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INTERVAL VALUED BIPOLAR FUZZY IDEALS IN ORDERED Γ-SEMIGROUPS

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Abstract. In this paper, we introduce the notion of interval valued $(\tilde{\eta}, \tilde{\delta})$ bipolar fuzzy ideal, bi-ideal,interior ideal, $(\epsilon, \epsilon \vee q)$-bipolar fuzzy ideal of ordered Γ-semigroups and discuss some properties with examples.

1. Introduction

Fuzzy set was introduced by Zadeh [**19**]. Interval valued fuzzy sets were introduced independently by Zadeh [**20**]. Ordered Γ-semigroup was studied by Kehayopula [**9**]. Pavel Pal [**13**] studied the Regularity of Po-Γ-semigroups in terms of fuzzy subsemigroups and fuzzy bi-ideals. Bipolar fuzzy set was first studied by Lee [**11**]. Bipolar fuzzy set is an extension of fuzzy set whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$. Faiz Muhammad Khan et al [**3**] introduced the concepts of (λ, θ)-fuzzy bi-ideal and (λ, θ)-fuzzy subsemigroup. Jun *et al* [**4**] provided some results on ordered semigroups characterized by their $(\epsilon, \epsilon \vee q)$-fuzzy bi-ideals. Kazanci and Yamak [**5**]introduced the concept of a generalized fuzzy bi-ideal in semigroup and established some properties of fuzzy bi-ideals in terms of $(\epsilon, \epsilon \vee q)$-fuzzy bi-ideals. Kehayopula and Tsingelies [**8**] initiated the study of fuzzy ordered semigroups. Bhakat and Das [**1**] introduced the concepts of $(\epsilon, \epsilon \vee q)$-fuzzy subgroups using the notion "belongingness (∈)" and "quasi-coincidence (y)". In this paper we define the new notions of interval valued $(\tilde{\eta}, \tilde{\delta})$ bipolar fuzzy ideal, bi-ideal, interior ideal, interval valued $(\tilde{\epsilon}, \tilde{\epsilon} \vee q)$-bipolar fuzzy ideal of ordered Γ-semigroup and discuss some properties with examples.

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2. Preliminaries

DEFINITION 2.1. [**15**] An ordered Γ-semigroup (shortly po-Γ-semigroup) is a Γ-semigroup S together with an order relation $\leqslant$ such that $a \leqslant b$ implies $a\gamma c \leqslant b\gamma c$ and $c\gamma a \leqslant c\gamma b$ for all $a, b, c \in S$ and $\gamma \in \Gamma$.

DEFINITION 2.2. [**15**] Let A and B be two non empty subsets of a Γ- semigroup S. We denote (i) $(A] = \{t \in S \,|\, t \leqslant h \, for\, some \, h \in A\}$,
(ii) $A\Gamma B = \{a\alpha b : a \in A,\, b \in B \; and \; \alpha \in \Gamma\}$,
(iii) $A_x = \{(y, z) \in S \times S \,|\, x \leqslant y\alpha z\}$.

DEFINITION 2.3. [**10**] A non-empty subset B of a po Γ-semigroup S is called a bi-ideal of S if (i) $a \in B$, $b \in S$ and $b \leqslant a$ implies $b \in B$, (ii) $B\Gamma S\Gamma B \subseteq B$.

DEFINITION 2.4. [**19**] Let X be a non-empty set. A mapping $\widetilde{\mu} : X \to D[0, 1]$ is called an *i-v fuzzy subset* of X. For any $x \in X$, $\widetilde{\mu}(x) = [\,\mu^-(x), \mu^+(x)]$, where μ^- and μ^+ are fuzzy subsets of X such that $\mu^-(x) \leqslant \mu^+(x)$. Thus $\widetilde{\mu}(x)$ is an interval(a closed subset of $[0, 1]$) and not a number from the interval $[0, 1]$ as in the case of a fuzzy set. Let $\tilde{\mu}, \tilde{\nu}$ be i-v fuzzy subsets of X. The following are defined by
(i) $\tilde{\mu} \leqslant \widetilde{\nu} \Longleftrightarrow \widetilde{\mu}(x) \leqslant \widetilde{\nu}(x)$, (ii) $\widetilde{\mu} = \widetilde{\nu} \Longleftrightarrow \widetilde{\mu}(x) = \widetilde{\nu}(x)$
(iii) $(\,\widetilde{\mu} \cup \widetilde{\nu}\,)(x) = \max^i \left\{\, \widetilde{\mu}(x), \widetilde{\nu}(x)\right\}$, (iv) $(\,\widetilde{\mu} \cap \widetilde{\nu}\,)(x) = \min^i \left\{\, \widetilde{\mu}(x), \widetilde{\nu}(x)\right\}$
(v) $\left(\bigcap_{j\in\Omega} \widetilde{A}_j\right)(x) = \inf^i\{\widetilde{A}_j(x) \,|\, j \in \Omega\}$,
(vi) $\left(\bigcup_{j\in\Omega} \widetilde{A}_j\right)(x) = \sup^i\{\widetilde{A}_j(x) \,|\, j \in \Omega\}$,
where $\inf^i\{\widetilde{A}_j(x) \,|\, j \in \Omega\} = \left[\inf_{j\in\Omega}\{A_j^-(x)\},\, \inf_{j\in\Omega}\{A_j^+(x)\}\right]$ is the interval valued infimum norm and $\sup^i\{\widetilde{A}_j(x) \,|\, j \in \Omega\} = \left[\sup_{j\in\Omega}\{A_j^-(x)\},\, \sup_{j\in\Omega}\{A_j^+(x)\}\right]$ is the interval valued supremum norm.

DEFINITION 2.5. [**18**] By an interval number $\widetilde{a}$ we mean an interval $[a^-, a^+]$ such that $0 \leqslant a^- \leqslant a^+ \leqslant 1$, where a^- and a^+ are the lower and upper limits of $\widetilde{a}$ respectively. The set of all closed subintervals of $[0, 1]$ is denoted by $D[0, 1]$. We also identify the interval $[a, a]$ by the number $a \in [0, 1]$. For any interval numbers $\widetilde{a}_j = [a_j^-, a_j^+], \widetilde{b}_j = [b_j^-, b_j^+] \in D[0, 1], j \in \Omega$ where Ω is any index set. We define
$\max^i \left\{\, \widetilde{a}_j, \widetilde{b}_j\right\} = [\max\left\{a_j^-, b_j^-\right\}, \max\left\{a_j^+, b_j^+\right\}]$,
$\min^i \left\{\, \widetilde{a}_j, \widetilde{b}_j\right\} = [\min\left\{a_j^-, b_j^-\right\}, \min\left\{a_j^+, b_j^+\right\}]$,
$\inf \widetilde{a}_j = \left[\bigcap_{j\in\Omega} a_j^-, \bigcap_{j\in\Omega} a_j^+\right], \sup \widetilde{a}_j = \left[\bigcup_{j\in\Omega} a_j^-, \bigcup_{i\in\Omega} a_j^+\right].$
Let (i) $\widetilde{a} \leqslant \widetilde{b} \Longleftrightarrow a^- \leqslant b^-$ and $a^+ \leqslant b^+$, (ii) $\widetilde{a} = \widetilde{b} \Longleftrightarrow a^- = b^-$ and $a^+ = b^+$.
(iii) $\widetilde{a} < \widetilde{b} \Longleftrightarrow \widetilde{a} \leqslant \widetilde{b}$ and $\widetilde{a} \neq \widetilde{b}$, (iv) $k\widetilde{a} = [ka^-, ka^+]$, whenever $0 \leqslant k \leqslant 1$.

DEFINITION 2.6. [**11**] A bipolar fuzzy set A in a universe U is an object having the form $A = \left\{\left\langle x, \mu_A^P(x), \mu_A^N(x)\right\rangle : x \in X\right\}$, where $\mu_A^P : X \to [0, 1]$ and $\mu_A^N : X \to [-1, 0]$. Here $\mu_A^P(x)$ represents the degree of satisfaction of the element

x to the property and $\mu_A^N(x)$ represents the degree of satisfaction of x to some implict counter property of A. For simplicity the symbol $\langle \mu_A^P, \mu_A^N \rangle$ is used for the bipolar fuzzy set $A = \{\langle x, \mu_A^P(x), \mu_A^N(x) \rangle : x \in X\}$.

Definition 2.7. [**3**] A fuzzy subset μ of an ordered Γ-semigroup S is called a (λ, θ)-fuzzy bi-ideal of S if it satisfies the following conditions
(i) If $x \leqslant y$, then $\mu(x) \geqslant \mu(y)$,
(ii) $\max\{\mu(xy), \lambda\} \geqslant \min\{\mu(x), \mu(y), \theta\}$,
(iii) $\max\{\mu(xyz), \lambda\} \geqslant \min\{\mu(x), \mu(z), \theta\}$, for all $x, y, z \in S$.

Definition 2.8. [**6**] A fuzzy subset μ of a po Γ-semigroup S is called a fuzzy bi-ideal of S if
(i) If $x \leqslant y$, then $\mu(x) \geqslant \mu(y)$ and
(ii) $\mu(x\alpha y\beta z) \geqslant \min\{\mu(x), \mu(z)\}$ for every $x, y, z \in S$ and every $\alpha, \beta \in \Gamma$.

Definition 2.9. [**12**] A fuzzy subset μ of an ordered Γ-semigroup S is called a fuzzy Γ-subsemigroup of S if
(i) $x \leqslant y \Rightarrow \mu(x) \geqslant \mu(y)$ for all $x, y \in S$, and
(ii) $\mu(x\alpha y) \geqslant \min\{\mu(x), \mu(y)\}$ for all $x, y \in S$ and $\alpha \in \Gamma$.

Definition 2.10. [**13**] A fuzzy subset μ of an ordered Γ-semigroup S is called a fuzzy right (resp. left) ideal of S if
(i) $x \leqslant y \Rightarrow \mu(x) \geqslant \mu(y)$ for all $x, y \in S$, and
(ii) $\mu(x\alpha y) \geqslant \mu(x)$ (resp. $\mu(x\alpha y) \geqslant \mu(y)$) for all $x, y \in S$ and $\alpha \in \Gamma$.
A fuzzy subset μ of an ordered Γ-semigroup S is called a fuzzy ideal of S, if it is both fuzzy left ideal and fuzzy right ideal.

Definition 2.11. Let A be a bipolar fuzzy set, if χ_A is the characteriatic function of A, then $(\chi_A)_\eta^{\tilde{\delta}}$ is defined as

$$(\chi_A)_\alpha^\beta(x) = \begin{cases} \beta \ if \ x \in A, \\ \alpha \ if \ x \notin A. \end{cases}$$

Definition 2.12. [**14**] For two bipolar fuzzy subsets $\mu = (\mu^P, \mu^N)$ and $\lambda = (\lambda^P, \lambda^N)$ of S, the product of two bipolar fuzzy subsets is denoted by $\mu \circ \lambda$ and is defined as

$$(\mu^P \circ \lambda^P)(x) = \begin{cases} \sup\limits_{(s,t)\in A_x} \{\mu^P(s) \wedge \lambda^P(t)\} \ if \ A_x \neq 0 \\ 0 \quad if \quad A_x = 0 \end{cases}$$

$$(\mu^N \circ \lambda^N)(x) = \begin{cases} \inf\limits_{(s,t)\in A_x} \{\lambda^N(s) \vee \lambda^N(t)\} \ if \ A_x \neq 0 \\ 0 \quad if \quad A_x = 0 \end{cases}$$

Definition 2.13. A bipolar $(\eta, \tilde{\delta})$ fuzzy sub Γ-semigroup $B = (\mu_B^P, \mu_B^N)$ of S is called a bipolar $(1, 2)$ fuzzy-Γ-ideal of S if
(i) $\max\{\mu_B^P(p\alpha q\beta(r\gamma s)), \eta^P\} \geqslant \min\{\mu_B^P(p), \mu_B^P(r), \mu_B^P(s), \tilde{\delta}^P\}$ and
(ii) $\min\{\tilde{\mu}_B^N(p\alpha q\beta(r\gamma s)), \eta^N\} \leqslant \max\{\tilde{\mu}_B^N(p), \tilde{\mu}_B^N(r), \mu_B^P(s), \tilde{\delta}^N\}$,
for all $p, q, r, s \in S$ and $\alpha, \beta, \gamma \in \Gamma$.

3. $(\tilde{\eta},\tilde{\delta})$-interval valued bipolar fuzzy bi-ideals of ordered Γ-semigroups

In this section S denote as ordered Γ-semigroup. In what follows, $(\tilde{\eta}^P,\tilde{\delta}^P) \in D[0,1]$ and $(\tilde{\eta}^N,\tilde{\delta}^N) \in D[-1,0]$ be such that $0 \leqslant \tilde{\eta}^P < \tilde{\delta}^P \leqslant 1$ and $-1 \leqslant \tilde{\delta}^N < \tilde{\eta}^N \leqslant 0$, both $(\tilde{\eta},\tilde{\delta}) \in D[0,1]$ are arbitrary but fixed.

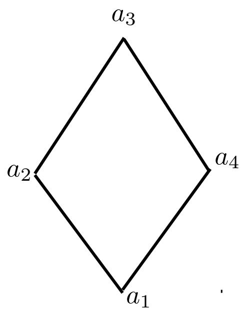
Definition 3.1. An interval valued fuzzy subset $\tilde{\mu}$ of S is called a $(\tilde{\eta},\tilde{\delta})$-interval valued (shortly i.v) bipolar fuzzy subsemigroup of S if it satisfies the following conditions: (i) $x \leqslant y \Rightarrow \tilde{\mu}^P(x) \geqslant \tilde{\mu}^P(y)$ and $x \leqslant y \Rightarrow \tilde{\mu}^N(x) \leqslant \tilde{\mu}^N(y)$.
(ii) $\max\{\tilde{\mu}^P(x\alpha y),\tilde{\eta}^P\} \geqslant \min\{\tilde{\mu}^P(x),\tilde{\mu}^P(y),\tilde{\tilde{\delta}}^P\}$ and
$\min\{\tilde{\mu}^N(x\alpha y),\tilde{\eta}^N\} \leqslant \max\{\tilde{\mu}^N(x),\tilde{\mu}^N(y),\tilde{\delta}^N\}$ for all $x, y \in S$.

Example 3.1. Let $S = \{a_1, a_2, a_3, a_4\}$ and $\Gamma = \{\alpha\}$ where α is defined on S with the following Cayley table:

α	a_1	a_2	a_3	a_4
a_1	a_1	a_1	a_1	a_1
a_2	a_1	a_2	a_3	a_4
a_3	a_1	a_3	a_3	a_3
a_4	a_1	a_3	a_3	a_3

$\leqslant := \{(a_1,a_1),(a_1,a_2),(a_1,a_3),(a_1,a_4),(a_2,a_2),(a_2,a_3),(a_2,a_4),(a_3,a_3),(a_4,a_3),$ $(a_4,a_4)\}$. We give the covering relation and the figure of S.

$$\prec = \{(a_1,a_2),(a_1,a_3),(a_1,a_4),(a_2,a_3),(a_2,a_4),(a_4,a_3)\}.$$



Define interval valued bipolar fuzzy subset $\tilde{\mu} = [\tilde{\mu}^P, \tilde{\mu}^N]$

$$\tilde{\mu}^P(x) = \begin{cases} [0.6,0.7] & if\ x = a_1 \\ [0.4,0.5] & if\ x = a_2 \\ [0.1,0.2] & if\ x = a_3 \\ [0.2,0.3] & if\ x = a_4 \end{cases} \quad \tilde{\mu}^N(x) = \begin{cases} [-0.9,-0.8] & if\ x = a_1 \\ [-0.7,-0.6] & if\ x = a_2 \\ [-0.3,-0.2] & if\ x = a_3 \\ [-0.6,-0.5] & if\ x = a_4 \end{cases}$$

Then $\tilde{\mu}$ is a $([05, 0.6], [07, 0.8])$ i.v bipolar fuzzy subsemigroup of S.

Definition 3.2. An i.v fuzzy subset $\tilde{\mu}$ of S is called a $(\tilde{\eta},\tilde{\delta})$-i.v bipolar fuzzy bi-ideal of S if it satisfies the following conditions

(i) if $x \leqslant y$, then $\tilde{\mu}^P(x) \geqslant \tilde{\mu}^P(y)$ and $x \leqslant y$, then $\tilde{\mu}^N(x) \leqslant \tilde{\mu}^N(y)$

(ii) $\max\{\tilde{\mu}^P(x\alpha y), \tilde{\eta}^+\} \geqslant \min\{\tilde{\mu}^P(x), \tilde{\mu}^P(y), \tilde{\delta}^P\}$,
$\min\{\tilde{\mu}^N(x\alpha y), \tilde{\eta}^N\} \leqslant \max\{\tilde{\mu}^N(x), \tilde{\mu}^N(y), \tilde{\delta}^N\}$.
(iii) $\max\{\tilde{\mu}^P(x\alpha y\beta z), \tilde{\eta}^P\} \geqslant \min\{\tilde{\mu}^P(x), \tilde{\mu}^P(z), \tilde{\delta}^P\}$,
$\min\{\tilde{\mu}^N(x\alpha y\beta z), \tilde{\eta}^N\} \leqslant \max\{\tilde{\mu}^N(x), \tilde{\mu}^N(z), \tilde{\delta}^N\}$, for all $x, y, z \in S$,$\alpha, \beta \in \Gamma$.

Example 3.2. Let $S = \{a_1, a_2, a_3, a_4\}$ and $\Gamma = \{\alpha, \beta\}$ where α, β is defined on S with the following Cayley tables:

α	a_1	a_2	a_3	a_4
a_1	a_1	a_1	a_1	a_1
a_2	a_1	a_2	a_3	a_4
a_3	a_1	a_3	a_3	a_3
a_4	a_1	a_3	a_3	a_3

β	a_1	a_2	a_3	a_4
a_1	a_1	a_1	a_1	a_1
a_2	a_1	a_2	a_3	a_4
a_3	a_1	a_3	a_3	a_3
a_4	a_1	a_2	a_3	a_4

$\leqslant := \{(a_1, a_1), (a_1, a_2), (a_1, a_3), (a_1, a_4), (a_2, a_2), (a_2, a_3), (a_2, a_4), (a_3, a_3), (a_4, a_3), (a_4, a_4)\}$.

Define i.v bipolar fuzzy subset $\tilde{\mu} = [\tilde{\mu}^P, \tilde{\mu}^N] : S \times \Gamma \times S \to [0, 1] \times [-1, 0]$ as

$$\tilde{\mu}^P(x) = \begin{cases} [0.81, 0.91] \; if \; x = a_1 \\ [0.62, 0.72] \; if \; x = a_2 \\ [0.34, 0.44] \; if \; x = a_3 \\ [0.43, 0.53] \; if \; x = a_4 \end{cases} \quad \tilde{\mu}^N(x) = \begin{cases} [-0.85, -0.75] \; if \; x = a_1 \\ [-0.65, -0.55] \; if \; x = a_2 \\ [-0.30, -0.20] \; if \; x = a_3 \\ [-0.50, -0.40] \; if \; x = a_4 \end{cases}$$

Then $\tilde{\mu}$ is a $([0.70, 0.80], [0.60, 0.90])$ i.v bipolar fuzzy bi-ideal of S

Theorem 3.1. *An i.v fuzzy subset $\tilde{\mu}_{\tilde{\eta}}$ is a $(\tilde{\eta}, \tilde{\delta})$-i.v bipolar fuzzy ordered Γ-subsemigroup (left, right, bi-ideal, interior ideal, (1, 2)-ideal) of S. Then the lower level set $\mu_{\tilde{\eta}} = [\mu^P_{\tilde{\eta}}, \mu^N_{\tilde{\eta}}]$ is an ordered Γ-subsemigroup (left, right, bi-ideal, interior ideal, (1, 2)-ideal) of S, where $\tilde{\mu}^P_{\tilde{\eta}} = \{p \in S | \tilde{\mu}^P(x) > \tilde{\eta}^P\}$ and $\tilde{\mu}^N_{\tilde{\eta}} = \{p \in S | \tilde{\mu}^N(x) < \tilde{\eta}^N\}$.*

Proof. Suppose that $\tilde{\mu}_{\tilde{\eta}}$ is a $(\tilde{\eta}, \tilde{\delta})$-i.v bipolar fuzzy ordered Γ-subsemigroup. Let $\mu^P_{\tilde{\eta}}$ is a $(\tilde{\eta}^P, \tilde{\delta}^P)$ fuzzy Γ-subsemigroup. Let $x, y \in S$ and $\alpha \in \Gamma$ such that $p, q \in \mu^P_{\tilde{\eta}}$. Then $\tilde{\mu}^P(x) > \tilde{\eta}^P, \tilde{\mu}^P(y) > \tilde{\eta}^P$. Since $\tilde{\mu}^P$ is a $(\tilde{\eta}^P, \tilde{\delta}^P)$ fuzzy subsemigroup, therefore $\max\{\tilde{\mu}^P(x\alpha y), \tilde{\eta}^P\} \geqslant \min\{\tilde{\mu}^P(x), \tilde{\mu}^P(y), \tilde{\delta}^P\} > \min\{\tilde{\eta}^P, \tilde{\eta}^P, \tilde{\delta}^P\} = \tilde{\eta}^P$. Hence $\tilde{\mu}^P(x\alpha y) > \tilde{\eta}^P$. It shows that $p\alpha q \in \mu^P_{\tilde{\eta}}$. Therefore $\mu^P_{\tilde{\eta}}$ is a Γ-subsemigroup of S. Let $\mu^N_{\tilde{\eta}}$ is a $(\tilde{\eta}^N, \tilde{\delta}^N)$ interval valued fuzzy ordered Γ-subsemigroup. Let $x, y \in S$ such that $x, y \in \mu^N_{\tilde{\eta}}$. Then $\tilde{\mu}^N(x) < \tilde{\eta}^N, \tilde{\mu}^N(y) < \tilde{\eta}^N$.Since $\tilde{\mu}^N$ is a $(\tilde{\eta}^N, \tilde{\delta}^N)$ interval valued fuzzy ordered Γ-subsemigroup. Therefore $\min\{\tilde{\mu}^N(p\alpha q), \tilde{\eta}^N\} \leqslant \max\{\tilde{\mu}^N(x), \tilde{\mu}^N(y), \tilde{\delta}^N\} < \max\{\tilde{\eta}^N, \tilde{\eta}^N, \tilde{\delta}^N\} = \tilde{\eta}^N$. Hence $\tilde{\mu}^N(p\alpha q) < \tilde{\eta}^N$. It shows that $p\alpha q \in \mu^N_{\tilde{\eta}}$. Therefore $\mu^N_{\tilde{\eta}}$ is a Γ-subsemigroup of S. Hence $\tilde{\mu}_{\tilde{\eta}} = [\tilde{\mu}^P_{\tilde{\eta}}, \tilde{\mu}^N_{\tilde{\eta}}]$ is a Γ-subsemigroup of S. □

Theorem 3.2. *A non-empty subset A of S is an ordered Γ-subsemigroup (left, right, bi-ideal, interior ideal, (1, 2)-ideal) of S if and only if the i. v bipolar fuzzy* *subset* $\tilde{\mu} = [\tilde{\mu}^P, \tilde{\mu}^N]$ *of* S *defined as*

$$\tilde{\mu}^P(x) = \begin{cases} \geqslant \tilde{\delta}^P \, for\, all\, p \in (A], \\ \tilde{\eta}^P \, for\, allp \notin (A], \end{cases} \qquad \tilde{\mu}^N(x) = \begin{cases} \leqslant \tilde{\delta}^N \, for\, all\, p \in (A], \\ \tilde{\eta}^N \, for\, all\, p \notin (A], \end{cases}$$

is a $(\tilde{\eta}, \tilde{\delta})$*-i.v bipolar fuzzy ordered* Γ*-subsemigroup (left, right, bi-ideal, interior ideal, (1, 2)-ideal) of* S*.*

Proof. Assume that A is an ordered Γ-subsemigroup of S. Let $p, q \in S$ be such that $p, q \in (A]$ then $p\alpha q \in (A]$ and $\alpha \in \Gamma$. Hence $\tilde{\mu}^P(p\alpha q) \geqslant \tilde{\delta}^P$ and $\tilde{\mu}^N(p\alpha q) \leqslant \tilde{\delta}^N$. Therefore $\max\{\tilde{\mu}^P(p\alpha q), \tilde{\eta}^P\} \geqslant \tilde{\delta}^P = \min\{\tilde{\mu}^P(x), \tilde{\mu}^P(y), \tilde{\delta}^P\}$ and $\min\{\tilde{\mu}^N(p\alpha q), \tilde{\eta}^N\} \leqslant \tilde{\delta}^N = \max\{\tilde{\mu}^N(x), \tilde{\mu}^N(y), \tilde{\delta}^N\}$. If $p \notin A$ or $q \notin (A]$ then $\min\{\tilde{\mu}^P(x), \tilde{\mu}^P(y), \tilde{\delta}^P\} = \tilde{\eta}^P$, $\max\{\tilde{\mu}^N(x), \tilde{\mu}^N(y), \tilde{\delta}^N\} = \tilde{\eta}^N$. That is $\max\{\tilde{\mu}^P(p\alpha q),\ \tilde{\eta}^P\} \geqslant \min\{\tilde{\mu}^P(x), \tilde{\mu}^P(y), \tilde{\delta}^P\}$ and $\min\{\tilde{\mu}^N(p\alpha q), \tilde{\eta}^N\} \leqslant \max\{\tilde{\mu}^N(x), \tilde{\mu}^N(y), \tilde{\delta}^N\}$. Therefore $\tilde{\mu} = [\tilde{\mu}^P, \tilde{\mu}^N]$ is an interval valued bipolar fuzzy Γ-subsemigroup of S.
Conversely assume that $\tilde{\mu} = [\tilde{\mu}^P, \tilde{\mu}^N]$ is a i.v bipolar fuzzy Γ-subsemigroup of S. Let $p, q \in (A]$. Then $\tilde{\mu}^P(x) \geqslant \tilde{\delta}^P, \tilde{\mu}^P(y) \geqslant \tilde{\delta}^P$ and $\tilde{\mu}^N(x) \leqslant \tilde{\delta}^N, \mu(y) \leqslant \tilde{\delta}^N$. Now $\tilde{\mu}^P$ is $(\tilde{\eta}^P, \tilde{\delta}^P)$ and $\tilde{\mu}^N$ is $(\tilde{\eta}^N, \tilde{\delta}^N)$- fuzzy Γ-subsemigroup of S. Therefore $\max\{\tilde{\mu}^P(p\alpha q), \tilde{\eta}^P\} \geqslant \min\{\tilde{\mu}^P(x), \tilde{\mu}^P(y), \tilde{\delta}^P\} \geqslant \min\{\tilde{\delta}^P, \tilde{\delta}^P, \tilde{\delta}^P, \} = \tilde{\delta}^P$ and $\min\{\tilde{\mu}^N(p\alpha q), \tilde{\eta}^N\} \leqslant \max\{\tilde{\mu}^N(x), \tilde{\mu}^N(y), \tilde{\delta}^N\} \leqslant \max\{\tilde{\delta}^N, \tilde{\delta}^N, \tilde{\delta}^N, \} = \tilde{\delta}^N$.
It follows that $p\alpha q \in (A]$. Therefore A is a ordered Γ-subsemigroup of S. □

Corollary 3.1. *A non-empty subset* A *of* S *is an ordered* Γ*-subsemigroup(left, right, bi-ideal, interior ideal,(1, 2)-ideal) of* S *if and only if the fuzzy subset* μ *of* S *defined as*

$$\tilde{\mu}^P(x) = \begin{cases} \geqslant [0.5, 0.5] \, for\, all\, x \in (A] \\ [0, 0] \, for\, all\, x \notin (A] \end{cases} \qquad \tilde{\mu}^N(x) = \begin{cases} \leqslant [-0.5, -0.5] \, for\, all\, x \in (A] \\ [0, 0] \, for\, all\, x \notin (A] \end{cases}$$

is a $(\epsilon, \epsilon \vee q)$*-i. v bipolar fuzzy subsemigroup(left, right, bi-ideal, interior ideal, (1, 2)-ideal) of* S*.*

Proof. The proof follows by taking $\tilde{\eta}^P = [0, 0], \tilde{\delta}^P = [0.5, 0.5]$ and $\tilde{\eta}^N = [0, 0], \tilde{\delta}^N = [-0.5, -0.5]$ in Theorem 3.2 □

Theorem 3.3. *An i.v fuzzy subset* $\tilde{\mu}$ *of* S *is a* $(\tilde{\eta}, \tilde{\delta})$*-i.v bipolar fuzzy subsemigroup (left, right, bi-ideal, interior ideal, (1, 2)-ideal) of* S *if and only if each non-empty level subset* $(\tilde{\mu}^{(t,s)})$ *is a subsemigroup (left, right, bi-ideal, interior ideal, (1, 2)-ideal) of* S *for all* $t \in (\tilde{\eta}^P, \tilde{\delta}^P]$ *and* $s \in (\tilde{\eta}^N, \tilde{\delta}^N]$*.*

Proof. Assume that $\tilde{\mu}^{(t,s)}$ is an ordered Γ-subsemigroup over S for each $t \in [0, 1]$ and $s \in [-1, 0]$. For each $p_1, p_2 \in S$ and $a \in (A]$, let $t = \min\{\tilde{\mu}^P(p_1), \tilde{\mu}^P(p_2)\}$ and $s = \max\{\tilde{\mu}^N(p_1), \tilde{\mu}^N(p_2)\}$, then $p_1, p_2 \in \tilde{\mu}^{(t,s)}$. That is $\max\{\tilde{\mu}^P((p\gamma q), \tilde{\eta}^P\} \geqslant t = \min\{\tilde{\mu}^P(p_1), \tilde{\mu}^P(p_2), \tilde{\delta}^P\}$ and $\min\{\tilde{\mu}^N(p_1\gamma p_2), \tilde{\eta}^N\} \leqslant s = \max\{\tilde{\mu}^N(p_1), \tilde{\mu}^N(p_2), \tilde{\delta}^N\}$. This shows that $\tilde{\mu}$ is i. v bipolar fuzzy Γ-subsemigroup over S.
Conversely, assume that $\tilde{\mu}$ is a i.v bipolar fuzzy ordered Γ-subsemigroup of S. For each $a \in (A], t \in [0, 1]$ and $s \in [-1, 0]$ and $p_1, p_2 \in \tilde{\mu}^{(t,s)}$.

We have $\tilde{\mu}^P(p_1) \geqslant t, \tilde{\mu}^P(p_2) \geqslant t$ and $\tilde{\mu}^N(p_1) \leqslant s$, $\tilde{\mu}^N(p_2) \leqslant s$. Since $\tilde{\mu}$ is a i.v bipolar fuzzy Γ-subsemigroup of S,

$$\max\{\tilde{\mu}^P(p_1\gamma p_2), \tilde{\eta}^P\} \geqslant \min\{\tilde{\mu}^P(p_1), \tilde{\mu}^P(p_2, \tilde{\delta}^P)\} \geqslant t$$

$$\min\{\tilde{\mu}^N(p_1\gamma p_2), \tilde{\eta}^N\} \leqslant \max\{\tilde{\mu}^N(p_1), \tilde{\mu}^N(p_2, \tilde{\delta}^N)\} \leqslant s,$$

$\gamma \in \Gamma$. Therefore $\tilde{\mu}^{(t,s)}$, this implies that $p_1\gamma p_2 \in \tilde{\mu}^{(t,s)}$. Therefore $\tilde{\mu}^{(t,s)}$ is a Γ-subsemigroup of S for each $t \in [0,1]$ and $s \in [-1,0]$. Similar proofs holds for left, right, bi-ideal, interior ideal, (1, 2)-ideal also. □

EXAMPLE 3.3. Every i.v bipolar fuzzy subsemigroup $\tilde{\mu} = [\tilde{\mu}^P, \tilde{\mu}^N]$ of ordered Γ-semigroup S is a $(\tilde{\eta}, \tilde{\delta})$-i.v bipolar fuzzy subsemigroup of S, but converse is not true.

For the Example 3.1, we define i.v bipolar fuzzy subset $\tilde{\mu} = [\tilde{\mu}^P, \tilde{\mu}^N]$ by

$$\tilde{\mu}^P(x) = \begin{cases} [0.65, 0.75] & if\ x = a_1 \\ [0.58, 0.68] & if\ x = a_2 \\ [0.51, 0.61] & if\ x = a_3 \\ [0.53, 0.63] & if\ x = a_4 \end{cases} \quad \tilde{\mu}^N(x) = \begin{cases} [-0.85, -0.75] & if\ x = a_1 \\ [-0.81, -0.71] & if\ x = a_2 \\ [-0.68, -0.58] & if\ x = a_3 \\ [-0.75, -0.65] & if\ x = a_4 \end{cases}$$

Then $\tilde{\mu}$ is a $([0.46, 0.56], [0.60, 0.70])$ i.v bipolar fuzzy ordered Γ-subsemigroup of S, but not a i.v bipolar fuzzy subsemigroup.
Since $\tilde{\mu}^P(a_4\alpha a_2) = \tilde{\mu}^P(a_3) = [0.51, 0.61] \ngeqslant \min\{\tilde{\mu}^P(a_4), \tilde{\mu}^P(a_4)\} = [0.53, 0.63]$

COROLLARY 3.2. *Every $(\epsilon, \epsilon \vee q)$ i.v bipolar fuzzy ordered Γ-subsemigroup of S is a $(\tilde{\eta}, \tilde{\delta})$-i.v bipolar fuzzy ordered Γ-subsemigroup of S, but converse is not true.*

For the Example 3.1, define i.v bipolar fuzzy subset $\tilde{\mu} = [\tilde{\mu}^P, \tilde{\mu}^N]$

$$\tilde{\mu}^P(x) = \begin{cases} [0.42, 0.52] & if\ x = a_1 \\ [0.38, 0.48] & if\ x = a_2 \\ [0.26, 0.36] & if\ x = a_3 \\ [0.30, 0.40] & if\ x = a_4 \end{cases} \quad \tilde{\mu}^N(x) = \begin{cases} [-0.33, -0.43] & if\ x = a_1 \\ [-0.30, -0.40] & if\ x = a_2 \\ [-0.20, -0.30] & if\ x = a_3 \\ [-0.24, -0.34] & if\ x = a_4 \end{cases}$$

Then $\tilde{\mu}$ is a $([0.35, 0.45], [0.45, 0.55])$ i.v bipolar fuzzy ordered Γ-subsemigroup of S, but not a $(\epsilon, \epsilon \vee q)$ i.v bipolar fuzzy ordered Γ-subsemigroup.
Since $\tilde{\mu}^P(a_4\alpha a_2) = \tilde{\mu}^P(a_3) = [0.26, 0.36] \ngeqslant \min\{\tilde{\mu}^P(a_4), \tilde{\mu}^P(a_2)\} = [0.30, 0.40]$

EXAMPLE 3.4. Every i.v bipolar fuzzy bi-ideal $\tilde{\mu} = [\tilde{\mu}^P, \tilde{\mu}^N]$ of an ordered Γ-semigroup S is a $(\tilde{\eta}, \tilde{\delta})$-i.v bipolar fuzzy bi-ideal of S, but converse is not true.

For the Example 3.2, we define i.v bipolar fuzzy subset $\tilde{\mu} = [\tilde{\mu}^P, \tilde{\mu}^N]$

$$\tilde{\mu}^P(x) = \begin{cases} [0.71, 0.81] & if\ x = a_1 \\ [0.52, 0.62] & if\ x = a_2 \\ [0.24, 0.34] & if\ x = a_3 \\ [0.33, 0.43] & if\ x = a_4 \end{cases} \quad \tilde{\mu}^N(x) = \begin{cases} [-0.85, -0.75] & if\ x = a_1 \\ [-0.65, -0.55] & if\ x = a_2 \\ [-0.30, -0.20] & if\ x = a_3 \\ [-0.50, -0.40] & if\ x = a_4 \end{cases}$$

Then $\tilde{\mu}$ is a $([0.60,0.70],[0.75,0.85)$ i.v bipolar fuzzy bi-ideal of S, but not a i. v bipolar fuzzy bi-ideal, since $\tilde{\mu}^P(a_2\alpha a_3\beta a_2) = \tilde{\mu}^P(a_3) = [0.24,0.34] \ngeqslant \min\{\tilde{\mu}^P(a_2), \tilde{\mu}^P(a_2)\} = [0.52,0.62]$

Corollary 3.3. *Every $(\tilde{\epsilon}, \tilde{\epsilon} \vee q)$ i.v bipolar fuzzy bi-ideal of S is a $(\tilde{\eta}, \tilde{\delta})$-i.v bipolar fuzzy bi-ideal of S, but converse is not true.*

For the Example 3.2, we define i.v bipolar fuzzy subset $\tilde{\mu} = [\tilde{\mu}^P, \tilde{\mu}^N]$

$$\tilde{\mu}^P(x) = \begin{cases} [0.33,0.43] & if\ x = a_1 \\ [0.28,0.38] & if\ x = a_2 \\ [0.15,0.25] & if\ x = a_3 \\ [0.20,0.30] & if\ x = a_4 \end{cases} \quad \tilde{\mu}^N(x) = \begin{cases} [-0.35,-0.32] & if\ x = a_1 \\ [-0.30,-0.28] & if\ x = a_2 \\ [-0.20,-0.18] & if\ x = a_3 \\ [-0.25,-0.22] & if\ x = a_4 \end{cases}$$

Then $\tilde{\mu}$ is a $([0.30,0.40],[0.37,0.47])$ fuzzy bi-ideal of S, but not a fuzzy bi-ideal, since $\tilde{\mu}^P(a_2\alpha a_3\beta a_2) = \tilde{\mu}^P(a_3) = [0.15,0.25] \ngeqslant \min\{\tilde{\mu}^P(a_2), \tilde{\mu}^P(a_2\} = [0.28,0.38]$

Definition 3.3. If χ_A is the characteristic function of A, then $(\chi_A)_{\tilde{\eta}}^{\tilde{\delta}}$ is defined as

$$(\chi_A^P)_{\tilde{\eta}}^{\tilde{\delta}}(x) = \begin{cases} \tilde{\delta}^P\ if\ x \in (A], \\ \tilde{\eta}^P\ if\ x \notin (A]. \end{cases} \quad (\chi_A^N)_{\tilde{\eta}}^{\tilde{\delta}}(x) = \begin{cases} \tilde{\delta}^N\ if\ x \in (A], \\ \tilde{\eta}^N\ if\ x \notin (A]. \end{cases}$$

Theorem 3.4. *A non empty subset A of S is a subsemigroup (left, right, bi-ideal, interior ideal, (1, 2)-ideal) of S if and only if fuzzy subset $\tilde{\chi}_A = [\chi_{(A]}^P, \chi_{(A]}^N]$ is a $(\tilde{\eta}, \tilde{\delta})$-i.v bipolar fuzzy subsemigroup(left, right, bi-ideal, interior ideal, (1, 2)-ideal) of S.*

Proof. Assume that A is a subsemigroup of S. Then $\tilde{\chi}_{(A]}$ is a i.v bipolar fuzzy subsemigroup of S and hence $\tilde{\chi}_{(A]}$ is an $(\tilde{\eta}, \tilde{\delta})$-i.v bipolar fuzzy subsemigroup of S. Conversely, let $p, q \in S$ be such that $p, q \in (A]$. Then $\chi_{(A]}^P(x) = \tilde{\delta}^P = \chi_{(A]}^P(y) = \tilde{\delta}^P$ and $\chi_{(A]}^N(x) = \tilde{\delta}^N = \chi_{(A]}^N(y) = \tilde{\delta}^N$. Since $\tilde{\chi}_{(A]}$ is a $(\tilde{\eta}, \tilde{\delta})$-i. v bipolar fuzzy subsemigroup. Consider

$$\begin{aligned} \max\{\chi_{(A]}^P(p\alpha q), \tilde{\eta}^P\} &\geqslant \min\{\chi_{(A]}^P(x), \chi_{(A]}^P(y), \tilde{\delta}^P\} \\ &= \min\{\tilde{\delta}^P, \tilde{\delta}^P, \tilde{\delta}^P\} \\ &= \tilde{\delta}^P \end{aligned}$$

as $\tilde{\eta}^P < \tilde{\delta}^P$, this implies that $\{\chi_{(A]}^P(p\alpha q)\} \geqslant \tilde{\delta}^P$. Thus $p\alpha q \in (A]$.
Therefore A is a subsemigroup of S. And

$$\begin{aligned} \min\{\chi_{(A]}^N(p\alpha q), \tilde{\eta}^N\} &\leqslant \max\{\chi_{(A]}^N(x), \chi_{(A]}^N(y), \tilde{\delta}^N\} \\ &= \max\{\tilde{\delta}^N, \tilde{\delta}^N, \tilde{\delta}^N\} \\ &= \tilde{\delta}^N \end{aligned}$$

as $\tilde{\delta}^N < \tilde{\eta}^N$, this implies that $\{\chi_{(A]}^N(p\alpha q)\} \leqslant \tilde{\delta}^N$. Thus $p\alpha q \in (A]$.
Therefore $(A]$ is a subsemigroup of S.

Let $p, q \in S$ be such that $p, q \notin (A]$. Then $\chi^P_{(A]}(x) = \tilde{\eta}^P = \chi^P_{(A]}(y) = \tilde{\eta}^P$ and $\chi^N_{(A]}(x) = \tilde{\eta}^P = \chi^N_{(A]}(y) = \tilde{\eta}^P$. Since $\tilde{\chi}_{(A]}$ is a $(\tilde{\eta}, \tilde{\delta})$-i. v bipolar fuzzy subsemigroup.

$$\begin{aligned}\max\{\chi^P_{(A]}(p\alpha q), \tilde{\eta}^P\} &\geqslant \min\{\chi^P_{(A]}(x), \chi^P_{(A]}(y), \tilde{\delta}^P\}\\ &= \min\{\tilde{\eta}^P, \tilde{\eta}^P, \tilde{\delta}^P\}\\ &= \tilde{\eta}^P\end{aligned}$$

as $\tilde{\eta}^P < \tilde{\delta}^P$, this implies that $\{\chi^P_{(A]}(p\alpha q)\} \geqslant \tilde{\eta}^P$. Thus $p\alpha q \in (A]$. Therefore $(A]$ is a subsemigroup of S. And

$$\begin{aligned}\min\{\chi^N_{(A]}(p\alpha q), \tilde{\eta}^N\} &\leqslant \max\{\chi^N_{(A]}(x), \chi^N_{(A]}(y), \tilde{\delta}^N\}\\ &= \max\{\tilde{\eta}^N, \tilde{\eta}^N, \tilde{\delta}^N\}\\ &= \tilde{\eta}^N\end{aligned}$$

as $\tilde{\delta}^N < \tilde{\eta}^N$, this implies that $\{\chi^N_{(A]}(p\alpha q)\} \leqslant \tilde{\eta}^N$. Thus $p\alpha q \in (A]$. Therefore $(A]$ is a subsemigroup of S. Similar to proof holds for left, right, bi-ideal, interior ideal, (1, 2)-ideal also. □

Definition 3.4. Let $\tilde{\mu}$ be a i.v bipolar fuzzy subset of an ordered semigroup S. We define the i.v bipolar fuzzy subsets $(\tilde{\mu}^P)^{\tilde{\delta}}_{\tilde{\eta}}(x) = \{\tilde{\mu}^P(x) \wedge \tilde{\delta}^P\} \vee \tilde{\eta}^P$ and $(\tilde{\mu}^N)^{\tilde{\delta}}_{\tilde{\eta}}(x) = \{\tilde{\mu}^N(x) \vee \tilde{\delta}^N\} \wedge \tilde{\eta}^N$ for all $p \in S$.

Definition 3.5. Let $\tilde{\mu}_1$ and $\tilde{\mu}_2$ be two i. v bipolar fuzzy subsets of an ordered semigroup S. Then we define the interval valued bipolar fuzzy subset

(i) $(\tilde{\mu}^P_1 \wedge^{\tilde{\delta}}_{\tilde{\eta}} \tilde{\mu}^P_2)(x) = \{\tilde{\mu}^P_1 \wedge \tilde{\mu}^P_2(x) \wedge \tilde{\delta}^P\} \vee \tilde{\eta}^P$
(ii) $(\tilde{\mu}^N_1 \wedge^{\tilde{\delta}}_{\tilde{\eta}} \tilde{\mu}^N_2)(x) = \{\tilde{\mu}^N_1 \wedge \tilde{\mu}^N_2(x) \vee \tilde{\delta}^N\} \wedge \tilde{\eta}^N$
(iii) $(\tilde{\mu}^P_1 \vee^{\tilde{\delta}}_{\tilde{\eta}} \tilde{\mu}^P_2)(x) = \{\tilde{\mu}^P_1 \vee \tilde{\mu}^P_2(x) \wedge \tilde{\delta}^P\} \vee \tilde{\eta}^P$
(iv) $(\tilde{\mu}^N_1 \vee^{\tilde{\delta}}_{\tilde{\eta}} \tilde{\mu}^N_2)(x) = \{\tilde{\mu}^N_1 \vee \tilde{\mu}^N_2(x) \vee \tilde{\delta}^N\} \wedge \tilde{\eta}^N$
(v) $(\tilde{\mu}^P_1 \circ^{\tilde{\delta}}_{\tilde{\eta}} \tilde{\mu}^P_2)(x) = \{\tilde{\mu}^P_1 \circ \tilde{\mu}^P_2(x) \wedge \tilde{\delta}^P\} \vee \tilde{\eta}^P$
(vi) $(\tilde{\mu}^N_1 \circ^{\tilde{\delta}}_{\tilde{\eta}} \tilde{\mu}^N_2)(x) = \{\tilde{\mu}^N_1 \circ \tilde{\mu}^N_2(x) \vee \tilde{\delta}^N\} \wedge \tilde{\eta}^N$.

Lemma 3.1. *Let A and B be non-empty subsets of S. Then the following hold:*
(i) $((\tilde{\mu}^P_1) \wedge^{\tilde{\delta}}_{\tilde{\eta}} (\tilde{\mu}^P_2))(x) = ((\tilde{\mu}^P_1)^{\tilde{\delta}}_{\tilde{\eta}} \wedge (\tilde{\mu}^P_2)^{\tilde{\delta}}_{\tilde{\eta}})$, $((\tilde{\mu}^N_1) \wedge^{\tilde{\delta}}_{\tilde{\eta}} (\tilde{\mu}^N_2))(x) = ((\mu^N_1)^{\tilde{\delta}}_{\tilde{\eta}} \wedge (\tilde{\mu}^N_2)^{\tilde{\delta}}_{\tilde{\eta}})$
(ii) $((\tilde{\mu}^P_1) \vee^{\tilde{\delta}}_{\tilde{\eta}} (\tilde{\mu}^P_2))(x) = ((\tilde{\mu}^P_1)^{\tilde{\delta}}_{\tilde{\eta}} \vee (\tilde{\mu}^P_2)^{\tilde{\delta}}_{\tilde{\eta}})$, $((\tilde{\mu}^N_1) \vee^{\tilde{\delta}}_{\tilde{\eta}} (\tilde{\mu}^N_2))(x) = ((\tilde{\mu}^N_1)^{\tilde{\delta}}_{\tilde{\eta}} \vee (\tilde{\mu}^N_2)^{\tilde{\delta}}_{\tilde{\eta}})$
(iii) $((\tilde{\mu}^P_1) \circ^{\tilde{\delta}}_{\tilde{\eta}} (\tilde{\mu}^P_2))(x) = ((\tilde{\mu}^P_1)^{\tilde{\delta}}_{\tilde{\eta}} \circ (\tilde{\mu}^P_2)^{\tilde{\delta}}_{\tilde{\eta}})$, $((\tilde{\mu}^N_1) \circ^{\tilde{\delta}}_{\tilde{\eta}} (\tilde{\mu}^N_2))(x) = ((\tilde{\mu}^N_1)^{\tilde{\delta}}_{\tilde{\eta}} \circ (\tilde{\mu}^N_2)^{\tilde{\delta}}_{\tilde{\eta}})$.

Lemma 3.2. *Let A and B be non-empty subsets of S. Then the following hold:*
(i) $(\chi^P_{(A]} \wedge^{\tilde{\delta}}_{\tilde{\eta}} \chi^P_{(B]}) = (\chi^P_{A\cap B})^{\tilde{\delta}}_{\tilde{\eta}}$ *and* $(\chi^N_{(A]} \wedge^{\tilde{\delta}}_{\tilde{\eta}} \chi^N_{(B]}) = (\chi^N_{A\cap B})^{\tilde{\delta}}_{\tilde{\eta}}$
(ii) $(\chi^P_{(A]} \vee^{\tilde{\delta}}_{\tilde{\eta}} \chi^P_{(B]}) = (\chi^P_{A\cup B})^{\tilde{\delta}}_{\tilde{\eta}}$ *and* $(\chi^N_{(A]} \vee^{\tilde{\delta}}_{\tilde{\eta}} \chi^N_{(A]}) = (\chi^N_{A\cup B})^{\tilde{\delta}}_{\tilde{\eta}}$
(iii) $(\chi^P_{(A]} \circ \Gamma^{\tilde{\delta}}_{\tilde{\eta}} \chi^P_B) = (\chi^P_{(A\Gamma B]})^{\tilde{\delta}}_{\tilde{\eta}}$ *and* $(\chi^N_A \circ \Gamma^{\tilde{\delta}}_{\tilde{\eta}} \chi^N_{(B]}) = (\chi^N_{(A\Gamma B]})^{\tilde{\delta}}_{\tilde{\eta}}$.

Proof. (i) and (ii) Straightforward.

(iii) Let $p \in S$. If $p \in (A\Gamma B]$, then $(\chi^P_{(A\Gamma B]})(x) = \tilde{\delta}^P$ and $(\chi^N_{(A\Gamma B]})(x) = \tilde{\delta}^N$. Since $p \leqslant a\alpha b$ for some $a \in (A]$, $b \in (B]$ and $\alpha \in \Gamma$, we have $(a, b) \in A_p$ and $A_p \neq 0$. We have

$$\begin{aligned}(\chi^P_{(A]} \circ_\Gamma \chi^P_{(A]})(x) &= \sup_{p=y\alpha z} \min\{\chi^P_{(A]}(y), \chi^P_{(A]}(z)\} \\ &\geqslant \min\{\chi^P_{(A]}(x), \chi^P_{(A]}(y)\} \\ &= \tilde{\delta}^P\end{aligned}$$

$$\begin{aligned}(\chi^N_A \circ_\Gamma \chi^N_{(B]})(x) &= \inf_{p=y\alpha z} \min\{\chi^N_{(A]}(y), \chi^N_{(A]}(z)\} \\ &\leqslant \max\{\chi^N_{(A]}(x), \chi^N_{(A]}(y)\} \\ &= \tilde{\delta}^N\end{aligned}$$

Therefore $(\chi^P_{(A]} \circ_\Gamma \chi^P_{(B]})(x) = \tilde{\delta}^P = (\chi^P_{(A\Gamma B]}(x)$ and $(\chi^N_{(A]} \circ_\Gamma \chi^N_{(B]})(x) = \tilde{\delta}^N = (\chi^N_{(A\Gamma B]}(x)$. If $p \notin (A\Gamma B]$ then $(\chi^P_{(A\Gamma B]})(x) = \tilde{\eta}^P$ and $(\chi^N_{(A\Gamma B]})(x) = \tilde{\eta}^N$. Since $p \leqslant a\alpha b$ for some $a \notin (A]$, $b \notin (B]$ and $\alpha \in \Gamma$. We have

$$\begin{aligned}(\chi^P_{(A]} \circ_\Gamma \chi^P_{(B]})(x) &= \sup_{p=y\alpha z} \min\{\chi^P_{(A]}(y), \chi^P_{(A]}(z)\} \\ &\geqslant \min\{\chi^P_{(A]}(x), \chi^P_{(A]}(y)\} \\ &= \tilde{\eta}^P \\ (\chi^N_{(A]} \circ_\Gamma \chi^N_{(A]})(x) &= \inf_{p=y\alpha z} \min\{\chi^N_{(A]}(y), \chi^N_{(A]}(z)\} \\ &\leqslant \max\{\chi^P_{(A]}(x), \chi^P_{(A]}(y)\} \\ &= \tilde{\eta}^N\end{aligned}$$

Hence $(\chi^P_{(A]} \circ_\Gamma \chi^P_{(B]})(x) = \tilde{\eta}^P = (\chi^P_{(A\Gamma B]})(x)$, $(\chi^N_{(A]} \circ_\Gamma \chi^N_{(B]})(x) = \tilde{\eta}^N = (\chi^N_{(A\Gamma B]})(x)$ □

THEOREM 3.5. *Let S be an $(\tilde{\eta}, \tilde{\delta})$ ordered Γ-semigroup. Let $A, B \subseteq S$ and $\{A_i | i \in I\}$ be a family of subsets of S then*
(i) $(A] \subseteq (B]$ if and only if $(\chi^P_{(A]})^{\tilde{\delta}}_{\tilde{\eta}} \leqslant (\chi^P_{(B]})^{\tilde{\delta}}_{\tilde{\eta}}$ and $(\chi^N_{(A]})^{\tilde{\delta}}_{\tilde{\eta}} \geqslant (\chi^N_{(B]})^{\tilde{\delta}}_{\tilde{\eta}}$
(ii) $(\cap_{i\in I}\chi^P_{(A_i]})^{\tilde{\delta}}_{\tilde{\eta}} = (\chi^P_{\cap_{i\in I}(A_i]})^{\tilde{\delta}}_{\tilde{\eta}}$ and $(\cup_{i\in I}\chi^N_{(A_i]})^{\tilde{\delta}}_{\tilde{\eta}} = (\chi^N_{\cap_{i\in I}(A_i]})^{\tilde{\delta}}_{\tilde{\eta}}$
(iii) $(\cup_{i\in I}\chi^P_{(A_i]})^{\tilde{\delta}}_{\tilde{\eta}} = (\chi^P_{\cup_{i\in I}(A_i]})^{\tilde{\delta}}_{\tilde{\eta}}$ and $(\cap_{i\in I}\chi^N_{(A_i]})^{\tilde{\delta}}_{\tilde{\eta}} = (\chi^N_{\cup_{i\in I}(A_i]})^{\tilde{\delta}}_{\tilde{\eta}}$.

PROOF. The proof follows from Proposition 2.4 [17]. □

PROPOSITION 3.1. *If A is a $(\tilde{\eta}, \tilde{\delta})$- i.v bipolar fuzzy left(subsemigroup, right, interior, (1, 2)-ideal)ideal of S, then $A = [(\tilde{\mu}^P)^{\tilde{\delta}}_{\tilde{\eta}}, (\tilde{\mu}^N)^{\tilde{\delta}}_{\tilde{\eta}}]$ is a i.v bipolar fuzzy left(subsemigroup, right, interior, (1, 2)-ideal) ideal of S.*

Proof. Assume that A is a $(\tilde{\eta}, \tilde{\delta})$-i.v bipolar fuzzy left ideal of S. If there exist $p, q \in S$, and $\alpha \in \Gamma$ then

$$\begin{aligned}
\max\{(\tilde{\mu}^P)_{\tilde{\eta}}^{\tilde{\delta}}(p\alpha q), \tilde{\eta}^P\} &= \max\{(\{\tilde{\mu}^P(p\alpha q) \wedge \tilde{\delta}^P\} \vee \tilde{\eta}^P), \tilde{\eta}^P\} \\
&= \{\tilde{\mu}^P(p\alpha q) \wedge \tilde{\delta}^P\} \vee \tilde{\eta}^P \\
&= \{\tilde{\mu}^P(p\alpha q) \vee \tilde{\eta}^P\} \wedge \{\tilde{\delta}^P \vee \tilde{\eta}^P\} \\
&= \{(\tilde{\mu}^P(p\alpha q) \vee \tilde{\eta}^P) \vee \tilde{\eta}^P\} \wedge \tilde{\delta}^P \\
&\geqslant \{(\tilde{\mu}^P(y) \wedge \tilde{\delta}^P) \vee \tilde{\eta}^P\} \wedge \tilde{\delta}^P \\
&\geqslant (\tilde{\mu}^P)_{\tilde{\eta}}^{\tilde{\delta}}(y) \wedge \tilde{\delta}^P. \\
\text{and } \min\{(\tilde{\mu}^N)_{\tilde{\eta}}^{\tilde{\delta}}(p\alpha q), \tilde{\eta}^N\} &= \min\{(\{\tilde{\mu}^N(p\alpha q) \vee \tilde{\delta}^N\} \wedge \tilde{\eta}^N), \tilde{\eta}^N\} \\
&= \{\tilde{\mu}^N(p\alpha q) \vee \tilde{\delta}^N\} \wedge \tilde{\eta}^N \\
&= \{\tilde{\mu}^N(p\alpha q) \wedge \tilde{\eta}^N\} \vee \{\tilde{\delta}^N \wedge \tilde{\eta}^N\} \\
&= \{(\tilde{\mu}^N(p\alpha q) \wedge \tilde{\eta}^N) \wedge \tilde{\eta}^N\} \vee \tilde{\delta}^N \\
&\leqslant \{(\tilde{\mu}^N(y) \vee \tilde{\delta}^N) \wedge \tilde{\eta}^N\} \vee \tilde{\delta}^N \\
&\leqslant (\tilde{\mu}^N)_{\tilde{\eta}}^{\tilde{\delta}}(y) \vee \tilde{\delta}^N\}
\end{aligned}$$

Hence $A = [(\tilde{\mu}^P)_{\tilde{\eta}}^{\tilde{\delta}}, (\tilde{\mu}^N)_{\tilde{\eta}}^{\tilde{\delta}}]$ is a i.v bipolar fuzzy left ideal of S.
Similar to proofs hold for subsemigroup, right ideals and interior ideal, (1, 2)-ideal also. □

Proposition 3.2. *If A is a $(\tilde{\eta}, \tilde{\delta})$-interval valued bipolar fuzzy bi-ideal,then $A = [(\tilde{\mu}^P)_{\tilde{\eta}}^{\tilde{\delta}}, (\tilde{\mu}^N)_{\tilde{\eta}}^{\tilde{\delta}}]$ is a interval valued bipolar fuzzy bi-ideal of S.*

Proof. Assume that A is a $(\tilde{\eta}, \tilde{\delta})$-i. v bipolar fuzzy bi-ideal of S. If there exist $p, q, r \in S$, and $\alpha, \beta \in \Gamma$ then

$$\begin{aligned}
\max\{(\tilde{\mu}^P)_{\tilde{\eta}}^{\tilde{\delta}}(p\alpha q\beta r), \tilde{\eta}^P\} &= \max\{(\{\tilde{\mu}^P(p\alpha q\beta r) \wedge \tilde{\delta}^P\} \vee \tilde{\eta}^P), \tilde{\eta}^P\} \\
&= \{\tilde{\mu}^P(p\alpha q\beta r) \wedge \tilde{\delta}^P\} \vee \tilde{\eta}^P \\
&= \{\tilde{\mu}^P(p\alpha q\beta r) \vee \tilde{\eta}^P\} \wedge \{\tilde{\delta}^P \vee \tilde{\eta}^P\} \\
&= \{\tilde{\mu}^P(p\alpha q\beta r) \vee \tilde{\eta}^P\} \wedge \tilde{\delta}^P \\
&= \{(\tilde{\mu}^P(p\alpha q\beta r) \vee \tilde{\eta}^P) \vee \tilde{\eta}^P\} \wedge \tilde{\delta}^P \\
&\geqslant \{(\tilde{\mu}^P(x) \wedge \tilde{\mu}^P(r) \wedge \tilde{\delta}^P) \vee \tilde{\eta}^P\} \wedge \tilde{\delta}^P \\
&= \{(\tilde{\mu}^P(x) \wedge \tilde{\mu}^P(r) \wedge \tilde{\delta}^P \wedge \tilde{\delta}^P) \vee \tilde{\eta}^P \vee \tilde{\eta}^P\} \wedge \tilde{\delta}^P \\
&= \{\{(\tilde{\mu}^P(x) \wedge \tilde{\delta}^P) \vee \tilde{\eta}^P\} \wedge \{(\tilde{\mu}^P(r) \wedge \tilde{\delta}^P) \vee \tilde{\eta}^P\}\} \wedge \tilde{\delta}^P \\
&= \{(\tilde{\mu}^P)_{\tilde{\eta}}^{\tilde{\delta}}(x) \wedge (\tilde{\mu}^P)_{\tilde{\eta}}^{\tilde{\delta}}(r)\} \wedge \tilde{\delta}^P.
\end{aligned}$$

$$
\begin{aligned}
\min\{(\tilde{\mu}^N)_{\tilde{\eta}}^{\tilde{\delta}}(p\alpha q\beta r), \tilde{\eta}^N\} &= \min\{(\{\tilde{\mu}^N(p\alpha q\beta r) \vee \tilde{\delta}^N\} \wedge \tilde{\eta}^N), \tilde{\eta}^N\} \\
&= \{\tilde{\mu}^N(p\alpha q\beta r) \vee \tilde{\delta}^N\} \wedge \tilde{\eta}^N \\
&= \{\tilde{\mu}^N(p\alpha q\beta r) \wedge \tilde{\eta}^N\} \vee \{\tilde{\delta}^N \wedge \tilde{\eta}^N\} \\
&= \{\tilde{\mu}^N(p\alpha q\beta r) \wedge \tilde{\eta}^N\} \vee \tilde{\delta}^N \\
&= \{(\tilde{\mu}^N(p\alpha q\beta r) \wedge \tilde{\eta}^N) \wedge \tilde{\eta}^N\} \vee \tilde{\delta}^N \\
&\leqslant \{(\tilde{\mu}^N(x) \vee \tilde{\mu}^N(r) \vee \tilde{\delta}^N) \wedge \tilde{\eta}^N\} \vee \tilde{\delta}^N \\
&= \{(\tilde{\mu}^N(x) \vee \tilde{\mu}^N(r) \vee \tilde{\delta}^N \vee \tilde{\delta}^N) \wedge \tilde{\eta}^N \wedge \tilde{\eta}\} \vee \tilde{\delta}^N \\
&= \{\{(\tilde{\mu}^N(x) \vee \tilde{\delta}^N) \wedge \tilde{\eta}^N\} \vee \{(\tilde{\mu}^N(r) \vee \tilde{\delta}^N) \wedge \tilde{\eta}^N\}\} \vee \tilde{\delta}^N \\
&= \{(\tilde{\mu}^N)_{\tilde{\eta}}^{\tilde{\delta}}(x) \vee (\tilde{\mu}^N)_{\tilde{\eta}}^{\tilde{\delta}}(r)\} \vee \tilde{\delta}^N.
\end{aligned}
$$

By similar way we can show the remaining part of the proposition. □

THEOREM 3.6. *Let* $A = (\tilde{\mu}_A^P, \tilde{\mu}_A^N)$ *be a i. v bipolar* $(\tilde{\eta}, \tilde{\delta})$*- fuzzy right ideal and* $B = (\tilde{\mu}_B^P, \tilde{\mu}_B^N)$ *be a i. v bipolar* $(\tilde{\eta}, \tilde{\delta})$*- fuzzy left ideal of* S *then* $((A \circ_\Gamma B])_{\tilde{\eta}}^{\tilde{\delta}} \subseteq A \cap_{\tilde{\eta}}^{\tilde{\delta}} B$ *and* $((A \circ_\Gamma B])_{\tilde{\eta}}^{\tilde{\delta}} \supseteq A \cup_{\tilde{\eta}}^{\tilde{\delta}} B$.

PROOF. Let $A = (\tilde{\mu}_A^P, \tilde{\mu}_A^N)$ be a i. v bipolar $(\tilde{\eta}, \tilde{\delta})$- fuzzy right ideal and $B = (\tilde{\mu}_B^P, \tilde{\mu}_B^N)$ be a i. v bipolar $(\tilde{\eta}, \tilde{\delta})$- fuzzy left ideal of S. Let $(p, q) \in I_r$. If $I_r \neq \emptyset$, then $r \leqslant p\gamma q$. Thus $\tilde{\mu}_A^P(r) \geqslant \tilde{\mu}_A^P(p\alpha q) \geqslant \tilde{\mu}_A^P(x)$ and $\tilde{\mu}_A^N(r) \leqslant \tilde{\mu}_A^N(p\alpha q) \leqslant \tilde{\mu}_A^N(x)$. Similarly $\tilde{\mu}_B^P(r) \geqslant \tilde{\mu}_B^P(p\alpha q) \geqslant \tilde{\mu}_B^P(y)$ and $\tilde{\mu}_B^N(r) \leqslant \tilde{\mu}_B^N(p\alpha q) \leqslant \tilde{\mu}_B^N(y)$. we have

$$
\begin{aligned}
(\tilde{\mu}_{(A\circ_\Gamma B]}^P)_{\tilde{\eta}}{}^{\tilde{\delta}}(r) &= (\tilde{\mu}_{(A\circ_\Gamma B]}^P(r) \wedge \tilde{\delta}^P) \vee \tilde{\eta}^P \\
&= (\max\{\tilde{\mu}_A^P(x) \wedge \tilde{\mu}_B^P(y)\} \wedge \tilde{\delta}^P) \vee \tilde{\eta}^P \\
&= (\max\{\tilde{\mu}_A^P(x) \wedge \tilde{\mu}_B^P(y)\} \wedge \tilde{\delta}^P \wedge \tilde{\delta}^P) \vee \tilde{\eta}^P \\
&= (\max\{(\tilde{\mu}_A^P(x) \wedge \tilde{\delta}^P) \wedge (\tilde{\mu}_B^P(y) \wedge \tilde{\delta}^P)\} \wedge \tilde{\delta}^P) \vee \tilde{\eta}^P \\
&\leqslant (\{(\tilde{\mu}_A^P(r) \vee \tilde{\eta}^P) \wedge (\tilde{\mu}_A^P(r) \vee \tilde{\eta}^P)\} \wedge \tilde{\delta}^P) \vee \tilde{\eta}^P \\
&= ((\tilde{\mu}_A^P(r) \vee \tilde{\eta}^P) \wedge (\tilde{\mu}_B^P(r) \vee \tilde{\eta}^P) \wedge \tilde{\delta}^P) \vee \tilde{\eta}^P \\
&= \{((\tilde{\mu}_A^P(r) \wedge \tilde{\mu}_B^P(r)) \vee \tilde{\eta}^P) \wedge \tilde{\delta}^P\} \vee \tilde{\eta}^P \\
&= \{((\tilde{\mu}_A^P \wedge \tilde{\mu}_B^P)(r) \wedge \tilde{\delta}^P\} \vee \tilde{\eta}^P \\
&= (\tilde{\mu}_{A\cap_{\tilde{\eta}}^{\tilde{\delta}} B}^P)(r).
\end{aligned}
$$

$$
\begin{aligned}
\text{and } (\tilde{\mu}_{(A\circ_\Gamma B]}^N)_{\tilde{\eta}}^{\tilde{\delta}}(r) &= (\tilde{\mu}_{(A\circ_\Gamma B]}^N(r) \vee \tilde{\delta}^N) \wedge \tilde{\eta}^N \\
&= (\min\{\tilde{\mu}_A^N(x) \vee \tilde{\mu}_B^N(y)\} \vee \tilde{\delta}^N) \vee \tilde{\eta}^N \\
&= (\min\{\tilde{\mu}_A^N(x) \vee \tilde{\mu}_B^N(y)\} \vee \tilde{\delta}^N \vee \tilde{\delta}^N) \wedge \tilde{\eta}^N \\
&= (\min\{(\tilde{\mu}_A^N(x) \vee \tilde{\delta}^N) \vee (\tilde{\mu}_B^N(y) \vee \tilde{\delta}^N)\} \vee \tilde{\delta}^N) \wedge \tilde{\eta}^N
\end{aligned}
$$

$$\begin{aligned}&\geqslant (\{(\tilde{\mu}_A^N(r)\wedge\tilde{\eta}^N)\vee(\tilde{\mu}_A^N(r)\wedge\tilde{\eta}^N)\}\vee\tilde{\delta}^N)\wedge\tilde{\eta}^N\\ &=((\tilde{\mu}_A^N(r)\wedge\tilde{\eta}^N)\vee(\tilde{\mu}_B^N(r)\wedge\tilde{\eta}^N)\vee\tilde{\delta}^N)\wedge\tilde{\eta}^N\\ &=\{((\tilde{\mu}_A^N(r)\vee\tilde{\mu}_B^N(r))\wedge\tilde{\eta}^N)\vee\tilde{\delta}^N\}\wedge\tilde{\eta}^N\\ &=\{((\tilde{\mu}_A^N\vee\tilde{\mu}_B^N)(r)\vee\tilde{\delta}^N\}\wedge\tilde{\eta}^N\\ &=(\tilde{\mu}^N_{A\cup^{\tilde{\delta}}_{\tilde{\eta}}B})(r)\end{aligned}$$

Let $p,q\notin I$. If $I_r=\emptyset$, then $(\tilde{\mu}_A^P\circ_\Gamma\tilde{\mu}_B^P)(r)=0=(\tilde{\mu}_A^N\circ_\Gamma\tilde{\mu}_B^N)(r)$ and $\alpha\in\Gamma$ such that $r\leqslant p\alpha q$.We have

$$\begin{aligned}(\tilde{\mu}^P_{(A\circ_\Gamma B]})^{\tilde{\delta}}_{\tilde{\eta}}(r)&=(\tilde{\mu}^P_{(A\circ_\Gamma B]}(x)\wedge\tilde{\delta}^P)\vee\tilde{\eta}^P\\ &=0\vee\tilde{\eta}^P\\ &=\tilde{\eta}^P\\ &\leqslant(\tilde{\mu}^P_{A\cap B}(x)\wedge\tilde{\delta}^P)\vee\tilde{\eta}^P\\ &=(\tilde{\mu}^P_{A\cap B}(x)\wedge\tilde{\delta}^P)\\ \text{and } (\tilde{\mu}^N_{(A\circ_\Gamma B]})^{\tilde{\delta}}_{\tilde{\eta}}(r)&=(\tilde{\mu}^N_{(A\circ_\Gamma B]}(r)\vee\tilde{\delta}^N)\wedge\tilde{\eta}^N\\ &=0\wedge\tilde{\eta}^N\\ &=\tilde{\eta}^N\\ &\geqslant(\tilde{\mu}^N_{A\cup B}(x)\vee\tilde{\delta}^N)\wedge\tilde{\eta}^N\\ &=(\tilde{\mu}^N_{A\cup B}(x)\vee\tilde{\delta}^N)\end{aligned}$$

Therefore $((A\circ_\Gamma B])^{\tilde{\delta}}_{\tilde{\eta}}\subseteq A\cap^{\tilde{\delta}}_{\tilde{\eta}}B$ and $((A\circ_\Gamma B])^{\tilde{\delta}}_{\tilde{\eta}}\supseteq A\cup^{\tilde{\delta}}_{\tilde{\eta}}B$. □

Corollary 3.4. *Let $A=(\tilde{\mu}_A^P,\tilde{\mu}_A^N)$ be a i. v bipolar $(\epsilon,\epsilon\vee q)$- fuzzy right ideal and $B=(\tilde{\mu}_B^P,\tilde{\mu}_B^N)$ be a i. v bipolar $(\epsilon,\epsilon\vee q)$- fuzzy left ideal of S then $((A\circ_\Gamma B])\subseteq A\cap B$ and $((A\circ_\Gamma B])\supseteq A\cup B$.*

Proof. The proof follows taking $\tilde{\eta}^P=0,\tilde{\delta}^P=0.5$ and $\tilde{\eta}^N=0,\tilde{\delta}^N=-0.5$ in Theorem 3.6. □

Corollary 3.5. [**7**] *Let S be an ordered Γ-semigroup is regular if and only if every right ideal A and every left ideal B of S then $A\cap B=(A\circ_\Gamma B]$.*

Theorem 3.7. *An ordered Γ-semigroup S is regular,let $A=(\tilde{\mu}_A^P,\tilde{\mu}_A^N)$ be a i. v bipolar $(\tilde{\eta},\tilde{\delta})$-fuzzy right ideal and $B=(\tilde{\mu}_B^P,\tilde{\mu}_B^N)$ be a i. v bipolar $(\tilde{\eta},\tilde{\delta})$- fuzzy left ideal of S if and only if $((A\circ_\Gamma B])^{\tilde{\delta}}_{\tilde{\eta}}=A\cap^{\tilde{\delta}}_{\tilde{\eta}}B$ and $((A\circ_\Gamma B])^{\tilde{\delta}}_{\tilde{\eta}}=A\cup^{\tilde{\delta}}_{\tilde{\eta}}B$.*

Proof. Let S be an ordered Γ-regular semigroup and $A=(\tilde{\mu}_A^P,\tilde{\mu}_A^N)$ be a i. v bipolar $(\tilde{\eta},\tilde{\delta})$-fuzzy right ideal and $B=(\tilde{\mu}_B^P,\tilde{\mu}_B^N)$ be a i. v bipolar $(\tilde{\eta},\tilde{\delta})$- fuzzy left ideal of S. Let I be a non-empty set, then $I_r=\{(p,q)\in S\times S|r\leqslant p\gamma q\}$ from definition 2.2 in (iii). Thus $\tilde{\mu}_A^P(r)\geqslant\tilde{\mu}_A^P(p\alpha q)\geqslant\tilde{\mu}_A^P(x)$ and $\tilde{\mu}_A^N(r)\leqslant\tilde{\mu}_A^N(p\alpha q)\leqslant$

$\tilde{\mu}_A^N(x)$. Similarly $\tilde{\mu}_B^P(r) \geqslant \tilde{\mu}_B^P(p\alpha q) \geqslant \tilde{\mu}_B^P(y)$ and $\tilde{\mu}_B^N(r) \leqslant \tilde{\mu}_B^N(p\alpha q) \leqslant \tilde{\mu}_B^N(y)$.

$$\begin{aligned}
(\tilde{\mu}^P_{(A\circ_\Gamma B]})^{\tilde{\delta}}_{\tilde{\eta}}(r) &= (\tilde{\mu}^P_{(A\circ_\Gamma B]}(r) \wedge \tilde{\delta}^P) \vee \tilde{\eta}^P \\
&= (\max\{\tilde{\mu}_A^P(x) \wedge \tilde{\mu}_B^P(y)\} \wedge \tilde{\delta}^P) \vee \tilde{\eta}^P \\
&= (\max\{\tilde{\mu}_A^P(x) \wedge \tilde{\mu}_B^P(y)\} \wedge \tilde{\delta}^P \wedge \tilde{\delta}^P) \vee \tilde{\eta}^P \\
&= (\max\{(\tilde{\mu}_A^P(x) \wedge \tilde{\delta}^P) \wedge (\tilde{\mu}_B^P(y) \wedge \tilde{\delta}^P)\} \wedge \tilde{\delta}^P) \vee \tilde{\eta}^P \\
&\geqslant (\{(\tilde{\mu}_A^P(r\alpha x) \vee \tilde{\eta}^P) \wedge (\tilde{\mu}_A^P(r) \vee \tilde{\eta}^P)\} \wedge \tilde{\delta}^P) \vee \tilde{\eta}^P \\
&\geqslant ((\tilde{\mu}_A^P(r) \vee \tilde{\eta}^P) \wedge (\tilde{\mu}_B^P(r) \vee \tilde{\eta}^P) \wedge \tilde{\delta}^P) \vee \tilde{\eta}^P \\
&= \{((\tilde{\mu}_A^P(r) \wedge \tilde{\mu}_B^P(r)) \vee \tilde{\eta}^P) \wedge \tilde{\delta}^P\} \vee \tilde{\eta}^P \\
&= \{((\tilde{\mu}_A^P \wedge \tilde{\mu}_B^P)(r) \wedge \tilde{\delta}^P\} \vee \tilde{\eta}^P \\
&= (\tilde{\mu}^P_{A\cap^{\tilde{\delta}}_{\tilde{\eta}} B})(r).
\end{aligned}$$

$$\begin{aligned}
\text{and } (\tilde{\mu}^N_{(A\circ_\Gamma B]})^{\tilde{\delta}}_{\tilde{\eta}}(r) &= (\tilde{\mu}^N_{(A\circ_\Gamma B]}(r) \vee \tilde{\delta}^N) \wedge \tilde{\eta}^N \\
&= (\min\{\tilde{\mu}_A^N(x) \vee \tilde{\mu}_B^N(y)\} \vee \tilde{\delta}^N) \vee \tilde{\eta}^N \\
&= (\min\{\tilde{\mu}_A^N(x) \vee \tilde{\mu}_B^N(y)\} \vee \tilde{\delta}^N \vee \tilde{\delta}^N) \wedge \tilde{\eta}^N \\
&= (\min\{(\tilde{\mu}_A^N(x) \vee \tilde{\delta}^N) \vee (\tilde{\mu}_B^N(y) \vee \tilde{\delta}^N)\} \vee \tilde{\delta}^N) \wedge \tilde{\eta}^N \\
&\leqslant (\{(\tilde{\mu}_A^N(r\alpha x) \wedge \tilde{\eta}^N) \vee (\tilde{\mu}_A^N(r) \wedge \tilde{\eta}^N)\} \vee \tilde{\delta}^N) \wedge \tilde{\eta}^N \\
&\leqslant ((\tilde{\mu}_A^N(r) \wedge \tilde{\eta}^N) \vee (\tilde{\mu}_B^N(r) \wedge \tilde{\eta}^N) \vee \tilde{\delta}^N) \wedge \tilde{\eta}^N \\
&= \{((\tilde{\mu}_A^N(r) \vee \tilde{\mu}_B^N(r)) \wedge \tilde{\eta}^N) \vee \tilde{\delta}^N\} \wedge \tilde{\eta}^N \\
&= \{((\tilde{\mu}_A^N \vee \tilde{\mu}_B^N)(r) \vee \tilde{\delta}^N\} \wedge \tilde{\eta}^N \\
&= (\tilde{\mu}^N_{A\cup^{\tilde{\delta}}_{\tilde{\eta}} B})(r)
\end{aligned}$$

Thus $((A \circ {}_\Gamma B])^{\tilde{\delta}}_{\tilde{\eta}} \supseteq A \cap^{\tilde{\delta}}_{\tilde{\eta}} B$ and $((A \circ {}_\Gamma B])^{\tilde{\delta}}_{\tilde{\eta}} \subseteq A \cup^{\tilde{\delta}}_{\tilde{\eta}} B$, by Theorem 3.7 and hence $((A \circ {}_\Gamma B])^{\tilde{\delta}}_{\tilde{\eta}} = A \cap^{\tilde{\delta}}_{\tilde{\eta}} B$ and $((A \circ {}_\Gamma B])^{\tilde{\delta}}_{\tilde{\eta}} = A \cup^{\tilde{\delta}}_{\tilde{\eta}} B$. Conversely assume that $((A \circ {}_\Gamma B])^{\tilde{\delta}}_{\tilde{\eta}} = A \cap^{\tilde{\delta}}_{\tilde{\eta}} B$ and $((A \circ {}_\Gamma B])^{\tilde{\delta}}_{\tilde{\eta}} = A \cup^{\tilde{\delta}}_{\tilde{\eta}} B$.
Let $A = (\tilde{\mu}_A^P, \tilde{\mu}_A^N)$ be a i. v bipolar $(\tilde{\eta}, \tilde{\delta})$-fuzzy right ideal and $B = (\tilde{\mu}_B^P, \tilde{\mu}_B^N)$ be a i. v bipolar $(\tilde{\eta}, \tilde{\delta})$- fuzzy left ideal of S. Then by Theorem 3.4, χ_A be a i. v bipolar $(\tilde{\eta}, \tilde{\delta})$-fuzzy right ideal and χ_A be a i. v bipolar $(\tilde{\eta}, \tilde{\delta})$- fuzzy left ideal of S. By Lemma 3.2 and Theorem 3.5, we have $(\chi^P_{(A\cap B]})^{\tilde{\delta}}_{\tilde{\eta}} = (\chi^P_A \cap^{\tilde{\delta}}_{\tilde{\eta}} \chi^P_B) = (\chi^P_A \circ_\Gamma \chi^P_B)^{\tilde{\delta}}_{\tilde{\eta}} = (\chi^P_{(A\circ_\Gamma B]})^{\tilde{\delta}}_{\tilde{\eta}}$ and $(\chi^N_{(A\cap B]})^{\tilde{\delta}}_{\tilde{\eta}} = (\chi^N_A \cup^{\tilde{\delta}}_{\tilde{\eta}} \chi^N_B) = (\chi^N_A \circ_\Gamma \chi^N_B)^{\tilde{\delta}}_{\tilde{\eta}} = (\chi^N_{(A\circ_\Gamma B]})^{\tilde{\delta}}_{\tilde{\eta}}$. This implies $(A \cap B)^{\tilde{\delta}}_{\tilde{\eta}} = ((A \circ_\Gamma B])^{\tilde{\delta}}_{\tilde{\eta}}$. Hence by Corollary 3.5 S is regular. □

Corollary 3.6. *Let $A = (\tilde{\mu}_A^P, \tilde{\mu}_A^N)$ be a i. v bipolar $(\epsilon, \epsilon \vee q)$-fuzzy right ideal and $B = (\tilde{\mu}_B^P, \tilde{\mu}_B^N)$ be a i. v bipolar $(\epsilon, \epsilon \vee q)$- fuzzy left ideal of an ordered Γ-semigroup S. S is regular if and only if $((A \circ_\Gamma B]) = A \cap B$ and $((A \circ_\Gamma B]) = A \cup B$.*

Proof. Taking $\tilde{\eta}^P = 0, \tilde{\delta}^P = [0.5, 0.5]$ and $\tilde{\eta}^N = [0,0], \tilde{\delta}^N = [-0.5, -0.5]$ in Theorem 3.7 the proof follows. □

Theorem 3.8. *Let $A = (\tilde{\mu}_A^P, \tilde{\mu}_A^N)$ be a i.v bipolar $(\tilde{\eta}, \tilde{\delta})$-fuzzy bi-ideal and $B = (\tilde{\mu}_B^P, \tilde{\mu}_B^N)$ be a i. v bipolar $(\tilde{\eta}, \tilde{\delta})$- fuzzy left ideal of an ordered Γ-semigroup S. S is regular if and only if $(A \circ_\Gamma B)_{\tilde{\eta}}^{\tilde{\delta}} = A \cap_{\tilde{\eta}}^{\tilde{\delta}} B$ and $(A \circ_\Gamma B)_{\tilde{\eta}}^{\tilde{\delta}} = A \cup_{\tilde{\eta}}^{\tilde{\delta}} B$.*

Proof. Let S be an ordered Γ-regular semigroup and $A = (\tilde{\mu}_A^P, \tilde{\mu}_A^N)$ be a i. v bipolar $(\tilde{\eta}, \tilde{\delta})$-fuzzy bi-ideal and $B = (\tilde{\mu}_B^P, \tilde{\mu}_B^N)$ be a i. v bipolar $(\tilde{\eta}, \tilde{\delta})$- fuzzy left ideal of S. Let I be a non-empty set, then $I_r = \{(p,q) \in S \times S | r \leqslant p\gamma q\}$. Thus $\tilde{\mu}_A^P(r) \geqslant \tilde{\mu}_A^P(p\alpha q) \geqslant \tilde{\mu}_A^P(x)$ and $\tilde{\mu}_A^N(r) \leqslant \tilde{\mu}_A^N(p\alpha q) \leqslant \tilde{\mu}_A^N(x)$. Similarly $\tilde{\mu}_B^P(r) \geqslant \tilde{\mu}_B^P(p\alpha q) \geqslant \tilde{\mu}_B^P(y)$ and $\tilde{\mu}_B^N(r) \leqslant \tilde{\mu}_B^N(p\alpha q) \leqslant \tilde{\mu}_B^N(y)$.
For $r \in S$, there exists $x \in S$ such that $r \leqslant r\alpha x\beta r = r\alpha(x\beta r) \leqslant (r\alpha x\beta r)\alpha(x\beta r)$. Then $(r\alpha x\beta r), (x\beta r) \in I_r$. We have

$$\begin{aligned}
(\tilde{\mu}_{A\circ_\Gamma B}^P)_{\tilde{\eta}}^{\tilde{\delta}}(r) &= (\tilde{\mu}_{A\circ_\Gamma B}^P(r) \wedge \tilde{\delta}^P) \vee \tilde{\eta}^P \\
&= (\max\{\tilde{\mu}_A^P(x) \wedge \tilde{\mu}_B^P(y)\} \wedge \tilde{\delta}^P) \vee \tilde{\eta}^P \\
&= (\max\{\tilde{\mu}_A^P(x) \wedge \tilde{\mu}_B^P(y)\} \wedge \tilde{\delta}^P \wedge \tilde{\delta}^P) \vee \tilde{\eta}^P \\
&= (\max\{(\tilde{\mu}_A^P(x) \wedge \tilde{\delta}^P) \wedge (\tilde{\mu}_B^P(y) \wedge \tilde{\delta}^P)\} \wedge \tilde{\delta}^P) \vee \tilde{\eta}^P \\
&\geqslant (\{(\tilde{\mu}_A^P(r\alpha x\beta r) \vee \tilde{\eta}^P) \wedge (\tilde{\mu}_A^P(x\beta r) \vee \tilde{\eta}^P)\} \wedge \tilde{\delta}^P) \vee \tilde{\eta}^P \\
&\geqslant ((\tilde{\mu}_A^P(r) \vee \tilde{\eta}^P) \wedge (\tilde{\mu}_B^P(r) \vee \tilde{\eta}^P) \wedge \tilde{\delta}^P) \vee \tilde{\eta}^P \\
&= \{((\tilde{\mu}_A^P(r) \wedge \tilde{\mu}_B^P(r)) \vee \tilde{\eta}^P) \wedge \tilde{\delta}^P\} \vee \tilde{\eta}^P \\
&= \{((\tilde{\mu}_A^P \wedge \tilde{\mu}_B^P)(r) \wedge \tilde{\delta}^P\} \vee \tilde{\eta}^P \\
&= (\tilde{\mu}_{A\cap_{\tilde{\eta}}^{\tilde{\delta}} B}^P)(r).
\end{aligned}$$

$$\begin{aligned}
\text{and } (\tilde{\mu}_{A\circ_\Gamma B}^N)_{\tilde{\eta}}^{\tilde{\delta}}(r) &= (\tilde{\mu}_{A\circ_\Gamma B}^N(r) \vee \tilde{\delta}^N) \wedge \tilde{\eta}^N \\
&= (\min\{\tilde{\mu}_A^N(x) \vee \tilde{\mu}_B^N(y)\} \vee \tilde{\delta}^N) \vee \tilde{\eta}^N \\
&= (\min\{\tilde{\mu}_A^N(x) \vee \tilde{\mu}_B^N(y)\} \vee \tilde{\delta}^N \vee \tilde{\delta}^N) \wedge \tilde{\eta}^N \\
&= (\min\{(\tilde{\mu}_A^N(x) \vee \tilde{\delta}^N) \vee (\tilde{\mu}_B^N(y) \vee \tilde{\delta}^N)\} \vee \tilde{\delta}^N) \wedge \tilde{\eta}^N \\
&\leqslant (\{(\tilde{\mu}_A^N(r\alpha x\beta r) \wedge \tilde{\eta}^N) \vee (\tilde{\mu}_A^N(x\beta r) \wedge \tilde{\eta}^N)\} \vee \tilde{\delta}^N) \wedge \tilde{\eta}^N \\
&\leqslant ((\tilde{\mu}_A^N(r) \wedge \tilde{\eta}^N) \vee (\tilde{\mu}_B^N(r) \wedge \tilde{\eta}^N) \vee \tilde{\delta}^N) \wedge \tilde{\eta}^N \\
&= \{((\tilde{\mu}_A^N(r) \vee \tilde{\mu}_B^N(r)) \wedge \tilde{\eta}^N) \vee \tilde{\delta}^N\} \wedge \tilde{\eta}^N \\
&= \{((\tilde{\mu}_A^N \vee \tilde{\mu}_B^N)(r) \vee \tilde{\delta}^N\} \wedge \tilde{\eta}^N \\
&= (\tilde{\mu}_{A\cup_{\tilde{\eta}}^{\tilde{\delta}} B}^N)(r)
\end{aligned}$$

Thus $(A \circ_\Gamma B)_{\tilde{\eta}}^{\tilde{\delta}} \supseteq A \cap_{\tilde{\eta}}^{\tilde{\delta}} B$ and $(A \circ_\Gamma B)_{\tilde{\eta}}^{\tilde{\delta}} \subseteq A \cup_{\tilde{\eta}}^{\tilde{\delta}} B$.
Conversely assume that $(A \circ_\Gamma B)_{\tilde{\eta}}^{\tilde{\delta}} \supseteq A \cap_{\tilde{\eta}}^{\tilde{\delta}} B$ and $(A \circ_\Gamma B)_{\tilde{\eta}}^{\tilde{\delta}} \subseteq A \cup_{\tilde{\eta}}^{\tilde{\delta}} B$.
Let $A = (\tilde{\mu}_A^P, \tilde{\mu}_A^N)$ be a i.v bipolar $(\tilde{\eta}, \tilde{\delta})$-fuzzy bi ideal and $B = (\tilde{\mu}_B^P, \tilde{\mu}_B^N)$ be a

i.v bipolar $(\tilde{\eta}, \tilde{\delta})$- fuzzy left ideal of S. Since every i.v bipolar fuzzy $(\tilde{\eta}, \tilde{\delta})$-right ideal of S is a i.v bipolar fuzzy $(\tilde{\eta}, \tilde{\delta})$-bi-ideal of S. Thus $(A \circ_\Gamma B)^{\tilde{\delta}}_{\tilde{\eta}} \supseteq A \cap^{\tilde{\delta}}_{\tilde{\eta}} B$ and $(A \circ_\Gamma B)^{\tilde{\delta}}_{\tilde{\eta}} \subseteq A \cup^{\tilde{\delta}}_{\tilde{\eta}} B$, by Theorem 3.7. Hence S is regular. □

Corollary 3.7. *Let $A = (\tilde{\mu}^P_A, \tilde{\mu}^N_A)$ be a i. v bipolar $(\epsilon, \epsilon \vee q)$-fuzzy bi-ideal and $B = (\tilde{\mu}^P_B, \tilde{\mu}^N_B)$ be a i. v bipolar $(\epsilon, \epsilon \vee q)$- fuzzy left ideal of an ordered Γ-semigroup S. S is regular if and only if $(A \circ_\Gamma B) = A \cap^{\tilde{\delta}}_{\tilde{\eta}} B$ and $(A \circ_\Gamma B) = A \cup^{\tilde{\delta}}_{\tilde{\eta}} B$.*

Proof. Follows from Theorem 3.8 □

Theorem 3.9. *Let $A = (\tilde{\mu}^P_A, \tilde{\mu}^N_A)$ be a i. v bipolar $(\tilde{\eta}, \tilde{\delta})$-fuzzy right ideal and $B = (\tilde{\mu}^P_B, \tilde{\mu}^N_B)$ be a i. v bipolar $(\tilde{\eta}, \tilde{\delta})$- fuzzy left ideal of an ordered Γ-semigroup S. S weakly regular if and only if $(A \circ_\Gamma B)^{\tilde{\delta}}_{\tilde{\eta}} \supseteq A \cap^{\tilde{\delta}}_{\tilde{\eta}} B$ and $(A \circ_\Gamma B)^{\tilde{\delta}}_{\tilde{\eta}} \subseteq A \cup^{\tilde{\delta}}_{\tilde{\eta}} B$.*

Proof. Straightforward. □

Corollary 3.8. *Let $A = (\tilde{\mu}^P_A, \tilde{\mu}^N_A)$ be a i. v bipolar $(\epsilon, \epsilon \vee q)$-fuzzy right ideal and $B = (\tilde{\mu}^P_B, \tilde{\mu}^N_B)$ be a i. v bipolar $(\epsilon, \epsilon \vee q)$- fuzzy left ideal of an ordered Γ-semigroup S. S is weakly regular if and only if $(A \circ_\Gamma B) \supseteq A \cap B$ and $(A \circ_\Gamma B) \subseteq A \cup B$.*

Proof. Straightforward. □

Theorem 3.10. *Let $A = (\tilde{\mu}^P_A, \tilde{\mu}^N_A)$ be a i. v bipolar $(\tilde{\eta}, \tilde{\delta})$-fuzzy bi-ideal and $B = (\tilde{\mu}^P_B, \tilde{\mu}^N_B)$ be a i. v bipolar $(\tilde{\eta}, \tilde{\delta})$- fuzzy left ideal of an ordered Γ-semigroup S. S is weakly regular if and only if $(A \circ_\Gamma B)^{\tilde{\delta}}_{\tilde{\eta}} \supseteq A \cap^{\tilde{\delta}}_{\tilde{\eta}} B$ and $(A \circ_\Gamma B)^{\tilde{\delta}}_{\tilde{\eta}} \subseteq A \cup^{\tilde{\delta}}_{\tilde{\eta}} B$.*

Proof. Straightforward. □

Corollary 3.9. *Let $A = (\tilde{\mu}^P_A, \tilde{\mu}^N_A)$ be a i. v bipolar $(\epsilon, \epsilon \vee q)$-fuzzy bi-ideal and $B = (\tilde{\mu}^P_B, \tilde{\mu}^N_B)$ be a i. v bipolar $(\epsilon, \epsilon \vee q)$- fuzzy left ideal of an ordered Γ-semigroup S. S is weakly regular if and only if $(A \circ_\Gamma B) \supseteq A \cap B$ and $(A \circ_\Gamma B) \subseteq A \cup B$.*

Proof. Straightforward. □

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