BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS/BULLETIN
Bull. Int. Math. Virtual Inst., **14**(2)(2024), 347–354
DOI: 10.7251/BIMVI2402347D

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

SOLUTION OF HERMITE AND LAGUERRE DIFFERENTIAL EQUATIONS BY FOURIER TRANSFORM

Murat Düz

Abstract. In this study, we tried to solve the Hermite and Laguerre differential equations, whose solutions are orthogonal polynomials, with the Fourier transform.

1. Introduction

The differential equation given as

$$y''-2xy'+2ny=0 \tag{1}$$

is called Hermite differential equation. Solutions of Hermite differential equation is called Hermite polynomial and is shown $H_n(x)$. Hermite polynomials are a classical orthogonal polynomial sequence. First few terms of Hermite polynomials

$$H_0(x)=1$$
$$H_1(x)=2x$$
$$H_2(x)=4x^2-2$$
$$H_3(x)=8x^3-12x$$
$$H_4(x)=16x^4-48x^2+12$$
$$H_5(x)=32x^5-160x^3+120x$$

As can be seen, every $H_n(x)$ is a polynomial of degree n. Hermite polynomials satisfy the following equalities:

$$H_{n+1}(x)=2xH_n(x)-(H_n(x))'$$

2020 *Mathematics Subject Classification.* Primary 33C45.
Key words and phrases. Fourier transform, orthogonal polynomials.
Communicated by Dusko Bogdanic.

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$
$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}$$

Hermite polynomials are widely used in various fields of mathematics, physics, engineering due to their properties. For example, the polynomials play a central role in the calculation of Gaussian integrals. In probability theory, the polynomials are used to compute moments of distributions. In control theory, the polynomials are used in the analysis of system stability and transfer functions. In signal processing, the polnomials are used to construct wavelet transforms. The polynomials are used in Fourier analysis for their orthogonal properties.

The differential equation given as

$$xy''+(1-x)y'+ny = 0 \tag{2}$$

is called Laguerre differential equation. Solutions of Laguerre differential equation are called Laguerre polynomials, denoted by $L_n(x)$. First few terms of Laguerre polynomials

$$L_0(x) = 1$$
$$L_1(x) = -x + 1$$
$$L_2(x) = \frac{1}{2}(x^2 - 4x + 2)$$
$$L_3(x) = \frac{1}{6}(-x^3 + 9x^2 - 18x + 6)$$
$$L_4(x) = \frac{1}{24}(x^4 - 16x^3 + 72x^2 - 96x + 24)$$
$$L_5(x) = \frac{1}{120}(-x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120)$$
$$L_n(x) = \frac{1}{n!}((-x)^n + n^2(-x)^{n-1} + \cdots + n.n!(-x) + n!)$$

As can be seen every $L_n(x)$ is a polynomial of degree n. The Laguerre polynomials satisfy following equalities:

$$(n+1)L_{n+1}(x) = (2n + 1 - x)L_n(x) - nL_{n-1}(x)$$
$$xL'_n(x) = nL_n(x) - nL_{n-1}(x)$$
$$L_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k!} x^k$$
$$\sum_{n=0}^{\infty} t^n L_n(x) = \frac{e^{-tx/(1-t)}}{1-t}$$
$$L_n(x) = \frac{1}{n!} e^x \frac{d^n}{dx^n}(x^n e^{-x})$$

The Laguerre polynomials arise in quantum mechanics, in the radial part of the solution of the Schrödinger equation for a one-electron atom. They also describe the static Wigner functions of oscillator systems in quantum mechanics in phase space. They further enter in the quantum mechanics of the Morse potential and of the 3D isotropic harmonic oscillator. For solutions of the Hermit and Laguerre equations the following references can be investigated [**2**, **3**].

1.1. Basic definitions and theorems. Let's give some definitions and theorems necessary to solve the equations with the Fourier transform.

THEOREM 1.1. *Let f and g be n.th order differentiable functions. In this case*

$$(f.g)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) g^{(k)}(x).$$

THEOREM 1.2. *Let f be a analytic function in and on simple closed cycle C which is oriented in the positive direction. If z_0 is any point in C, then*

$$f^{(n)}(z_0) = \frac{n!}{2i\pi} \oint \frac{f(z)}{(z-z_0)^{n+1}} dz.$$

The Fourier transform of $f(t)$ is given by

$$\mathcal{F}(f(t)) = \int_{-\infty}^{\infty} f(t) e^{-iwt} dt = F(w).$$

The inverse Fourier transform of $F(w)$ is given by

$$\mathcal{F}^{-1}(F(w)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(w) e^{iwt} dt = f(t).$$

THEOREM 1.3. *The Fourier transform is linear.*

The Fourier transform satisfies the following equalities [**1**]:

$$\mathcal{F}(y^{(n)}) = (iw)^n \mathcal{F}(y)$$

$$\mathcal{F}(xy) = i \frac{d\mathcal{F}(y)}{dw}.$$

2. Solution of Laguerre and Hermite equation by using Fourier transform method.

THEOREM 2.1. *Laguerre equation $xy'' + (1-x)y' + ny = 0$ can be solved by the Fourier transform, and if the solution which has been obtained is $Lf_n(x)$, then $Lf_n(x) = iL_n(x)$.*

PROOF. Let's apply the Fourier transform to this equation.

$$\mathcal{F}(xy'' + (1-x)y' + ny) = 0$$

$$\mathcal{F}(xy'') + \mathcal{F}(y') - \mathcal{F}(xy') + n\mathcal{F}(y) = 0$$

From Theorem 1.3 following equalities are obtained:

$$i\frac{d}{dw}(-w^2Y) + iwY - i\frac{d}{dw}(iwY) + nY = 0$$

$$-i(2wY + w^2Y') + iwY + Y + wY' + nY = 0$$

$$(-iw^2 + w)Y' + (1 + n - iw)Y = 0$$

$$\frac{dY}{Y} + \frac{iw - n - 1}{iw^2 - w} dw = 0$$

$$\frac{dY}{Y} + (\frac{n+1}{w} - \frac{n}{w+i})dw = 0$$

$$Y = \frac{(w+i)^n}{w^{n+1}}$$

$$y = \mathcal{F}^{-1}(Y) = \mathcal{F}^{-1}(\frac{(w+i)^n}{w^{n+1}}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(w+i)^n}{w^{n+1}} e^{iwx} dw$$

$$= \frac{1}{2\pi}\frac{2i\pi}{n!}\frac{d^n}{dw^n}((w+i)^n e^{iwx})(0)$$

If $n = 0$, then $y = i$.
If $n = 1$, then $y = i\frac{d}{dw}((w+i)e^{iwx})(0) = i(e^{iwx} + ix(w+i)e^{iwx})(0) = i(1-x)$.
If $n = 2$, then

$$y = \frac{i}{2}\frac{d^2}{dw^2}((w+i)^2 e^{iwx})(0)$$

$$= \frac{i}{2}\frac{d}{dw}(2(w+i)e^{iwx} + ix(w+i)^2 e^{iwx})(0)$$

$$= \frac{i}{2}(2e^{iwx} + 4(w+i)ixe^{iwx} + (ix)^2(w+i)^2 e^{iwx})(0)$$

$$= \frac{i}{2}(x^2 - 4x + 2).$$

If $n = 3$, then

$$y = \frac{i}{6}\frac{d^3}{dw^3}((w+i)^3 e^{iwx})(0)$$

$$= \frac{i}{6}\frac{d^2}{dw^2}(3(w+i)^2 e^{iwx} + ix(w+i)^3 e^{iwx})(0)$$

$$= \frac{i}{6}\frac{d}{dw}(6(w+i)e^{iwx} + 6ix(w+i)^2 e^{iwx} + (ix)^2(w+i)^3 e^{iwx})(0)$$

$$= \tfrac{i}{6}(6e^{iwx} + 18ix(w+i)e^{iwx} - 6x^2(w+i)^2 e^{iwx} - 3x^2(w+i)^2 e^{iwx} + (ix)^3(w+i)^3 e^{iwx})(0)$$

$$= \frac{i}{6}(-x^3 + 9x^2 - 18x + 6).$$

If $n = 4$, then

$$y = \frac{i}{24}\frac{d^4}{dw^4}((w+i)^4 e^{iwx})(0)$$

$$= \frac{i}{24}(\binom{4}{0}24e^{iwx} + \binom{4}{1}24(w+i)ixe^{iwx} + \binom{4}{2}12(w+i)^2(ix)^2 e^{iwx}$$

$$+ \binom{4}{3}4(w+i)^3(ix)^3 e^{iwx} + \binom{4}{4}(w+i)^4(ix)^4 e^{iwx})(0)$$

$$= \frac{i}{24}(24 - 96x + 72x^2 - 16x^3 + x^4).$$

□

THEOREM 2.2. *The Hermite equation*

$$y'' - 2xy' + 2ny = 0$$

can be solved by the Fourier transform, and if the solution which has been obtained is $Hf_n(x)$, then $Hf_n(x) = \frac{i^{n+1}}{2^n n!}H_n(x)$.

PROOF. Let us consider the Hermite equation

$$y'' - 2xy' + 2ny = 0.$$

Let's apply the Fourier transform to this equation:

$$\mathcal{F}(y'' - 2xy' + 2ny) = 0$$
$$\mathcal{F}(y'') - 2\mathcal{F}(xy') + 2n\mathcal{F}(y) = 0.$$

From Theorem 1.3 the following equalities are obtained:

$$-w^2Y - 2i\frac{d}{dw}(iwY) + 2nY = 0$$
$$-w^2Y + 2(Y + wY') + 2nY = 0$$
$$2wY' + (2n + 2 - w^2)Y = 0$$
$$\frac{dY}{Y} = \frac{w^2 - 2n - 2}{2w}dw$$
$$lnY = \frac{w^2}{4} - (n + 1)lnw$$
$$Y = \frac{e^{\frac{w^2}{4}}}{w^{n+1}}$$
$$y = \mathcal{F}^{-1}(Y) = \mathcal{F}^{-1}(\frac{e^{\frac{w^2}{4}}}{w^{n+1}}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{\frac{w^2}{4}}}{w^{n+1}}e^{iwx}dw$$
$$= \frac{1}{2\pi}\frac{2\pi i}{n!}\frac{d^n}{dw^n}(e^{\frac{w^2}{4}+iwx})(0).$$

If $n = 0$, then

$$y = i.$$

If $n = 1$, then

$$y = i\frac{d}{dw}(e^{\frac{w^2}{4}+iwx}))(0) = i(\frac{w}{2} + ix)e^{\frac{w^2}{4}+iwx})(0) = -x.$$

If $n = 2$, then

$$y = \frac{i}{2}\frac{d^2}{dw^2}(e^{\frac{w^2}{4}+iwx})(0) = \frac{i}{2}\frac{d}{dw}((\frac{w}{2} + ix)e^{\frac{w^2}{4}+iwx})(0)$$
$$= \frac{i}{2}(\frac{1}{2}e^{\frac{w^2}{4}+iwx} + (\frac{w}{2} + ix)^2e^{\frac{w^2}{4}+iwx})(0)$$
$$= \frac{i}{2}(\frac{1}{2} - x^2).$$

If $n = 3$, then

$$y = \frac{i}{6}\frac{d^3}{dw^3}(e^{\frac{w^2}{4}+iwx})(0)$$

$$= \frac{i}{6}\frac{d}{dw}(\frac{1}{2}e^{\frac{w^2}{4}+iwx} + (\frac{w}{2}+ix)^2 e^{\frac{w^2}{4}+iwx})(0)$$

$$= \frac{i}{6}(\frac{1}{2}(\frac{w}{2}+ix)e^{\frac{w^2}{4}+iwx} + (\frac{w}{2}+ix)e^{\frac{w^2}{4}+iwx} + (\frac{w}{2}+ix)^3 e^{\frac{w^2}{4}+iwx})(0)$$

$$= \frac{i}{6}(\frac{ix}{2} + \frac{2ix}{2} + (ix)^3) = \frac{2x^3-3x}{12}.$$

If $n = 4$, then

$$y = \frac{i}{24}\frac{d^4}{dw^4}(e^{\frac{w^2}{4}+iwx})(0)$$

$$= \frac{i}{24}\frac{d}{dw}(\frac{3}{2}(\frac{w}{2}+ix)e^{\frac{w^2}{4}+iwx} + (\frac{w}{2}+ix)^3 e^{\frac{w^2}{4}+iwx})(0)$$

$$= \frac{i}{24}(\frac{3}{4}e^{\frac{w^2}{4}+iwx} + \frac{3}{2}(\frac{w}{2}+ix)^2 e^{\frac{w^2}{4}+iwx} + \frac{3}{2}(\frac{w}{2}+ix)^2 e^{\frac{w^2}{4}+iwx} + (\frac{w}{2}+ix)^4 e^{\frac{w^2}{4}+iwx})(0)$$

$$= \frac{i(4x^4-12x^2+3)}{96}.$$

□

n	$L_n(x)$	$L_{fn}(x)$
$n=0$	1	i
$n=1$	$-x+1$	$i(1-x)$
$n=2$	$\frac{1}{2}(x^2-4x+2)$	$\frac{i}{2}(x^2-4x+2)$
$n=3$	$\frac{1}{6}(-x^3+9x^2-18x+6)$	$\frac{i}{6}(-x^3+9x^2-18x+6)$
$n=4$	$\frac{1}{24}(x^4-16x^3+72x^2-96x+24)$	$\frac{i}{24}(x^4-16x^3+72x^2-96x+24)$

TABLE 1. Table of $L_n(x)$ and $Lf_n(x)$ for different values of n.

n	$H_n(x)$	$Hf_n(x)$
$n=0$	1	i
$n=1$	$2x$	$-x$
$n=2$	$4x^2-2$	$\frac{i}{2}(\frac{1}{2}-x^2)$
$n=3$	$8x^3-12x$	$\frac{2x^3-3x}{12}$
$n=4$	$16x^4-48x^2+12$	$\frac{i}{96}(4x^4-12x^3+3)$

TABLE 2. Table of $H_n(x)$ and $Hf_n(x)$ for different values of n.

2.1. Recurrence relations for $Lf_n(x)$.

THEOREM 2.3. *Since* $Lf_n(x) = iL_n(x)$*, the recurrence ralations for* $L_n(x)$ *are valid for* $Lf_n(x)$ *:*

$$(n+1)Lf_{(n+1)}(x) = (2n+1-x)Lf_n(x) - nLf_{(n-1)}(x)$$
$$xLf_n'(x) = nLf_n(x) - nLf_{(n-1)}(x)$$
$$Lf_n(x) = i\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{k!}x^k$$
$$\sum_{n=0}^{\infty} t^n Lf_n(x) = \frac{i}{1-t}e^{\frac{-tx}{1-t}}$$
$$Lf_n(x) = \frac{i}{n!}e^x\frac{d^n}{dx^n}(x^n e^{-x}).$$

2.2. Recurrence relations for $Hf_n(x)$.

THEOREM 2.4.

$$Hf_{n+1}(x) = \frac{ix}{n+1}Hf_n(x) - \frac{i}{2(n+1)}(Hf_n(x))'.$$

PROOF.

$$\begin{aligned} Hf_{n+1}(x) &= \frac{i^{n+2}}{(n+1)!2^{n+1}}H_{n+1}(x) \\ &= \frac{i^{n+2}}{(n+1)!2^{n+1}}(2xH_n(x) - (H_n(x))') \\ &= \frac{i^{n+2}}{(n+1)!2^{n+1}}(\frac{2x2^n n!}{i^{n+1}}Hf_n(x) - \frac{2^n n!}{i^{n+1}}(Hf_n(x))') \\ &= \frac{ix}{n+1}Hf_n(x) - \frac{i}{2(n+1)}(Hf_n(x))'. \end{aligned}$$

□

THEOREM 2.5.

$$Hf_{n+1}(x) = \frac{ix}{n+1}Hf_n(x) + \frac{1}{2(n+1)}Hf_{n-1}(x).$$

PROOF.

$$\begin{aligned} Hf_{n+1}(x) &= \frac{i^{n+2}}{(n+1)!2^{n+1}}H_{n+1}(x) \\ &= \frac{i^{n+2}}{(n+1)!2^{n+1}}(2xH_n(x) - 2nH_{n-1}(x)) \\ &= \frac{i^{n+2}}{(n+1)!2^{n+1}}(\frac{2x2^n n!}{i^{n+1}}Hf_n(x) - \frac{2n2^{n-1}(n-1)!}{i^n}Hf_{n-1}(x)) \\ &= \frac{ix}{n+1}Hf_n(x) + \frac{1}{2(n+1)}Hf_{n-1}(x). \end{aligned}$$

□

THEOREM 2.6.
$$e^{2xt-t^2} = -i\sum_{n=0}^{\infty} Hf_n(x)(-2it)^n.$$

PROOF.
$$\begin{aligned} e^{2xt-t^2} &= \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{n!2^n}{i^{n+1}} Hf_n(x)t^n/n! \\ &= -i\sum_{n=0}^{\infty} Hf_n(x)(-2it)^n. \end{aligned}$$

□

References

1. R. Bracewell, *The Fourier Transform and Its Applications*, Boston: McGraw-HillBook Company, 1965.
2. M. Mazmumy and A. Alsulami, *Solution of Laguerres Differential Equations via Modified Adomian Decomposition Method*, J. appl. math. phys.11,1:85-100, 2023.
3. O.Yürekli and S. Wilson, *A new method of solving Hermites differential equation using the L_2 transform*, Appl. Math. Comput. 145:495-500, 2003.

Received by editors 18.7.2024; Revised version 10.12.2024; Available online 30.12.2024.

MURAT DÜZ, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KARABK UNIVERSITY, 78050 KARABK, TURKEY

Email address: mduz@karabuk.edu.tr