BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS/BULLETIN
Bull. Int. Math. Virtual Inst., **14**(2)(2024), 355–361
DOI: 10.7251/BIMVI2402355R

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

PRIME BI-INTERIOR IDEALS OF SEMIGROUPS

M. Murali Krishna Rao and Noorbhasha Rafi

ABSTRACT. In this paper, we introduce the notion of prime bi-interior ideal, semiprime bi-interior ideal, irreducible bi-interior ideal, and strongly prime bi-interior ideal of semigroups. We study properties of these ideals, relations between them, and also characterize regular semigroups using prime bi-interior ideals.

1. Introduction

Many mathematicians proved important results and charecterization of algebraic structures by using the concept and the properties of generalization of ideals in algebraic structures. During 1950-1980, the concepts of bi-ideals, quasi ideals and interior ideals were studied by many mathematicians.Then the author [**9**, **14**, **15**, **17**, **18**] introduced and studied weak interior ideals, tri-ideals , bi-interior ideals, bi quasi ideals,quasi interior ideals and bi quasi interior ideals and tri quasi of $\Gamma-$semirings, semirings, $\Gamma-$semigroups, semigroups and semirings as a generalization of bi-ideal, quasi ideal and interior ideal of algebraic structures and charecterized regular algebraic structures as well as simple algebraic structures using these ideals. Semiring is the algebraic structure which is a common generalization of rings and distributive lattices, was first introduced by Vandiver [**20**] in 1934 but non-trivial examples of semirings had appeared in the studies on the theory of commutative ideals of rings by Dedekind in 19th century. We know that the notion of a one sided ideal of any algebraic structure is a generalization of an ideal. The quasi ideals are generalization of left ideal and right ideal whereas the bi-ideals are generalization of quasi ideals. In 1952, the concept of bi-ideals was introduced by Good and Hughes [**3**] for semigroups. The notion of bi-ideals in rings

2020 *Mathematics Subject Classification.* Primary 20M12; Secondary 06F05.
Key words and phrases. Semigroups, bi-interior ideal, prime ideals, prime bi-interior ideal.
Communicated by Dusko Bogdanic.

and semigroups were introduced by Lajos and Szasz [**10**–**12**]. The concept of interior ideals was introduced by Lajos for semigroups. Steinfeld [**19**] first introduced the notion of quasi ideals for semigroups and then for rings. Iseki [**2**] introduced the concept of quasi ideal for a semiring. In this paper,as a further generalization of ideals, M.Shabir [**18**] studied the prime bi-ideals of semigroups. In this paper, we introduce the notion of prime bi-interior ideals of semigroups.

2. Preliminaries

In this section we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

Definition 2.1. *[**18**] A bi-ideal B of S is called a strongly prime bi-ideal if $(B_1B_2)\bigcap(B_2B_1) \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$, for any bi-ideals B_1 and B_2 of S.*

Definition 2.2. *[**18**] A bi-ideal B of S is called a semiprime bi-ideal if for any bi-ideal B_1 of S, $B_1^2 = B_1B_1 \subseteq B$ implies $B_1 \subseteq B$.*

Obviously every strongly prime bi-ideal in S is a prime bi-ideal and every prime bi-ideal in S is a semiprime bi-ideal.

Definition 2.3. *[**18**] A bi-ideal B of S is called an irreducible bi-ideal if $B_1 \bigcap B_2 = B$ implies $B_1 = B$ or $B_2 = B$, for any bi-ideals B_1 and B_2 of S.*

Definition 2.4. *[**18**] A bi-ideal B of S is called a strongly irreducible bi-ideal if for any bi-idealsB_1 and B_2 of S, $B_1 \bigcap B_2 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$.*

Obviously every strongly irreducible bi-ideal is an irreducible bi-ideal.

Definition 2.5. *[**18**] Let B_1 and B_2 be bi- ideals of a semigroup M.*
- (i) *If $B_1B_2 \subseteq B \Rightarrow B_1 \subseteq B$ or $B_2 \subseteq B$ then B is prime bi- ideal .*
- (ii) *If $(B_1B_2) \cap (B_2B_1) \subseteq B \Rightarrow B_1 \subseteq B$ or $B_2 \subseteq B$ then strongly prime bi-ideal.*

Definition 2.6. *[**18**] A bi- ideal B of a semigroup M is called a semi prime bi-interior ideal if $B_1B_1 \subseteq B \Longrightarrow B_1 \subseteq B$, for any bi- ideal B_1 of M.*

3. Prime bi-interior ideals of semigroups

In this section, we introduction the notion of prime, strongly prime, semi prime, irreducible and strongyle irreducible bi-interior ideals of semigroups and we study the properties of prime ideals and relations between them.

Definition 3.1. *A bi-interior ideal B of a semigroup M is called a prime bi-interior ideal of M if $B_1B_2 \subseteq B \Rightarrow B_1 \subseteq B$ or $B_2 \subseteq B$.*

Definition 3.2. *A bi-interior ideal B of a semigroup M is called a semi prime bi-interior ideal of M if for any bi-interior ideal B_1 of M, $B_1B_2 \subseteq B \Longrightarrow B_1 \subseteq B$ and $B_2 \subseteq B$.*

Definition 3.3. *A bi-interior ideal B of M is called an irreducible bi-interior ideal B if bi-interior ideals B_1, B_2*
and $B_1 \cap B_2 \subseteq B \Longrightarrow B_1 \subseteq B$ or $B_2 \subseteq B$.

Definition 3.4. *A bi-ideal B of S is called a strongly prime bi-interior ideal if $(B_1B_2)\bigcap(B_2B_1) \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$, for any bi-ideals B_1 and B_2 of S.*

Remark 3.1. (i). Every strongly prime bi-interior ideal of a semigroup M is a prime bi-interior ideal of M.

(ii). Every prime bi-interior ideal B of a semigroup M is a semi prime bi-interior ideal of M.

Theorem 3.1. *A bi-interior ideal B of a semigroup M is a prime bi-interior ideal if and only if $RL \subseteq B \Longrightarrow R \subseteq B$ or $L \subseteq B$ where R is a right ideal and L is a left ideal of M.*

Proof. Suppose that a prime bi-interior ideal B of the semigroup M and $RL \subseteq B$. Since R and L are bi-interior ideals $R \subseteq B$ or $L \subseteq B$. Conversely suppose that $RL \subseteq B$ where R is a right ideal and L is a left ideal of M. $\Longrightarrow R \subseteq B$ or $L \subseteq B$. Suppose $AC \subseteq B$, A and C are bi-interior ideals and $(a)_r, (c)_l$ are right and left ideals generated by a and c respectively,where $a \in A$ and $c \in B$. Then

$$\begin{aligned}&(a)_r(c)_l \subseteq AC \subseteq B,\\ \Rightarrow&(a)_r \subseteq B \text{ or } (c)_l \subseteq B\end{aligned}$$

Then $a \in B$ or $c \in B$. Therefore $A \subseteq B$ or $C \subseteq B$.
Hence a bi-interior ideal B is a prime bi-interior ideal of the semigroup M. □

Theorem 3.2. *If B_1, B_2 are bi-interior ideals of a semigroup M and $(B_1B_2)\cap(B_2B_1) = B_1 \cap B_2$ then every bi-interior ideal of a semigroup M is a semi prime ideal of M..*

Proof. Let B be any bi-interior ideal of M and $B_1B_1 \subseteq B$, B_1 is a bi-interior ideal of M. Then

$$\begin{aligned}B_1 =&B_1 \cap B_1\\ =&(B_1B_1) \cap (B_1B_1)\\ \subseteq&B \cap B\\ =&B.\end{aligned}$$

Hence every bi-interior ideal of M is semi prime. □

Theorem 3.3. *M is a regular semigroup if and only if $AB = A \cap B$ for any right ideal A and left ideal B of M.*

Proof. Let A, B be a right ideal and a left ideal of a regular semigroup M respectively. Obviously $AB \subseteq A \cap B$. Let $x \in A \cap B$. Since M is a regular, there exist $y \in M$ such that $x = xyx$. Since $xy \in A$ and $x \in B$, $xyx \in AB$. Thus $x \in AB$. Hence $AB = A\cap B$. Conversely, suppose that $AB = A\cap B$ for any right ideal A and left ideal B of M. Let $x \in M$ and I be the right ideal generated by x and J be the left ideal generated by x. We have $x \in I \cap J = IJ$. Therefore $x = xy = zx,\ y, z \in M$ which implies that $x = xyzx$, Hence M is a regular ordered semigroup. □

THEOREM 3.4. *If $BB = B$, for all bi-interior ideals of a semigroup M, then semigroup M is regular and $B_1 \cap B_2 = (B_1B_2)(B_2B_1)$, for any bi-interior ideals B_1 and B_2 of M.*

PROOF. Suppose $BB = B$, for all bi-interior ideal B of M. Let R be a right ideal and L be a left ideal of M. Then $R \cap L$ is a bi-interior ideal of M. Therefore

$$\begin{aligned}(R \cap L)(R \cap L)) &= (R \cap L)\\ \Rightarrow R \cap L &\subseteq RL.\end{aligned}$$

We have $RL \subseteq R \cap L$. Therefore $R \cap L = RL$.

Hence by Theorem 3.3, M is a regular semigroup.
Let B_1 and B_2 be a bi-interior ideals of M. Then $B_1 \cap B_2$ is a bi-interior ideal of M

$$\begin{aligned}B_1 \cap B_2 = (B_1 \cap B_2) &= (B_1 \cap B_2)(B_1B_2)\\ &\subseteq (B_1B_2).\end{aligned}$$

Similarly we can prove $(B_1 \cap B_2) \subseteq B_2B_1$.
Therefore $B_1 \cap B_2) \subseteq (B_1B_2) \cap ((B_2B_1)$.

$$\begin{aligned}(B_1B_2) \cap (B_2B_1) &= (B_1B_2)(B_2B_1) \cap (B_1B_2)(B_2B_1)\\ &\subseteq B_1MB_1 \cap MB_1M\\ &\subseteq B_1\end{aligned}$$

Similarly we can prove that $(B_1B_2) \cap (B_2B_1) \subseteq B_2$.
Therefore $(B_1B_2) \cap (B_2B_1) \subseteq B_1 \cap B_2$.
Hence $(B_1B_2) \cap (B_2B_1) = B_1 \cap B_2$.

□

The following theorem is similar to Theorem in [17], so we omit the proof.

THEOREM 3.5. *If B is a bi-interior ideal of M and $a \in M$ such that $a \notin B$ then there exists an irreducible bi-interior ideal I of M such that $B \subseteq I$ and $a \in I$.*

THEOREM 3.6. *Let M be a regular semigroup and $BB = B$, for all bi-interior ideal B of M. Then any bi-interior ideal B of M is strongly irreducible bi-interior ideal if and only if B is a strongly prime bi-interior ideal.*

PROOF. Let M be a regular semigroup and $BB = B$, for any bi-interior ideal B of M. Suppose B is a strongly irreducible bi-interior ideal of M. Then by Theorem 3.4, $(B_1B_2) \cap (B_2B_1) = B_1 \cap B_2$, where B_1, B_2 are bi-interior ideals of M.

$$\begin{aligned}&(B_1B_2) \cap (B_2B_1) \subseteq B\\ \Rightarrow &B_1 \cap B_2 \subseteq B\\ \Rightarrow &B_1 \subseteq B \text{ or } B_2 \subseteq B\end{aligned}$$

Thus B is a strongly prime bi-interior ideal of M.

Conversely suppose B is a strongly prime bi-interior ideals of M. Let B_1 ,B_1 are bi-interior ideals of M. such that $B_1B_2 \subseteq B$,

$$\begin{aligned}(B_1B_2) \cap (B_2B_1) &= (B_1B_2)(B_2B_1) \cap (B_1B_2)(B_2B_1)\\ &\subseteq B_1MB_1 \cap MB_1M\\ &\subseteq B_1\end{aligned}$$

Similarly we can prove that $(B_1B_2) \cap (B_2B_1) \subseteq B_2$.
Therefore $(B_1B_2) \cap (B_2B_1) \subseteq B_1 \cap B_2 \subseteq B$.
As B is a strongly prime bi-interior ideals of M.,$B_1 \subseteq B$ or $B_2 \subseteq B$ Hence B is a strongly irreducible bi-interior ideal of M. □

THEOREM 3.7. *If B is an irreducible bi-interior ideal and $BB = B$ of a regular semigroup M then B is a strongly irreducible ideal of M.*

PROOF. Let B_1 and B_2 be bi-interior ideals of M such that $B_1 \cap B_2 \subseteq B$. Then By Theorem 3.9, $(B_1B_2) \cap (B_2B_1) = B_1 \cap B_2$.

$$\begin{aligned}(B_1B_2) \cap (B_2B_1) = B_1 \cap B_2 \subseteq B\\ \Longrightarrow B_1 \subseteq B \text{ or } B_2 \subseteq B.\end{aligned}$$

Therefore B is a strongly irreducible ideal of M. □

THEOREM 3.8. *Any proper bi-interior ideal B of M is the intersection of all irreducible bi-interior ideals M containing B.*

PROOF. Let B be a bi-interior ideal of M and $\{B_i/i \in \wedge\}$ be the collection of irreducible ideals containing B. Then $B \subseteq \bigcap\limits_{i\in\wedge} B_i$.

Suppose that $a \notin B$. By Theorem3.15, there exists an irreducible bi-interior ideal I such that $B \subseteq I$ and $a \in I$. Then

$$a \notin \bigcap_{i\in\wedge} B_i, \Rightarrow \bigcap_{i\in\wedge} B_i \subseteq I.\text{Hence } I = \bigcap_{i\in\wedge} B_i.$$

□

THEOREM 3.9. *The intersection of any family of prime bi- interior ideals of a semigroup M is a semiprime bi- interior ideal.*

PROOF. Let $\{P_i \mid i \in \Delta\}$ be the family of prime bi- interior ideals of M.
For any bi- interior ideal B of $M, B^2 \subseteq \bigcap\limits_i P_i$ implies $B^2 \subseteq P_i$, for all $i \in \Delta$.
As P_i are prime bi-interior ideals, P_i are semiprime bi -interior ideals .
Therefore $B \subseteq P_i$, for all $i \in \Delta$.
Hence $B \subseteq \bigcap\limits_i P_i$. □

Remark: Arbitrary intersection of bi- interior ideals of S is also a bi- interior ideal of M and hence the set of all bi- interior ideals of M forms a complete lattice.

THEOREM 3.10. *Every strongly irreducible, semiprime bi- interior ideal of a semigroup M is a strongly prime bi- interior ideal.*

PROOF. Let B be a strongly irreducible and semiprime bi- interior ideal of a semigroup M.
For any bi- interior ideals B_1 and B_2 of M,

$$(B_1B_2) \cap (B_1B_2) \subseteq B.$$

Hence, by Remark, $B_1 \cap B_2$ bi- interior ideal of M. Since

$$(B_1 \cap B_2)^2 = (B_1 \cap B_2)(B_1 \cap B_2) \subseteq (B_1 \cap B_2).$$

Similarly, we get $(B_1 \cap B_2)^2 \subseteq (B_2 \cap B_1)$.
Therefore $(B_1 \cap B_2)^2 \subseteq (B_1 \cap B_2) \cap (B_2 \cap B_1) \subseteq B$.

As B is a semiprime bi- interior ideal of $M, B_1 \cap B_2 \subseteq B$.
But B is a strongly irreducible bi- interior ideal. Therefore $B_1 \subseteq B$ or $B_2 \subseteq B$.
Hence B is a strongly prime bi- interior ideal of M. □

THEOREM 3.11. *Following statements are equivalents in a semigroup M :*

(1). *The set of bi- interior ideals of M is totally ordered set under inclusion of sets.*
(2). *Each bi- interior ideal of M is strongly irreducible.*
(3). *Each bi- interior ideal of M is irreducible.*

PROOF. Let M be a semigroup M.

(1) ⇒ (2) : Suppose that the set of bi- interior ideals of M is a totally ordered set under inclusion of sets.

Let B be any bi- interior ideal of M. To show that B is a strongly irreducible bi- interior ideal of M.

Let B_1 and B_2 be any two bi- interior ideals of M such that $B_1 \cap B_2 \subseteq B$.

But by the hypothesis we have either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$.
Therefore $B_1 \cap B_2 = B_1$ or $B_1 \cap B_2 = B_2$.
Hence $B_1 \subseteq B$ or $B_2 \subseteq B$.
Thus B is a strongly irreducible bi- interior ideal of M.

(2) ⇒ (3): Suppose that each bi- interior ideal of M is strongly irreducible.
Let B be any bi- interior ideal of M such that $B = B_1 \cap B_2$, for any bi-interior ideals B_1 and B_2 of M.
Hence by (2), we have $B_1 \subseteq B$ or $B_2 \subseteq B$.
As $B \subseteq B_1$ and $B \subseteq B_2$, we have $B_1 = B$ or $B_2 = B$.
Hence B is an irreducible bi- interior ideal of M.

(3) ⇒ (1) : Suppose that each bi- interior ideal of M is an irreducible bi-ideal.
Let B_1 and B_2 be any two bi- interior ideals of M.
Then $B_1 \cap B_2$ is also a bi- interior ideals of M, (from remark).
Hence $B_1 \cap B_2 = B_1 \cap B_2$ implies $B_1 \cap B_2 = B_1$ or $B_1 \cap B_2 = B_2$, by our assumption.
Therefore either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$.
This shows that the set of bi-interior ideals of M is a totally ordered set under inclusion of sets.

□

References

1. G. L. Booth and N. I. Groenewald, On prime one-sided bi-ideals and quasi ideals of a $\Gamma-$Ring J. Asstral. Math. Soc., 53(1)(1992), 55-63,
2. T. K. Dutta and S. K. Sardar, Semi-prime Ideals and Irreducible Ideals of $\Gamma-$semirings. Novi Sad J. Math., 30(1)(2000), 97–108.
3. R. A. Good and D. R. Hughes, Associated groups for a semigroup, Bull. Amer. Math. Soc., 58(1952), 624-625.
4. M. Henriksen, Ideals in semirings with commutative addition, Amer. Math. Soc. Notices, 5 (1958), p. 321.
5. K. Iseki, Quasi-ideals in semirings without zero, Proc. Japan Acad., 34 (1958), 79-84.
6. K. Iseki, Ideal theory of semirings, Proc. Japan Acad., 32(1956), 554-559.
7. K. Iseki, Ideal in semirings Proc. Japan Acad., 34(1958), 29-31.
8. R. D. Jagatap and Y.S. Pawar, Quasi-ideals and minimal quasi-ideals in $\Gamma-$semirings, Novi Sad J. Math., 39(2) (2009), 79–87.
9. P. Khamrot, A. Phukhaengst, and T. Gaketem, Ordered Almost n-Interior Ideals in Semigroups Class Fuzzifications, IAENG International Journal of Computer Science, Volume 51, Issue 11, November 2024, 1711–1719.
10. S. Lajos, Generalized ideals in semigroups, Acta. Sci. Math. Szeged., 22(1961),217–222.
11. S. Lajos, On the bi-ideals in semigroups, Proc. Japan Acad., 45(1969), 710–712.
12. S. Lajos and F. A. Szasz, On the bi-ideals in associative ring, Proc. Japan Acad., 46 (1970), 505–507.
13. M. Murali Krishna Rao, $\Gamma-$semirings-I, Southeast Asian Bull. Math. 19 (1)(1995), 49–54.
14. M. Murali Krishna Rao, Left bi-quasi ideals ofsemirings, Bull. Int. Math. Virtual Inst., 45-53.
15. M.Murali Krishna Rao. Bi-quasi-ideals and fuzzy bi-quasiideals of $\Gamma-$semigroup, Bull. Int. Math. Virtual Inst., Vol. 7(2)(2017), 231–242.
16. M. Murali Krishna Rao, Prime bi-interior ideals of Γ-semirings, Journal of Hyperstructures, 11(1) (2022), 20–30
17. M. Murali Krishna Rao, B. Venkateswarlu, and N.Rafi, Left bi-quasi-ideals of $\Gamma-$semirings,Asia Pacific Journal of Mathematics, Vol. 4, No. 2 (2017), 144–153.
18. M.Shabir and N.Kanwal, Prime bi Ideals in semigroups, South East Asian Bulletin of Mathematics,27(2004) 923-928
19. O. Steinfeld, Uher die quasi ideals, Von halbgruppend Publ. Math., Debrecen, 4 (1956), 262–275.
20. H. S. Vandiver, Note on a simple type of algebra in which cancellation law of addition does not hold,Bull. Amer. Math. Soc.(N.S.), 40 (1934), 914–920.

Received by editors 21.6.2024; Revised version 11.12.2024; Available online 30.12.2024.

M. Murali Krishna Rao, Department of Mathematics, Sankethika Institute of Tech. and Management,, Visakhapatnam-530 041, India.
Email address: mmarapureddy@gmail.com

Noorbhasha Rafi, Department of Mathematics, Bapatla Engineering College, Bapatla, 522 102, India.
Email address: rafimaths@gmail.com