BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS/BULLETIN
Bull. Int. Math. Virtual Inst., **14**(2)(2024), 363–367
DOI: 10.7251/BIMVI2402363N

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

ON SOME NEW RESULTS IN DOMINATION OF $d-$GRAPH

Nayaka S. R., Raju S., and Puttaswamy

ABSTRACT. The relation between the graph theory and number theory is very strong where there are many numbers that deal with it in graph theory such as chromatic, independence, domination, the clique number, and others. A graph G with the vertex set V labeled by a set of integers and for each edge $xy \in E$ either $f(x)|f(y)$ or vice versa is called a divisor graph. We call the divisor graph a d-graph if the vertices can be labeled with distinct integers $1, 2, \ldots, |V|$. In this paper, we discuss domination and independence numbers of $d-$graphs.

1. Introduction

The graph theory depends on two sets, one of them is a non-empty and it is called the vertex set and the other is called the edge set. An edge is called directed edge if it assigned a direction. The number vertices and edges in a graph are respectively called order and size of the graph. A graph is simple if it contains no multiple edges and loops. Moreover, in this paper, we consider graphs that are un-directed, finite and simple. Nowadays, Graph theory participates in finding solutions for most sciences through new technologies, such as medicine, engineering, computers, chemistry and others.

Number theory is one of the branch in Mathematics that has find many applications in cryptography and coding theory. The relation between the graph theory and number theory is very strong where there are many numbers that deal with it in graph theory such as chromatic, independence, domination, the clique

2020 *Mathematics Subject Classification.* Primary 05C69.
Key words and phrases. Domination number, d-graph, maximum d-graph, prime number.
Communicated by Dusko Bogdanic.

number and others. In this article, we shall develop the relation between number theory and graph theory by considering one of the labeled graphs called d-graph.

A subset S of vertices of a graph Γ is called the dominating set if every vertex not in S is adjacent to at least one vertex in S. A dominating set is said to be minimal if removal of any vertex yields the set non - dominating. A minimal dominating of of least cardinality called the minimum dominating set, order of which is called domination number denoted by $\gamma(\Gamma)$. A subset S of vertex set is called the independent set if every pair of vertices are non-adjacent. We call cardinality of maximum independent set as independence number of the graph, denoted by $\beta(\Gamma)$. A sub-graph of a graph Γ isomorphic to a complete graph K_p is called a clique sub-graph or simply clique of Γ. The cardinality of a maximum clique sub-graph is called the clique number or c-number.

The concept of divisor graph was introduced by G. Suresh Singh and G. Santhosh [**5**] in 2000. Later it was studied by Chartrand et. al. [**1**] in 2001 and Frayer [**2**] in 2003. A graph is said to be a divisor graph if its vertex set V is labeled by a set of integers and for each edge $xy \in E$ either $f(x)|f(y)$ or vice versa. We call the divisor graph a d-graph if the vertices can be labeled with distinct integers $1, 2, \dots, |V|$. The reader can find all the concepts that were not previously mentioned in [**3**].

Definition 1.1. *Let G be a $d-$graph and if addition of any edge to this graph yields a non $-$ $d-$graph. Then the graph G is called the maximum $d-$graph.*

Definition 1.2. *The number of primes not exceeding u is called the Gauss's function $\prod(u)$ and can be written as $\prod(u) = |p : p$ is prime $, 2 \leqslant p \leqslant u|$.*

We recall fundamental theorem of arithmetic which we use in next section.

Theorem 1.1. *Every positive integer n except 1 can be expressed as a product of primes. This prime factorization is unique up to the order.*

Theorem 1.2. *Let Γ be the maximum $d-$graph. Then*

$$\beta(\Gamma) = \frac{m}{2}$$
$$\gamma(\Gamma) = 1$$

Theorem 1.3. *Let Γ be the maximum $d-$graph. Then $\gamma^{(-1)}(\Gamma) = \prod(m)$.*

2. Main results

Theorem 2.1. *The $c-$number of the maximum d-graph of order m is equal to $m + 1$, where m is the maximum number such that $2^m \leqslant n$*

Proof. Let $S = \{v : f(v) = 2^i, 1 \leqslant i \leqslant m$, where m is the maximum number such that $2^m \leqslant n\}$. It is obvious that for every pair of vertices in the set U say x and y, these vertices are adjacent since x divides y or y divides x. Thus, the set U is a $c-$ sub-graph in Γ. The vertex v_1 is adjacent to other vertices, since $f(v_1) = 1$ and 1 divides all integer numbers. Therefore, the set $U \cup [v_1]$ is a c$-$sub-graph in Γ. For each vertex not in the set $U \cup [v_1]$, there exists at least one prime number say

$q \neq 2$ divides the labeled of this vertex so this vertex does not divide any vertex in the set U and vice versa. Thus, the set $U \cup [v_1]$ is the maximal c−sub-graph in Γ. The other maximal $c-$subset in being following form $\{v, f(v) = r^i; 1 \leqslant i \leqslant m$ and $3 \leqslant r \leqslant m$; where m is the maximum number such that $i^m \leqslant n\}$ number of vertices in each subset of this form is less than of the set U. Therefore, the set $U \cup [v_1]$ is the maximum $c-$subset, so the $c-$number is equal to $U \cup [v_1] = m + 1$. □

PROPOSITION 2.1. *Let Γ be the maximum $d-$ graph of order n. Then the set of pendent vertices are* $= \begin{cases} \{v_7, v_8\} & \text{if } m=7; \\ \{v_7, v_8\}, & \text{if } m=8 \end{cases}$ *also $\{p_i,\ p_i$ is prime number such that $p_i > \lceil \frac{m}{2} \rceil\}$ is prime if $n > 3$ and $p_i \geqslant \lceil \frac{m}{2} \rceil$, if $\lceil \frac{m}{2} \rceil$ is not prime.*

PROOF. **Case 1**. If $m = 1$, then $G \cong K_1$, so there is no pendant vertex.
Case 2. If $m = 2$, then $G \cong K_2$, so there are two pendant vertices.
Case 3. If $m = 3$, then $G \cong P_3$, so again there are two pendant vertices.
Case 4. If $m > 3$, then there are four sub-cases:
Sub-case 1. If $f(x_i) < \lceil \frac{m}{2} \rceil$, then the neighborhood of the vertex x_i contains at least two vertices one of them is x_1 and the other of labeled $2f(x_i)$. Thus, the vertex x is not a pendant
Sub-case 2. If $f(x_i) = \lceil \frac{m}{2} \rceil$, then there are two sub-cases:
i. If m is even, then the vertex of labeled $\frac{m}{2}$ is adjacent to at least two vertices one of them is v_1 and the other of labeled n.
ii. If m is odd, then there are two sub-cases:
a. if $\lceil \frac{m}{2} \rceil$ is not prime, then there are at least two primes number say p and q less than $\lceil \frac{m}{2} \rceil$. Therefore, the two vertices that have labeled p and q are adjacent to the vertex of labeled $\lceil \frac{m}{2} \rceil$. Thus, the vertex of labeled $\lceil \frac{m}{2} \rceil$ is not a pendant.
b. If $\lceil \frac{m}{2} \rceil$ is prime, then the vertex of label $\lceil \frac{m}{2} \rceil$ is adjacent to only one vertex v_1, since $2\lceil \frac{m}{2} \rceil > n$.
Sub-case 3. If $f(v_i) > \lceil \frac{m}{2} \rceil$, then there are two sub-cases:
i. If $\lceil \frac{m}{2} \rceil$ is not prime, then by the same manner in case $4(subcase2(ii(a)))$, the vertex of labeled $\lceil \frac{m}{2} \rceil$ is not pendant.
ii. If $\lceil \frac{m}{2} \rceil$ is prime, then the vertex of labeled $\lceil \frac{m}{2} \rceil$ is adjacent to only one vertex v_1, since $2\lceil \frac{m}{2} \rceil > n$.
Depending on the cases above, the proof is done. □

THEOREM 2.2. *Consider Γ be the maximum $d-$graph of order m, then $\bar{\Gamma}$ has the following properties:*

(1) *The set of isolated vertices contains two vertices if $m = 2$, otherwise contains one vertex.*
(2) *The graph $\bar{\Gamma} - V_1$ is Hamiltonian if m is odd and semi Hamiltonian if m is even.*

PROOF. 1. Two cases are discussed:
Case 1. If $n = 2$, then $\bar{\Gamma} \cong N_2$, so Γ has two isolated vertices.
Case 2. If $n \neq 2$, then the vertex v_1 is the only isolated vertex since for each vertex $v_i \neq v_1$, the vertex v_i is adjacent to at least one vertex $v_{(i-1)}$ or $v_{(i+1)}$.

2. The vertex v_2 is adjacent to v_3 and the vertex v_m is adjacent to the vertex $v_{(n-1)}$. For each vertex $v_2 < v_i < v_n$, the vertex v_i is adjacent to the two vertices $v_{(i-1)}$ and $v_{(i+1)}$. Thus, there is a path passing all vertices in the graph $\bar{\Gamma} - v_1$ and there are two cases as follows:
Sub-case 1. If m is even, then the vertex v_m is adjacent to the vertex v_2. Thus, there is a cycle passing all vertices in the graph $\bar{\Gamma} - \{v_1\}$, so this graph is Hamiltonian.
Sub-case 2. If m is odd, then the vertex v_n is not adjacent to the vertex v_2. Thus, there is no cycle passing all vertices in the graph $\bar{\Gamma} - \{v_1\}$, so this graph is semi Hamiltonian. Depending on the cases above, the proof is done. □

THEOREM 2.3. *Let $\bar{\Gamma}$ be the complement of the maximum $d-$ graph Γ of order n. Then $deg(v_2) = \prod(n) - 1 + |m = p_1^{(\alpha_1)} p_2^{(\alpha_2)} p_r^{(\alpha_r)} : 2$ does not divide $m|$.*

PROOF. In a graph $\bar{\Gamma}$, it is obvious that the labeled vertex v_2 is adjacent to all vertices which have labeled prime numbers, so $deg(v_2) \geqslant \prod(n) - 1$. Now, the other vertices that mean the vertices of labeled not prime is written by the form
$m = p_1^{(\alpha_1)} p_2^{(\alpha_2)} p_r^{(\alpha_r)}$, so there are two sub-cases to discuss this case:
Sub-case 1. If 2 divides m, then the vertex v_2 is not adjacent to these vertices.
Sub-case 2. If 2 does not divide m, then the vertex v_2 is adjacent to these vertices. Thus, depending on two sub-cases above,
$deg(v_2) = \prod(n) - 1 + |m = p_1^{(\alpha_1)} p_2^{(\alpha_2)} p_r^{(\alpha_r)} : 2$ *does not divide* $m|$. □

THEOREM 2.4. *Let $\bar{\Gamma}$ be the complement of the maximum $d-$graph of order n. Then the $c-$ number of the maximum $d-$ graph is $\prod(m)$.*

PROOF. Let S be the set of all vertices of labeled prime numbers. Then every two vertices in the set U are adjacent in the graph $\bar{\Gamma}$. Thus, the set U is a $c-$subgraph of $\bar{\Gamma}$. Therefore, the $c-$number greater than or equal to the cardinality of the set S. For each vertex say x not belong to the set U, there are at least two vertices having different prime labeled say p and q, so the vertex x not adjacent to at least two vertices which have the labeled p and q. Therefore, the $c-$number is equal to $|S| = \prod(m)$. □

THEOREM 2.5. *Let $\bar{\Gamma}$ be the complement of the maximum $d-$graph of order m.*

$$\text{Then } \gamma(\bar{\Gamma}) = \begin{cases} 1, & \text{if } m = 1; \\ 2, & \text{if } m = p; \quad \text{where } p \text{ is a prime number.} \\ 3, & \text{if } m \neq p; \end{cases}$$

PROOF. There are three cases to be discussed as follows:
Case 1. If $m = 1$, then it is obvious that $\gamma(\bar{\Gamma}) = 1$.
Case 2. If $n = p$, then the vertex v_n is adjacent to all other vertices except the isolated vertex v_1. Thus $\gamma(\bar{\Gamma}) = 2$.
Case 3. If $n \neq p$, then there is no vertex adjacent to all vertices in $\gamma(\bar{\Gamma})$. Therefore, $\gamma(\bar{\Gamma}) \geqslant 3$. Let $D = \{v_1, v_{n-1}, v_n\}$, so for each vertex in $\gamma(\bar{\Gamma})$ this vertex is adjacent to at least one vertex of the two vertices v_{n-1} or v_n. Thus, the set D is the minimum dominating set and $\gamma(\bar{\Gamma}) = 3$. □

THEOREM 2.6. *Let $\bar{\Gamma}$ be the complement of the maximum $d-$graph of order n. Then $\beta(\Gamma) = m$, where m is the maximum number such that $2^m \leqslant n$.*

PROOF. The maximum $c-$sub-graph in Γ isomorphic to the induced sub-graph spanning by the set $U = \{v : f(v) = 2^i; 1 \leqslant i \leqslant m$; where m is the maximum number such that $2^m \leqslant n\}$. Thus, in Γ the set U is independent. Now, if the vertex x does not belong to the set U, there is at least one prime not equal to two divides the labeled of the vertex x. Thus, the vertex x is adjacent to some vertices in the set U. Therefore, the set S is the maximum independent set and $\bar{\Gamma} = m$, where m is the maximum number such that $2^m \leqslant n$. □

References

1. G. Chartrand, R. Muntean, V. Saenpholphat, and P. Zhang, Which Graphs are Divisor Graphs? Congr. Numer. 151 (2001) 189-200.
2. C. Frayer, Properties of Divisor Graphs, Rose-Hulman Undergraduate Mathematics Journal: Vol. 4 : Issue 2, (2003).
3. A. A. Jabor and A. A. Omran, Topological domination in graph theory, AIP Conference Proceedings 2334., 020010 (2021).
4. A. A. Omran and T. A. Ibrahim, Fuzzy co-even domination of strong fuzzy graphs, Int. J. Nonlinear Anal. Appl.12, No. 1, (2021) 727-734.
5. G. S. Singh and G. Santhosh, Divisor graphs-I, Preprint.

Received by editors 8.8.2024; Revised version 13.12.2024; Available online 30.12.2024.

NAYAKA S. R., P.E.S. COLLEGE OF ENGINEERING, DEPARTMENT OF MATHEMATICS, MANDYA, KARNATAKA, INDIA
Email address: nayaka.abhi11@gmail.com

RAJU S., P.E.S. COLLEGE OF ENGINEERING, DEPARTMENT OF MATHEMATICS, MANDYA, KARNATAKA, INDIA
Email address: rajus24@gmail.com

PUTTASWAMY, P.E.S. COLLEGE OF ENGINEERING, DEPARTMENT OF MATHEMATICS, MANDYA, KARNATAKA, INDIA
Email address: prof.puttaswamy@gmail.com