$. Then p is a weakly 1-absorbing prime element if and only if p is a 1-absorbing prime element.*

PROOF. One side is clear. To see the other side, assume that $(1,1,1) \neq p = (p_1, p_2, p_3)$ is a weakly 1-absorbing prime element of $\mathcal{L}$. Since $p \leqslant (p_1, 0, 0) \vee_c (0, p_2, 0) \vee_c (0, 0, p_3)$ and p is a weakly 1-absorbing prime element, we conclude that either $p \leqslant (p_1, 0, 0) \vee_c (0, p_2, 0) = (p_1, p_2, 0)$ or $p \leqslant (0, 0, p_3)$. Therefore either $p_3 = 0$ or $p_1 = 0$ and $p_2 = 0$, and so $p = (p_1, p_2, 0)$ or $p = (0, 0, p_3)$. Hence, by Proposition 3.4, p is 1-absorbing prime. □

THEOREM 3.5. *Suppose that $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ is a decomposable lattice such that $\mathcal{L}_1, \mathcal{L}_2$ are not simples and let p be a nontrivial element of $\mathcal{L}$. The following statements are equivalent.*

(1) p is a weakly 1-absorbing prime element of $\mathcal{L}$;

(2) $p = (p_1, 0)$ for some prime element p_1 of $\mathcal{L}_1$ or $p = (0, p_2)$ for some prime element p_2 of $\mathcal{L}_2$;

(3) p is a prime element of $\mathcal{L}$;

(4) p is a weakly prime element of $\mathcal{L}$;

(5) p is a 1-absorbing prime element of $\mathcal{L}$.

PROOF. (1) ⇒ (2) Let $p = (p_1, p_2)$ be a nontrivial element of $\mathcal{L}$. Since $p \neq 1$, either $p_1 \neq 1$ or $p_2 \neq 1$. Without loss of generality, we may assume that $p_1 \neq 1$. Since p is a weakly 1-absorbing prime element and $p \leqslant (0,1) \vee_c (0,1) \vee_c (p_1, 0) = (p_1, 1) \neq (1,1)$, we conclude that either $p \leqslant (0,1)$ or $p \leqslant (p_1, 0)$ which implies that $p_1 = 0$ or $p_2 = 0$. Suppose that $p_1 = 0$. Now we will show that p_2 is a prime element of $\mathcal{L}_2$. Let $p \leqslant a \vee b$ for some $a, b \in \mathcal{L}_2$. If $a = 0$ or $b = 0$, then we are done. So suppose that $a, b \in \mathcal{L}_2^*$. Since $\mathcal{L}_1$ is not a simple lattice, there exists a non-zero element $s \in \mathcal{L}_1$ such that $s \neq 1$. This implies that $p \leqslant (s, 0) \vee_c (0, a) \vee_c (0, b) = (s, a \vee b) \neq (1,1)$. Since p is a weakly 1-absorbing prime element, we conclude that either $p \leqslant (s, 0) \vee_c (0, a) = (s, a)$ or $p \leqslant (0, b)$. Therefore we obtain that $p_2 \leqslant a$ or $p_2 \leqslant b$ and so p_2 is a prime element of $\mathcal{L}_2$. Similarly, we can show that $p = (p_1, 0)$ and p_1 is a prime element of $\mathcal{L}_1$.

(2) ⇒ (3) Without loss of generality, we may assume that $p = (p_1, 0)$, where p_1 is prime in $\mathcal{L}_1$. Let $p \leqslant (a, b) \vee_c (c, d) = (a \vee c, b \vee d)$ for some $(a, b), (c, d) \in \mathcal{L}$. Then $p_1 \leqslant a \vee c$ gives either $p_1 \leqslant a$ or $p_1 \leqslant c$ which implies that either $p \leqslant (a, b)$ or $p \leqslant (c, d)$.

(3) ⇔ (4) Clearly, every prime element is a weakly prime element. Conversely, suppose that $(1,1) \neq p = (p_1, p_2)$ is a weakly prime element of $\mathcal{L}$. Then $p \leqslant (p_1, 0) \vee_c (0, p_2) = (p_1, p_2) \neq (1,1)$ gives $p \leqslant (p_1, 0)$ or $p \leqslant (0, p_2)$. If $p \leqslant (p_1, 0)$, then $p_2 = 0$ which implies that $p = (p_1, 0)$. We show that p_1 is a prime element of $\mathcal{L}_1$; hence p is a prime element of $\mathcal{L}$. Let $p_1 \leqslant x \vee y$, where $x, y \in \mathcal{L}_1$. If $x = 1$ or $y = 1$, then we are done. So assume that $x \neq 1$ and $y \neq 1$. Then $p \leqslant (x, 0) \vee_c (y, 0) = (x \vee y, 0) \neq (1,1)$, so $p \leqslant (x, 0)$ or $p \leqslant (y, 0) \in$ and hence $p_1 \leqslant x$ or $p_1 \leqslant y$. The case where $p \leqslant (0, b)$ is similar. The implication (3) ⇒ (5) is clear by definition of a 1-absorbing prime filter. The implication (5) ⇒ (1) is clear by definition of a weakly 1-absorbing prime filter. □

THEOREM 3.6. *Let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ be a decomposable lattice, p_1 is a nontrivial element of $\mathcal{L}_1$ and p_2 is a proper element of $\mathcal{L}_2$. If (p_1, p_2) is a weakly 1-absorbing prime element of $\mathcal{L}$ that is not a 1-absorbing prime element, then p_1 is a weakly prime element of $\mathcal{L}_1$ that is not a prime element and $p_2 = 1$ is a prime element of $\mathcal{L}_2$.*

PROOF. Suppose that $p = (p_1, p_2)$ has the stated property and $p_2 \neq 1$. Therefore, by Theorem 3.11, (p_1, p_2) is a 1-absorbing prime element of $\mathcal{L}$ which is impossible, and so $p_2 = 1$. Suppose that $p_2 \leqslant a \vee b$ for some $a, b \in \mathcal{L}_2$. Then $p \leqslant (p_1, 0) \vee_c (0, a) \vee_c (0, b) = (p_1, a \vee b) \neq (1, 1)$. But p_1 is a nontrivial element gives $p \nleqslant (0, a)$ and $p \nleqslant (0, b)$. We may assume that $a, b \in \mathcal{L}_2^*$. Since p is a weakly 1-absorbing prime element, we conclude that $p \leqslant (c, 0) \vee_c (0, a) = (c, a)$; hence $p_2 = 1$ is a prime element of $\mathcal{L}_2$. Now, we show that p_1 is a weakly prime element of $\mathcal{L}_1$. Let $p_1 \leqslant a \vee b \neq 1$ for some $a, b \in \mathcal{L}_1$. We can assume that $a, b \in \mathcal{L}_1^*$. Since $p \leqslant (b, 0) \vee_c (0, 1) \vee_c (a \vee b, 0) = (a \vee b, 1) \neq (1, 1)$, $(p_1, p_2) = (p_1, 1) \nleqslant (a \vee b, 0)$ and p is a weakly 1-absorbing prime element, we conclude that $(p_1, 1) \leqslant (b, 1)$, and so $p_1 \leqslant b$. Hence p_1 is a weakly prime element of $\mathcal{L}_1$. It remains to show that p_1 is not a prime element. On the contrary, assume that p_1 is a prime element. Since p_1 is a nontrivial element, $p_1 \neq 1$. Then $p \leqslant (p_1, 0) \vee_c (p_1, 0) \vee_c (0, 1) = (p_1, 1) \neq (1, 1)$ gives either $p \leqslant (P_1, 0) \vee_c (p_1, 0) = (p_1, 0)$ or $p \leqslant (0, 1)$ which is impossible. □

The following remark shows that the converse of Theorem 3.6 need not be true.

REMARK 3.1. Let $\mathcal{L}, \mathcal{L}_1, \mathcal{L}_2$ and p_1, p_2 be as in Theorem 3.6. Suppose that p_1 is a weakly prime element of $\mathcal{L}_1$ that is not a prime element and $p_2 = 1$ is a prime element of $\mathcal{L}_2$. We claim that $p = (p_1, p_2)$ need not be a weakly 1-absorbing prime element of $\mathcal{L}$. Since p_1 is a nontrivial element, $p_1 \neq 1$, and so $p \leqslant (p_1, 0) \vee_c (p_1, 0) \vee_c (0, 1) = (p_1, 1) \neq (1, 1)$. Since $p \nleqslant (p_1, 0) \vee_c (p_1, 0) = (p_1, 0)$ and $p \nleqslant (0, 1)$ (as p_1 is nontrivial), we conclude that p is not a weakly 1-absorbing prime element of $\mathcal{L}$.

THEOREM 3.7. *Let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$ be a decomposable filter ($n \geqslant 2$). The following statements are equivalent.*

(1) Every proper element of $\mathcal{L}$ is a weakly 1-absorbing prime element;

(2) $n = 2$ and for each $i \in \{1, 2\}$, $|\mathcal{L}_i| = 2$.

PROOF. (1) $\Rightarrow$ (2) On the contrary, assume that $n \geqslant 3$. Set $p = (1, 1, 0, \cdots, 0)$. Consider $1 \neq a \in \mathscr{L}_3$. Since

$$p \leqslant (0, 1, 0, 0, \cdots, 0) \vee_c (0, 1, 0, 0, \cdots, 0) \vee_c (1, 0, a, 0, 0, \cdots 0) =$$

$(1, 1, a, 0, 0, \cdots, 0) \neq (1, 1, \cdots, 1)$ and p is a weakly 1-absorbing prime element by (1), we conclude that either $p \leqslant (0, 1, 0, 0, \cdots, 0)$ or

$$p \leqslant (1, 0, a, 0, 0, \cdots, 0)$$

which both of them are contradictions. Hence $n = 2$ and $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$. Now, we will show that $|\mathcal{L}_1| = 2$. It is enough to show that if $s \in \mathcal{L}_1$, then either $s = 0$ or $s = 1$. Assume to the contrary, that $0 < s < 1$. Suppose that $q = (s, 1)$. Since q is a weakly 1-absorbing prime element by (1), $q \leqslant (s, 0) \vee_c (s, 0) \vee_c (0, 1) = (s, 1) \neq (1, 1)$

and both $q \nleq (s,0) \vee_c (s,0) = (s,0)$ and $q \nleq (0,1)$, we have a contradiction. Hence $|\mathcal{L}_1| = 2$. Likewise, $|\mathcal{L}_2| = 2$.

(2) $\Rightarrow$ (1) Suppose that $n = 2$ and for each $i \in \{1,2\}$, $|\mathcal{L}_i| = 2$. Let p be a proper element of $\mathcal{L}$. Then $\mathcal{L}$ has exactly three proper elements, i.e., $(0,0)$, $(1,0)$ and $(0,1)$. If $p = (1,0)$ or $p = (0,1)$, then p is a weakly 1-absorbing prime element by Theorem 3.5. If $p = (0,0)$, then p is trivially a weakly 1-absorbing prime element of $\mathcal{L}$. □

We close this section with the following theorem:

Theorem 3.8. *If every proper element of a lattice $\mathcal{L}$ is a weakly 1-absorbing prime, then $|\mathcal{A}(\mathcal{L})| \leqslant 2$.*

Proof. Let $\mathcal{L}$ be a lattice such that every proper element is weakly 1-absorbing prime. On the contrary, assume that $|\mathcal{A}(\mathcal{L})| \geqslant 3$. We suppose that a_1, a_2 and a_3 are distinct atom elements of $\mathcal{L}$, and look for a contradiction. We split the proof into two cases.

Case 1: Suppose that $p = a_1 \vee a_2 \vee a_3 \neq 1$. Since $p \subseteq a_1 \vee a_2 \vee a_3$ and p is a weakly 1-absorbing prime element, we conclude that either $a_3 \leqslant p \leqslant a_1 \vee a_2$ or $a_1 \leqslant p \leqslant a_3$. This shows that either $a_3 = a_1$ or $a_3 = a_2$ by Proposition 2.5 which is impossible.

Case 2: Suppose that $p = 1$. We claim that $q = a_1 \vee a_2 = 1$. Assume to the contrary, that $p \neq 1$. Since $q \leqslant a_1 \vee a_1 \vee a_2$ and q is a weakly 1-absorbing prime element, we have either $q \leqslant a_1$ or $q \leqslant a_2$ which implies that $a_1 = a_2$, a contradiction. Thus $a_1 \vee a_2 = 1$ and $a_1 \wedge a_2 = 0$. Since $a_3 \wedge a_1 = 0 = a_3 \wedge a_2$, we conclude that $a_3 = a_3 \wedge (a_1 \vee a_2) = 0$ which is a contradiction, as required. □

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