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A note on I-sets in graphs

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Dedicated to Prof. E. Sampathkumar on his 76th birthday

Abstract. C.A. Barefoot, et. al. introduced the concept of the integrity of a graph. It is an useful measure of vulnerability and it is defined as follows. $I(G) = min\{|G| + m(G-S) : S \subset V(G)$ where $m(G-S)$ denotes the order of the largest component in $G-S\}$. Unlike the connectivity measures, integrity shows not only the difficulty to break down the network but also the damage that has been caused. A subset S of $V(G)$ is said to be an I-set if $I(G) = |S| + m(G-S)$. In this paper, we define the I-critical graphs , I-excellent graphs and Bondage Integrity number and we study these parameters.

1. Introduction

The stability of a communication network is of prime importance for network designers. In an analysis of the vulnerability of a communication network to disruption, two quantities that come to our mind are the number of elements that are not functioning and the size of the largest remaining sub network within which mutual communications can still occur. In adverse relationship, it would be desirable for an opponent's network to be such that the two quantities can be made simultaneously small. C.A. Barefoot, R.Entriger and H.Swart [**1**] introduced the concept of the integrity of a graph. It is an useful measure of vulnerability and it is defined as follows. $I(G) = min\{|G| + m(G-S) : S \subset V(G)\}$ where $m(G-S)$ denotes the order of the largest component in $G-S$. Unlike the connectivity measures, integrity shows not only the difficulty to break down the network but also the damage that has been caused.

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Key words and phrases. Integrity, I-excellent, I-critical, Bondage Number.

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2. I-critical graphs

Definition 2.1. [**1**] The vertex-integrity of G is defined as $I(G) = \min_{S \subset V(G)} \{|S| + m(G - S)\}$.

Definition 2.2. [**1**] A set of vertices S in a graph G is an I-set of G if $|S| + m(G - S) = I(G)$.

Definition 2.3. [**1**] A graph G is said to be I-minimal if $I(G - e) < I(G)$, for every edge $v \in V(G)$.

Remark 2.1. [**1**] Note that if G is I-minimal ,then $I(G - e) = I(G) - 1$, and because of the monotonicity property of integrity, $I(H) < I(G)$ for every proper subgraph H of G.

Remark 2.2. [**1**] Clearly, every graph has an I-minimal subgraph with the same integrity. K_n is $I-$minimal graph. K_2 is the only I-minimal graph of integrity 2.

Definition 2.4. [**1**] A graph G is said to be I-critical if $I(G - u) < I(G)$, for every vertex $v \in V(G)$.

Remark 2.3. [**1**] Clearly, I-critical graph can have no isolate vertices and an I-minimal graph without such vertices must be I-critical. Some graphs that are $I-$ critical but not I-minimal are the cycles of square order.

Definition 2.5.

For any graph G, $I^0(G) = \{u \in V(G) : I(G - u) = I(G)\}$;
$I^-(G) = \{u \in V(G) : I(G - u) < I(G)\}$.

Proposition 2.1. *For any graph G, $V(G) = I^0(G) \cup I^-(G)$. That is,*
$I(G - v) \leqslant I(G)$ for every $v \in V(G)$.

Proof. Let S be an I-set of G. Then $|S| + m(G - S) = I(G)$. Let $u \notin S$.

$$\begin{aligned} I(G - u) &\leqslant |S - \{u\}| + m((G - u) - (S - \{u\})) \\ &= |S - \{u\}| + m((G - u) - S) \\ &\leqslant |S - \{u\}| + m((G - (S - \{u\})) = I(G). \end{aligned}$$

Therefore, $I(G-u) \leqslant I(G)$. Let $u \in S$. Then $|S-\{u\}|+m((G-u)-(S-\{u\}) = |S| - 1 + m((G - S) = I(G) - 1$. Therefore, $I(G - u) \leqslant I(G)$. □

Remark 2.4. In a graph G, if v is a vertex for which $deg(v) \geqslant I(G - v)$, then $I(G - v) = I(G) - 1$. That is, $v \in I^-(G)$.

Proposition 2.2. *Let G be a simple and connected graph. Let S be an I-set of G. Let $u \in V(G)$ such that $u \in I^-(G)$. Then $I(G - u) = I(G) - 1$.*

Proof. Let S be an I-set of G containing u.
$|S - \{u\}| + m((G - u) - (S - \{u\})) = |S| - 1 + m(G - S) = I(G) - 1$. Therefore, $I(G - u) \leqslant I(G) - 1$. Let S_1 be an I-set of $G_1 = G - u$. Since $u \in I^-(G)$, $I(G_1) < I(G) - 1$. Therefore, $|S_1| + m((G - u) - S_1) \leqslant I(G) - 2$. Let $S_2 = S_1 \cup \{u\}$.
$I(G) \leqslant |S_2| + m(G - S_2) = |S_1 \cup \{u\}| + m(G - (S_1 \cup \{u\}))$.

$$= |S_1| + 1 + m((G-u) - S_1)$$
$$= |S_1| + 1 + m(G_1 - S_1) = I(G_1) - 1$$
$$\leqslant I(G) - 1, \text{ a contradiction.}$$

Therefore, $I(G-u) \geqslant I(G) - 1$. Hence, $I(G-u) = I(G) - 1$. □

COROLLARY 2.1. $\min\limits_{u \in V(G)} \{I(G-u)\} = I(G) - 1$.

PROOF. For any vertex $u \in V(G)$, either $u \in I^0(G)$ or $u \in I^-(G)$. If $u \in I^0$, then $I(G) = I(G-u)$. If $u \in I^-$, then $I(G) = I(G-u) + 1$. Since I^- is always nonempty (any vertex belonging to an I-set of G belongs to $I^-(G)$), there exists $u \in I^-$ and hence $I(G) = I(G-u) + 1$. □

PROPOSITION 2.3. *Let G be a simple and connected graph.*
Then $I(G) - 1 \leqslant I(G-u) \leqslant I(G)$, for every vertex $v \in V(G)$.

Proof follows from corollary 2.1 and proposition 2.2.

COROLLARY 2.2. *Let $\mathcal{F}$ be the union of all I-sets of G. Then $|\mathcal{F}| = I^-$ and $|V(G)| - |\mathcal{F}| = I^0(G)$.*

LEMMA 2.1. *Let G be a connected graph. Then every vertex of G is an I-set of G if and only if $G = K_n$.*

PROOF. Let every vertex of G constitute an I-set of G. Let $u \in V(G)$. Then $\{u\}$ is an I-set of G. Therefore, $u \in I^-$ and $1 + m(G-u) = I(G)$. Hence, $m(G-u) = I(G) - 1$, for every $u \in V(G)$. Since G is connected, there exists at least two non-cut vertices in G. Let u be a non-cut vertex of G. Then $G - u$ is connected and hence $I(G) = n$. Therefore, G is complete. If G is complete, then clearly, every vertex of G is an I-set of G. □

THEOREM 2.1. *Let G be a graph such that $\{u\}$ is an I-set of G for every $u \in V(G)$. Then G is connected and hence G is complete.*

PROOF. Suppose $\{u\}$ is an I-set for every $u \in V(G)$. Suppose that G is disconnected. Let $G_1, G_2, \cdots, G_k$ be the components of G.
Case(1): There exists exactly one G_i such that $|V(G_i)| = m(G)$. Then for every non cut vertex $u \in V(G_i)$, $m(G-u) = m(G) - 1$ and for any non cut vertex $v \in V(G_j)$, $j \neq i$, $m(G-v) = m(G)$, a contradiction (since for any vertex u, $m(G-u)$ is constant, namely $I(G) - 1$).
Case(2): There exists at least two components say G_{i1}, G_{i2} such that $|V(G_{i1})| = |V(G_{i2})| = m(G)$. Then empty set is the only I-set of G and hence no $\{u\}$ can be an I-set of G, a contradiction. Therefore, G is connected. By the above lemma 2.1, G is complete. □

OBSERVATION 2.1. If G is disconnected and $m(G) = k$ is attained by at least k components of G, then $I^0(G) = V(G)$ (For: In such a case, empty set is the only I-set of G).

THEOREM 2.2. *Let G be a graph. Let S be an I-set of G. Let $u \in V(G)$. Then $m(G-S) - m((G-u)-S) \leqslant 1$.*

Proof. If $u \in S$, then $m(G-S) = m((G-u)-S)$.
Otherwise, $(G-u)-S = (G-(S \cup \{u\})$.
Case(i): Suppose u is a cut vertex in $G-S$.
Subcase(a): Let T_1 and T_2 be two components of maximum order in $G-S$. If $u \in T_1$ or T_2, then also, $m((G-S)-u) = m(G-S)$. If $u \notin T_1$ and $u \notin T_2$, then $m((G-S)-u) = m(G-S)$.
Subcase(b): Suppose that $G-S$ has exactly one component, say T of maximum order. Then $m((G-S)-u) = m(G-S)$, if $u \notin T$. Suppose that $u \in T$. Let T_1 be a component of $G-S$ of next maximum order component. Suppose $|T_1| \leqslant |T|-2$. Let $G_1 = (G-S)-\{u\}$. In this case, there are at least two component resulting from $T-\{u\}$ and the cardinality of each component is less than or equal to $|T|-2$.

Hence, $|S \cup \{u\}| + m(G-S \cup \{u\}) \leqslant |S|+1+|T|-2$
$= |S|+|T|-1 < |S|+|T| = I(G)$. Therefore, S is not an I-set of G, a contradiction. Thus, $|T_1| \geqslant |T|-1$. Since T is a unique component of maximum cardinality of $G-S$, $|T_1| = |T|-1$. Therefore, $m(G-(S \cup \{u\})) = |T_1| = |T|-1 = m(G-S)-1$. Thus, $m(G-S)-m((G-u)-S) = 1$.
Case(ii): Suppose that u is not a cut vertex in $G-S$.
If $G-S$ has a unique component of maximum order say T and $u \notin T$, then $m((G-u)-S) = m(G-S)$. If $u \in T$, then $m((G-u)-S) = m(G-S)-1$. If $G-S$ has more than one component of maximum order , then $m((G-u)-S) = m(G-S)$. □

Proposition 2.4. *Let S be an I-set of G and $u \notin S$. Suppose $G-S$ has a unique component of maximum order, say T and let $u \in T$. Then $S \cup \{u\}$ is an I-set of G.*

Proof. By the hypothesis, $m(G-S) = m((G-u)-S)+1$.
$|S \cup \{u\}| + m((G-u)-S) = |S|+1+m(G-S)-1 = |S|+m(G-S) = I(G)$.
Therefore, $S \cup \{u\}$ is an I-set of G. □

Theorem 2.3. *Let G be a simple graph. Let S be an I-set of G. Let $u \in V(G)$. Then $u \in I^0(G)$ if and only if u does not belong to any I-set of G.*

Proof. Let S be an I-set of G. Let $u \in V(G)$. Let $u \in I^0(G)$. Then $I(G-u) = I(G)$. Suppose that $u \in S$. Then $|S|+m(G-S) = I(G) = |S|+m((G-u)-S)$.
$I(G-u) \leqslant |S-u| + m((G-u)-(S-u)) = I(G)-1$.
Therefore, $I(G-u) < I(G)$, a contradiction. Therefore, $u \notin S$.
Conversely, let S be any I-set of G and let u be not a vertex of any I-set of G. Suppose that $m(G-S)-m((G-u)-S) = 1$. Then $u \in T$, where T is a unique maximum order component of $G-S$. By the proposition 2.4, $S \cup \{u\}$ is an I-set of G, a contradiction. Therefore, $m(G-S) = m((G-u)-S)$. Now,

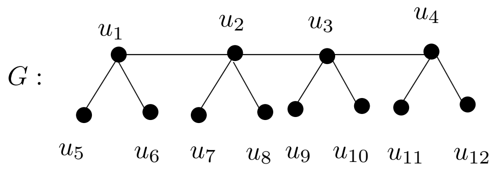
$I(G-u) \leqslant |S|+m((G-u)-S) = |S|+m(G-S) = I(G)$.

Let S_1 be an I-set of $G-u$. Then $I(G) \leqslant |S_1 \cup \{u\}| + m(G-(s \cup \{u\})) = |S_1|+1+m((G-u)-S_1) = I(G-u)+1$.
If $I(G) = I(G-u)+1$, then $S_1 \cup \{u\}$ is an I-set of G, a contradiction, since u does not belong to any I-set of G. As $I(G-u) \leqslant I(G) \leqslant I(G-u)+1$ and $I(G) \neq I(G-u)+1$, it follows that $I(G) = I(G-u)$. Hence $u \in I^0$. □

Corollary 2.3. *Let G be a simple graph. Let S be an I-set of G. Let $u \in V(G)$. Then $u \in I^-$ if and only if u belongs to some I-set of G.*

3. I-excellent graphs

Definition 3.1. A vertex of $V(G)$ is called *I-good* if it is contained in some I-set of G. A vertex of $V(G)$ is called *I-bad* if it does not belong to any I-set of G. A graph G is called *I-excellent* if every vertex in $V(G)$ is I-good.



$I(G) = 5$ and $\{u_1, u_2, u_3, u_4\}; \{u_1, u_2\}; \{u_2, u_4\}$ are the I-sets of G. Thus, u_1, u_2, u_3, u_4 are the I-good vertices of G and the remaining vertices of G are I-bad.

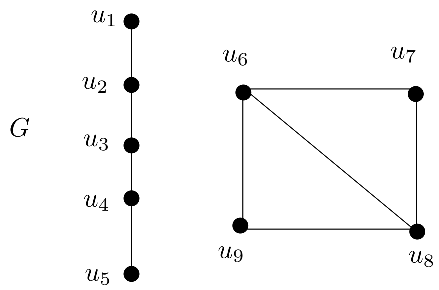
Remark 3.1. I-critical graphs are I-excellent.

Corollary 3.1. *For every vertex u in an I-set of G, $I(G) = I(G - u) + 1$. Hence, if G is I-excellent, then for any $u \in V(G)$, $I(G) = I(G - u) + 1$.*

Remark 3.2. G is I-excellent if and only if $V(G) = I^-$.

Remark 3.3. An I-excellent graph G may be disconnected.

Exmple 3.1.



The I-sets of G are $\{u_1\}, \{u_5\}, \{u_3, u_6, u_8\}, \{u_3, u_7, u_9\}, \{u_2, u_4, u_6, u_8\}$ and $I(G) = 5$. Thus, G is I-excellent.

Proposition 3.1. *The Path P_n is I-excellent if and only if $n = \lceil \frac{k+2}{2} \rceil \lceil \frac{k+3}{2} \rceil$, where $k = 1, 2, 3, \cdots$.*

Proof. Let $V(P_n) = \{u_1, u_2, \cdots, u_n\}$. Let $k = 2r - 1$ or $2r$, $r = 1, 2, 3 \cdots$. Consider the sets $\{u_1, u_{r+2}, u_{2r+3}, \cdots, u_{n-r}\}; \{u_2, u_{r+3}, u_{2r+4}, \cdots, u_{n-r+1}\}; \cdots$, $\{u_{r+1}, u_{2r+2}, u_{3r+3}, \cdots, u_n\}$. These are I-sets (for any of those sets S, $|S| = t$ or

$t+1$ according as $n = t^2$ or n is not a perfect square and $t^2 < n < (t+1)^2$ and $m(P_n - S) = r$ and $I(P_n) = t + r$ or $t + r + 1$). Clearly every vertex belongs to an I-set of P_n and hence P_n is I-excellent.
Claim : P_n is not I-excellent when $n \neq \lceil \frac{n+2}{2} \rceil \lceil \frac{n+3}{2} \rceil$, $k = 1, 2, \cdots$.
Proof of the claim: Let $n \neq \lceil \frac{n+2}{2} \rceil \lceil \frac{n+3}{2} \rceil$, $k = 1, 2, \cdots$. Then n is not a perfect square. Let $t^2 < n < (t+1)^2$. Then $n \neq t(t+1)$.
Therefore, $n = t^2+1, \cdots, t^2+t-1, t^2+t+1, \cdots, t^2+2t$. When $t^2+1 \leqslant n \leqslant t^2+t-1$, $I(P_n) = I(P_{t^2}) = 2t - 1$ and when $t^2 + t + 1 \leqslant n \leqslant t^2 + 2t$, $I(P_n) = 2t$.
Case(i): $t^2 + 1 \leqslant n \leqslant t^2 + t - 1$.
Suppose $V(P_n)$ is I-excellent. Then there exists an I-set S of P_n containing u_1. Let $|S| = l$. Since $I(P_n) = 2t - 1$, $m(G - S) = 2t - 1 - l$.
Suppose $n = t^2 + i, 1 \leqslant i \leqslant t - 1$. Then
$S = \{u_1, u_{2t-l+r}, u_{2(2t-l)+1}, \cdots, u_{s(2t-l)+1}\}$. Hence $|S| = s + 1 = l$. Therefore, $s = l - 1$.
$t^2 + i - s(2t - l) + 1 \leqslant 2t + l - 1$. That is, $t^2 + i + 2 \leqslant (s + 1)(2t - l) = l(2t - l)$.
$l^2 - 2tl + (t^2 + i + 2) \leqslant 0$. The roots of the quadratic equation in the L.H.S are $l = \frac{2t \pm \sqrt{4t^2 - 4(t^2+i+2)}}{2}$, which are imaginary, a contradiction. Therefore, there is no I-set containing u. Hence in this case, P_n is not I-excellent.
Case(ii): $t^2 + t + 1 \leqslant n \leqslant t^2 + 2t$.
Using a similar argument as in case (i), it can be proved that, P_n is not I-excellent. The converse is obvious. □

PROPOSITION 3.2. *K_n is I-excellent.*

PROOF. Since $I(K_n) = n$, any singleton of $V(K_n)$ is an I-set of K_n. □

PROPOSITION 3.3. *C_n is I-excellent.*

COROLLARY 3.2. *W_n is I-excellent. For: Any I-set of W_n is obtained by adding the center vertex to every I-set of C_n.*

PROPOSITION 3.4. *$K_{n,n}$ is I-excellent.*

PROOF. Since $I(K_{n,n}) = n + 1$, any one of the partite sets of $V(K_{n,n})$ is an I-set of $K_{n,n}$. □

THEOREM 3.1. *Every vertex transitive graph is I-excellent.*

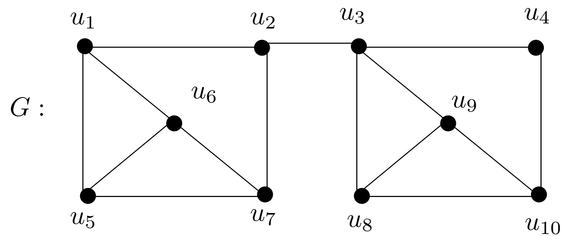
PROOF. Let G be a vertex transitive graph. Let S be an I-set of G. Let $u \in S$. Let v belong to $V(G)$ and $u \neq v$. Since G is vertex transitive, there exists an automorphism $\phi : V(G) \to V(G)$ such that $\phi(u) = v$. Consider $\phi(S)$. Let T be a maximum order component of $G - S$.
Claim: $\phi(T)$ is a maximum order component of $G - \phi(S)$.
As T is connected, $\phi(T)$ is connected. Suppose there exists a component T' in $G - \phi(S)$ such that $|V(T')| > |V(\phi(T))|$. Let $W = \phi^{-1}(T')$. Let $x \in V(W)$. Then $\phi(x) \in V(T') \subset G - \phi(S)$. Therefore, $x \in V - S$ (since $\phi(V - S) = V - \phi(S)$). Therefore, $W \subseteq V - S$. Clearly, W is a component of $V - S$. $|W| = |\phi^{-1}(T')| = |T'| > |\phi(T)| = |T|$. Therefore, $|V(W)| > |V(T)|$, a contradiction. Hence, $\phi(T)$

is a maximum order component of $G - \phi(S)$. Therefore, $|\phi(S)| + m(G - \phi(S)) = |S| + |\phi(T)| = |S| + |V(T)| = I(G)$, which implies, $\phi(S)$ is an I-set of G. $\square$

REMARK 3.4.

There exist I-excellent graphs which are not vertex transitive.

EXMPLE 3.2.



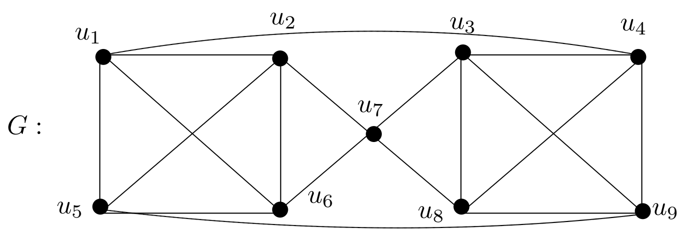
It can be easily seen that for any subset S of $V(G)$, $|S| + m(G - S) \geqslant 6$. The I-sets of G are $\{u_1, u_4, u_7, u_8\}$; $\{u_2, u_3\}$; $\{u_2, u_8\}$; $\{u_2, u_9\}$; $\{u_3, u_7\}$; $\{u_3, u_5\}$; $\{u_3, u_6\}$; $\{u_2, u_{10}\}$. For these sets, $|S|+m(G-S) = 6$. Therefore, $I(G) = 6$. Hence, G is I-excellent but G is not vertex transitive.

EXMPLE 3.3.

The path P_n with $n = \lceil \frac{k+2}{2} \rceil \lceil \frac{k+3}{2} \rceil$, where k is a non-negative integer, is I-excellent but not vertex transitive.

REMARK 3.5. There exist regular graphs which are not I-excellent.

EXMPLE 3.4.



G is regular but not I-excellent. Here the only I-sets are $\{u_4, u_5, u_7\}$ and $\{u_1, u_7, u_9\}$ with $I(G) = 6$.

PROPOSITION 3.5. *Let G be a disconnected graph with unique maximum order component. Then there exists at least one $u \in V(G)$ such that $\{u\}$ is not an I-set of G.*

PROOF. Suppose $\{u\}$ is an I-set of G for every $u \in V(G)$. Then $m(G - u) =$ constant $= I(G)-1$, for every $u \in V(G)$. Let $G_1, G_2, G_3, \cdots, G_r$ be the components of G. Let $u \in V(G)$.
Then $m(G - u) = \max\, \{m(G_1 - u), |V(G_2)|, \cdots, |V(G_r)|\}$. By hypothesis, there

exists a unique G_i say G_1 such that $|V(G_1)| > |V(G_j)|$, for every $j \neq 1$. For any vertex $u \in V(G_1)$, $m(G-u) < |V(G_1)|$ and for any vertex $u \notin V(G_1)$, $m(G-u) = |V(G_1)|$, a contradiction.
Hence, every vertex $u \notin V(G_1)$ is such that $\{u\}$ is not an I-set of G. $\square$

Lemma 3.1. *Let G be any graph of order n. Then $G + K_{n,n}$ has integrity $2n+1$ and any I-set of $G + K_{n,n}$ is obtained by taking an I-set of $K_{n,n}$ together with $V(G)$.*

Proof. Let $H = G+K_{n,n}$. Since $I(G_1+G_2) = min\{I(G_1)+|V(G_2)|, I(G_2)+|V(G_1)|\}$, $I(H) = min\{I(G) + 2n, 2n+1\} = 2n+1$. Let S be any I-set of H. Suppose there exists $x \in V(G)$ such that $x \notin S$. Then $m(H-S) = |V(H)| - |S|$. Therefore, $|S| + m(H-S) = |V(H)|$ and hence, S is not an I-set of H, which implies, any I-set of H contains $V(G)$. Let (V_1, V_2) be the bipartition of $K_{n,n}$. Suppose $S \cap V_1 \neq \emptyset$ and $V_1 \not\subset S$. Then $|S| + m(G-S) = |V(H)|$. Therefore, S is not an I-set. Similarly, $S \cap V_2 \neq \emptyset$ and $V_2 \not\subset S$ are not possible if S is an I-set of H. Therefore, either $V_1 \subset S$ and $S \cap V_2 = \emptyset$ or $V_2 \subset S$ and $S \cap V_1 = \emptyset$. $\square$

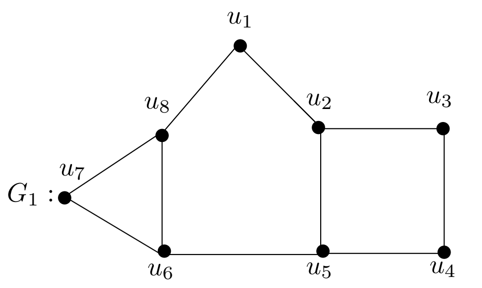
Corollary 3.3. *Any graph G is an induced subgraph of an I-excellent graph.*

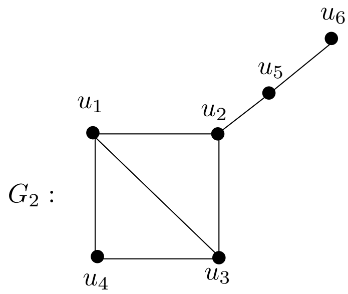
Corollary 3.4. *There is no forbidden subgraph characterization of the class of I-excellent graphs.*

Proposition 3.6. *Let G be a graph which is not I-excellent. Suppose G has a unique I-bad vertex, say u. If u belongs to a maximum order component in every I-set of G, then there exists a I-excellent graph H such that*
(i) $I(H) = I(G) + 1$
(ii) H is I-excellent.
(iii) G is an induced subgraph of H.

Proof. Add a pendent vertex v to u. Let H be the resulting graph. Then any I-set of G, say S, will satisfy $|S|+m(H-S) = |S|+m(G-S)+1 = I(G)+1$. Let S be any I-set of G. Then $|S\cup\{u\}|+m(H-(S\cup\{u\})) = |S|+1+m(G-S) = I(G)+1$. $|S\cup\{v\}|+m(H-(S\cup\{v\})) = |S|+1+m(G-S) = I(G)+1$. Thus $I(H) \leqslant I(G)+1$. Suppose that $I(H) = I(G)$. Let S_1 be any I-set of H.
Then $|S_1| + m(H-S_1) = I(H) = I(G) \leqslant |S_1| + m(G-S_1)$.
But $m(G-S_1) \leqslant m(H-S_1)$. Therefore, $|S_1| + m(G-S_1) \leqslant |S_1| + m(H-S_1) = I(G) \leqslant |S_1| + m(G-S_1)$. Therefore, S_1 is an I-set of G.
Then $|S_1|+m(H-S_1) = I(G)+1$, a contradiction. Therefore, $I(H) > I(G)$. Hence, $I(H) = I(G)+1$. Thus, H is I-excellent containing G as an induced subgraph and $I(H) = I(G)+1$. $\square$

Illustration 3.1.





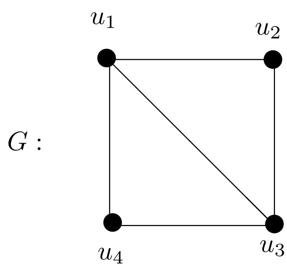
The I-sets of G_1 are $\{u_1, u_5\}; \{u_2, u_6\}; \{u_3, u_5, u_8\}; \{u_2, u_4, u_6, u_8\}$ and $I(G_1) = 5$. The I-sets of G_2 are $\{u_1, u_3, u_5\}; \{u_2, u_4\}$ and $I(G_2) = 4$. By adding a pendent vertex to u_7 in G_1 and u_6 in G_2, the resulting graph is an I-excellent graph containing G_1 and G_2.

PROPOSITION 3.7. *Let G be a graph with a unique I-bad vertex u. Then $G - u$ is I-excellent.*

PROOF. Since u is I-bad, $I(G - u) = I(G)$. Let S be any I-set of G. Then $|S| + m(G - S) = I(G) = I(G - u) \leqslant |S| + m((G - u) - S)$.
Case(1): $m((G - u) - S) = m(G - S)$. Then S is an I-set of $G - u$.
Case(2): $m((G - u) - S) = m(G - S) - 1$.
Then $I(G - u) \leqslant |S| + m(G - S) - 1 = I(G) - 1$, a contradiction, since $I(G - u) = I(G)$. Hence S is an I-set of $G - u$ and since every vertex in $G - u$ is an element of some I-set of G, $G - u$ is I-excellent. $\square$

REMARK 3.6. There exists a graph G containing two I-bad vertices and removal of one of them makes the resulting graph is I-excellent.

EXMPLE 3.5.



In G, u_2 and u_4 are I-bad vertices but $G - \{u_2\}$ and $G - \{u_4\}$ are I-excellent. $K_n - \{e\}$ is another example.

PROPOSITION 3.8. *Let G be a connected graph. Let $u \in V(G)$ satisfy the property that for every $v \in V(G)$, $v \neq u$, there exists an I-set of G containing u and v. Let H be the graph obtained from G by making v as a full degree vertex in G. Then every I-set of G containing u is an I-set of H.*

PROOF. Let S be an I-set of G containing u. Adding an edge in $< S >$ (or) an edge from S to $V - S$ will not affect the value of $|S| + m(G - S)$. Since for any

$v \in V(G)$, there exists an I-set containing u and v, joining v with every vertex of G will not change $|S| + m(G - S)$ for any I-set S containing u. $\square$

PROPOSITION 3.9. *Let T be a tree and S be an I-set of T. If x is a pendent vertex with support y, then x and y together can not belong to S.*

PROOF. If x and y belong to S, then $|S - \{x\}| + m(G - (S - \{x\})) < |S| + m(G - S)$, a contradiction. $\square$

PROPOSITION 3.10. *Let G be a graph. Let u be a non support and non pendent vertex belonging to an I-set of G. Attach a pendent vertex v to u. In the resulting graph H, v is a I-bad vertex and u is a I-good vertex.*

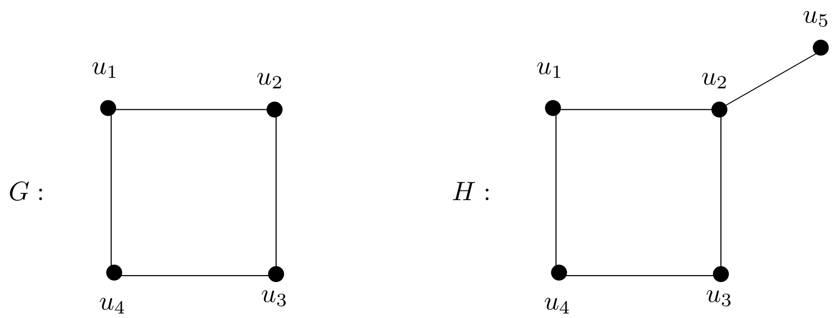
PROOF. Suppose that v is I-good in H. Let S be an I-set of H containing v. Suppose that $u \in S$. Then $m(H - S) = m(G - S)$.
$I(H) = |S| + m(H - S) = |S| + m(G - S)$.
Consider $S - \{v\}$. $m(H - (S - \{v\})) = m(G - S)$.
Therefore, $I(H) \leqslant |S - \{v\}| + m(H - (S - \{v\}))$
$= |S| - 1 + m(G - S)$
$= |S| - 1 + m(H - S)$, a contradiction. Therefore, $u \notin S$. Clearly, $I(G) \leqslant I(H)$. Let S_1 be an I-set of G containing u. In H, $m(H - S_1) = m(G - S_1)$, since v is a singleton component in $H - S_1$. Therefore, $I(H) \leqslant |S_1| + m(H - S_1) = |S_1| + m(G - S_1) = I(G)$. Hence, $I(H) = I(G)$. Therefore, u is a I-good vertex in H. $I(G) = I(H) = |S| + m(H - S) = |S| + m(G - S)$ (since $v \in S$, $< H - S >=< G - S >$). Therefore, $I(G) = |S| + m(G - S)$. Let S_2 be the subset obtained from S by removing v. Then $S_2 \subset V(G)$ and $< G - S_2 >$ with respect to G is the same as $< G - S >$ with respect H.
Therefore, $I(G) \leqslant |S_2| + m(G - S_2) = |S| - 1 + m(G - S) = I(G) - 1$,
a contradiction. Hence, v is a I-bad vertex in H. $\square$

PROPOSITION 3.11. *Let G and H be as in Proposition 3.10. Every vertex w of G which is I-good in G is I-good in H if and only if either u belongs to an I-set of G containing w (or) u does not belong to a maximum order component of $G - S$, where S is an I-set of G containing w.*

PROOF. Let $u \in V(G)$ be I-good in G. Suppose u belongs to an I-set of G containing w. Then $|S| + m(H - S) = |S| + m(G - S) = I(G) = I(H)$, which means, S is an I-set of H containing w. Therefore, w is I-good in H. Suppose u does not belong to any I-set, say S of G containing w and u does not belong to any maximum order component of $G - S$. Then, v being adjacent to u in $H - S$ will increase the order of the component of $H - S$ containing u. As u does not belong to any maximum order component of $G - S$, $m(G - S) = m(H - S)$ and $|S| + m(H - S) = |S| + m(G - S) = I(G) = I(H)$. Hence, w is I-good in H. Suppose u does not belong to any I-set of G containing w and u belongs to a maximum order component of $G - S$ for any I-set S of G containing w. Let S be an I-set of G containing w.Then $u \notin S$ and u belongs to a maximum order component of $G - S$. Therefore,
$|S| + m(H - S) = |S| + m(G - S) + 1 = I(G) + 1 = I(H) + 1$.

Therefore, S is not an I-set of H. Thus any I-set of G containing w is not an I-set of H. Suppose w is I-good in H. Let S_1 be an I-set of H containing w.
Case(i): Suppose that $v \notin S_1$. Then $S_1 \subset V(G)$.
Now $m(G - S_1) \leqslant m(H - S_1)$. Therefore, $I(G) \leqslant |S_1| + m(G - S_1) \leqslant |S_1| + m(H - S_1) = I(H) = I(G)$. Therefore, S_1 is an I-set of G containing w. But any I-set of G containing w is not an I-set of H, a contradiction.
*Case(ii):*Suppose that $v \in S_1$. Therefore, $u \notin S_1$ (since S_1 is an I-set of H). If u is not in a maximum order component of $H - S_1$, then $|S_1 - \{v\}| + m(H - (S_1 - \{v\}) = |S_1| - 1 + m(H - S_1) < |S_1| + m(H - S_1) = I(H)$, a contradiction. Therefore, u is in a maximum order component of $H - S_1$. Thus, $S_2 = S_1 - \{v\}$ is an I-set of H contained in $V(G)$. Since $w \in S_2$, S_2 is an I-set of H and $v \notin S_2$, a contradiction (by case(i)). Hence w is not I-good in H. $\square$

Exmple 3.6.



G is I-excellent with $I(G) = 3$. $I(H) = 3$ and the only I-good vertices in H are u_2 and u_4. Even though u_1 and u_3 are I-good in G, they become I-bad in H since u_2 belongs to maximum order component of the unique I-set containing u_1 and u_3 in G.

Proposition 3.12. *For any I-excellent graph G, every pendent vertex is in some I-set of G and no pendent vertex is in every I-set of G.*

Proof. Since G is I-excellent graph, every vertex and in particular every pendent vertex of G is in some I-set of G. Let x be an pendent vertex of G and let $x \in S$ where S is an I-set of G. Let y be the support of x. If $y \in S$, then $|S - \{x\}| + m(G - (S - \{x\})) = |S| - 1 + m(G - S) < I(G)$, a contradiction. Therefore, $y \notin S$. Let $S_1 = S \cup \{y\} - \{x\}$. Then $m(G - S_1) = m(G - S)$. Therefore, S_1 is an I-set of G not containing x. $\square$

Proposition 3.13. *Let T be a tree with order greater than or equal to 2. Let x be a pendent vertex of T belonging to an I-set of T. Then there exists an I-set of T such that S is not independent.*

Proof. Let x be a pendent vertex and y be its support.
If $T = K_2$, then $S = \{x, y\}$ is an I-set of T which is not independent. Let $|V(T)| \geqslant 3$. Let S be an I-set of T containing x. Clearly, $y \notin S$. If S is independent, then for any vertex $z \in S$, there exists a vertex $z_1 \notin S$ such that z and z_1 are adjacent

(note that $S = \{x\}$ is not possible, since, otherwise $|S + m(G - S) = |V(T)|$, a contradiction).
Case(i): Suppose that y is not in a maximum order component of $T - S$. Then $S_1 = S - \{x\} \cup \{z_1\}$ is an I-set of T which is not independent.
Case(ii): Suppose that y is in a maximum order component of $T - S$. Since T is connected, there exists a path from y to z_1. If every vertex in this path belongs to $T - S$, then z_1 belongs to the maximum order component of $T - S$ containing y. Then $S - \{x\} \cup \{z_1\}$ is an I-set of T containing the edge zz_1. Suppose that the path from y to z_1 intersects S. Let y_1 be the first vertex in this path belonging to S. Let y_2 be the adjacent vertex of y_1 in $T - S$. Then $S - \{x\} \cup \{y_2\}$ is an I-set of T containing the edge y_1y_2. In both cases, S contains an edge. Therefore, S is not independent. $\square$

Corollary 3.5. *If T is an I-excellent tree, then there exists an I-set S such that S is not independent.*

4. Bondage Integrity number in graphs

Definition 4.1. [**2**] The *bondage integrity number* of G is the minimum cardinality of a smallest set $E_1(G)$ of edges for which $I(G - E_1) < I(G)$ and is denoted by $b_I(G)$.

Remark 4.1.

For any connected graph G, $1 \leqslant b_I(G) \leqslant m$, where $m = |E(G)|$.

Theorem 4.1. *Let G be a connected graph. Then $b_I(G) = |E(G)|$ if and only if $G = K_{1,n}$.*

Proof. Let $G = K_{1,n}$. Then $I(K_{1,n}) = 2$. Let E_1 be a subset of $E(G)$ of cardinality $k \leqslant n$ such that $I(K_{1,n} - E_1) < 2$. That is, $I(K_{1,n} - E_1) = 1$. Therefore, $K_{1,n} - E_1$ is totally disconnected. Therefore, $E_1 = E(K_{1,n})$. Thus, $b_I(G) = |E(G)|$.
Conversely, let G be a connected graph with $b_I(G) = |E(G)|$. That is, $I(G - E(G)) < I(G)$ and for any proper subset E_1 of $E(G)$, $I(G - E_1) = I(G)$. Let E_1 be contain $|E(G)| - 1$ edges. Then $G - E_1$ is a subgraph of G with exactly one edge. Therefore, $I(G - E_1) = 2$ and hence, $I(G) = 2$. Therefore, $G = K_{1,n}$ (since $I(G) = 2$ if and only if $G = K_{1,n}$). $\square$

Proposition 4.1. *The bondage integrity number of the complete graph K_n $(n \geqslant 2)$ is $b_I(K_n) = 1$.*

Proof. Let $V(K_n) = \{v_1, v_2, \cdots, v_n\}$. Let $v_1v_2 \in E(K_n)$. Let $H = K_n - v_1v_2$. Then $I(H) = n - 1 < I(K_n)$ (since $\{v_3, v_4, \cdots, v_n\}$ is an I-set of $(K_n - v_1v_2)$, we get only isolated vertices v_1, v_2). Hence $b_I(K_n) = 1$. $\square$

Proposition 4.2. *The bondage integrity number of the path of order $n \geqslant 2$ is given by* $b_I(P_n) = \begin{cases} 2, & \text{if } n = \lceil \frac{k+2}{2} \rceil \lceil \frac{k+3}{2} \rceil - 1,\ k \geqslant 0 \\ 1 & \text{otherwise} \end{cases}$

PROOF. It can be easily seen that the removal of at most two edges reduces the integrity of P_2, P_3 and P_4.
Case(i): Let $n = \lceil\frac{k+2}{2}\rceil\lceil\frac{k+3}{2}\rceil - 1$, $k = 1, 2, \cdots$. For $5 \leqslant n \leqslant 23$, the removal of second and the last but one edge decreases the integrity. The removal $(k+2)^{th}$, $(k \geqslant 1)$ edge and last but one edge for all paths P_n, $n = \lceil\frac{k+2}{2}\rceil\lceil\frac{k+3}{2}\rceil - 1$, $k = 1, 2, \cdots$ with $k^2 + 8k + 15 \leqslant n \leqslant k^2 + 10k + 23$ results in a graph whose integrity is less than $I(P_n)$. Therefore, $b_I(P_n) \leqslant 2$ when $n = \lceil\frac{k+2}{2}\rceil\lceil\frac{k+3}{2}\rceil - 1$, $k = 1, 2, \cdots$.
It can be easily verified that the removal of any one edge does not decrease the integrity of P_n, when $n = \lceil\frac{k+2}{2}\rceil\lceil\frac{k+3}{2}\rceil - 1$, $k = 0, 1, 2, \cdots$. Therefore, $b_I(P_n) = 2$.
Case(ii): Let $n = \lceil\frac{k+2}{2}\rceil\lceil\frac{k+3}{2}\rceil$, $k = 1, 2, \cdots$. Then the removal of either the first edge or the last edge decrease the integrity. Therefore, $b_I(P_n) = 1$.
It can be easily seen that the removal of at most two edges reduces the integrity of P_6.
Case(ii): Let $n \neq \lceil\frac{k+2}{2}\rceil\lceil\frac{k+3}{2}\rceil$ and $n \neq \lceil\frac{k+2}{2}\rceil\lceil\frac{k+3}{2}\rceil - 1$, $k = 1, 2, \cdots$. For $5 \leqslant n \leqslant 23$, the removal of second and the last but one edge decreases the integrity. then the removal of $(k+2)^{th}$,$(k \geqslant 1)$-edge reduces the integrity of P_n, when $n \neq \lceil\frac{k+2}{2}\rceil\lceil\frac{k+3}{2}\rceil - 1$ and $n \neq \lceil\frac{k+2}{2}\rceil\lceil\frac{k+3}{2}\rceil - 1$.
Therefore, $b_I(P_n) = 1$. □

PROPOSITION 4.3. *The bondage integrity number of the cycle of order* $n \geqslant 3$ *is given by* $b_I(C_n) = \begin{cases} 2 & \text{if } n = \lceil\frac{k+2}{2}\rceil\lceil\frac{k+3}{2}\rceil,\ k \geqslant 0 \\ 1 & \text{otherwise} \end{cases}$

PROPOSITION 4.4. *Let G be a connected graph . Then $b_I(G) \neq m - 1$, where $m = |E(G)|$.*

PROOF. Let $b_I(G) = m - 1$. Then $I(G - (|E(G)| - 1)) < I(G)$, and for any proper subset $E_1(G)$ of $E(G)$ with $|E_1| \leqslant |E(G)| - 1$. Then $I(G - E_1) = I(G)$. Let $|E_1| = m - 2$. Then $G - E_1$ is a subgraph of G with exactly two edges. Therefore, $I(G - E_1) = 2$. Hence,$I(G) = 2$. Thus, $G - E_1$ is $K_{1,n}$. But G is $K_{1,n}$ if and only if $b_I(G) = |E(G)|$, a contradiction. Thus, there exists no graph G with $b_I(G) = m - 1$. Hence $b_I(G) \neq m - 1$. □

REMARK 4.2. Let $d = min\{deg(u) : u \in S$, for some I-set S of $G\}$. Then $b_I(G) \leqslant d$.

REMARK 4.3. If a vertex $v \in V(G)$ of degree $\delta(G)$ belongs to an I-set of G, then $b_I(G) \leqslant \delta(G)$.

PROPOSITION 4.5. *Let G be a connected graph. Let S be an I-set of G. Let $u \in S$. Then $b_I(G) \leqslant \Delta(G)$.*

PROOF. Let S be an I-set of G. Let $u \in S$. Remove all the edges incident at u. Let H be the resulting graph. Then $I(H) < I(G)$. Hence $b_I(G) \leqslant deg(u) \leqslant \Delta(G)$. □

REMARK 4.4. The bound is sharp as seen from $K_{m,n}$ or P_5.

REMARK 4.5. In K_n, $b_I(G) = 1$ and $\Delta(G) = n - 1$. Thus the difference between $b_I(G)$ and $\Delta(G)$ may be large. That is, given any positive integer k, $\Delta(G) - b_I(G) = k$ where $G = K_{k+2}$.

PROPOSITION 4.6. *If $b_I(G) = \Delta(G)$, then every I-set of G has at least two maximum order component in $G - S$.*

PROOF. If an I-set S has a unique maximum order component, say T then for some $u \in T$, $deg_{<T>}(u) < \Delta(G)$. (If $deg_{<T>}(u) = \Delta(G)$ for all $u \in T$, then T is a proper component of G, a contradiction, since G is connected.) Therefore, removing all the edges in T incident at u, $m(G-S)$ gets reduced and $b_I(G) < \Delta(G)$ in the resulting graph, a contradiction. Therefore, every I-set of G has at least two maximum order components. □

REMARK 4.6. If $b_I(G) = \Delta(G)$, then no I-set of T contains any pendent vertex.

OBSERVATION 4.1. Let S be a non I-set of G. Let $T_1, T_2, \cdots, T_k$ be the k maximum order components of $G - S$. Suppose $\sum_{i=1}^{k} \kappa'(T_i) = t$. Remove $\kappa'(T_i)$ edges in each component so that the maximum order of the remaining components is least. Let l be that order. Suppose $|S| + l = I(G) + k_1, k_1 \geqslant 0$. Choose vertices $u_1, u_2, \cdots, u_{(k_1+1)}$ such that $m(G-(S-\{u_1, u_2, \cdots, u_{(k_1+1)}\}))$ is minimum. Let k_2 be the minimum number of edges whose removal reduces the order of the maximum order component to l. Then the set $S - \{u_1, u_2, \cdots, u_{(k_1+1)}\}$ reduces the integrity in the resulting graph. Then $k_2 + \sum_{i=1}^{k} \kappa'(T_i) < \Delta(G)$ if and only if $b_I(G) < \Delta(G)$.

THEOREM 4.2. *Let S be an I-set of G. If $V - S$ has k maximum order components say $T_1, T_2, \cdots, T_k$ then for every $u_i \in T_i$, $1 \leqslant i \leqslant k$.*
$b_I(G) = \Delta(G)$ if and only if

1. *for any vertex u in any I-set S of G, every vertex in $N(u)$ is contained in some maximum order component of $G - S$. That is, for every vertex $u \in S$, $|N(u) \cap (V - S)| = \Delta(G)$.*
2. *for any vertex $u \in S$, every vertex in $N(u)$ is contained in some maximum order component of $G - S$.*
3. $\sum_{i=1}^{k} \kappa'(T_i) = \Delta(G)$.
4. *for any non I-set of $V(G)$, $k_2 + \sum_{i=1}^{k} \kappa'(T_i) = \Delta(G)$.*

PROOF. Let $b_I(G) = \Delta(G)$.

1. Let S be an I-set of G such that among all the I-sets of G, S contains a vertex u such that $|N(u) \cap (V - S)|$ is minimum.
Then $b_I(G) \leqslant |N(u) \cap (V - S)|$.
Therefore, $\Delta(G) = b_I(G) \leqslant |N(u) \cap (V - S)| \leqslant \Delta(G)$.
Thus, $|N(u) \cap (V - S)| = \Delta(G)$. Therefore, u is of degree $\Delta(G)$ and u has $\Delta(G)$ neighbours in $V - S$. Hence, every vertex in every I-set of G has $\Delta(G)$ neighbours in $V - S$.

(2) Suppose there exists a vertex u in an I-set S such that there exists a vertex v in $N(u)$ which is not contained in any maximum order component of $G - S$. Then remove all the edges incident at u except uv.
In the resulting graph H, $S - \{u\}$ is a subset of $V(H)$ such that
$|S - \{u\}| + m(H - (S - \{u\})) = |S| - 1 + m(G - S) < I(G)$.
Therefore, $b_I(G) \leqslant \Delta(G) - 1$, a contradiction. Hence, for every vertex $u \in S$, $|N(u) \cap (V - S)| = \Delta(G)$.

(3) Suppose $\sum\limits_{i=1}^{k} \kappa'(T_i) < \Delta(G) - 1$. Then remove all that edges which reduces the integrity of the graph. Therefore, $b_I(G) < \Delta(G)$, a contradiction.

(4) Suppose, there exists a non I-set of $V(G)$ such that
$k_2 + \sum\limits_{i=1}^{k} \kappa'(T_i) < \Delta(G)$. By observation 4.1, $b_I(G) < \Delta(G)$, we get a contradiction.

Conversely, let the conditions in the theorem hold. Suppose that $b_I(G) = k < \Delta(G)$. Suppose that there exists an I-set S such that the removal of k edges in G decreases the integrity of G.
Then either $\sum\limits_{i=1}^{k} \kappa'(T_i) < \Delta(G)$ or $|N(u) \cap (V - S)| < \Delta(G)$ for some $u \in S$ or $|N(u) \cap (V - S)| = \Delta(G)$ and there exists a vertex $u \in S$ which is not contained in any maximum order component of $G - S$, a contradiction. Therefore, $b_I(G) = \Delta(G)$. Suppose that there exists a non I-set S_1 of $V(G)$ such that removal of k edges reduces the integrity of G. Then $k_2 + \sum\limits_{i=1}^{k} \kappa'(T_i) = \Delta(G)$, a contradiction. Hence the theorem. □

Proposition 4.7. *Let S be a subset of $V(G)$. If there exists $u \in S$ such that every vertex in $N(u) \cap (V - S)$ does not belong to any maximum order component in $G - S$ and if $N(u) \cap (V - S)$ intersects components $T_1, T_2, \cdots, T_k$ in $G - S$, then $\sum\limits_{i=1}^{k} |T_i| < m(G - S)$, then S is not an I-set of G.*

Proof. Suppose the condition in the proposition is true.
Then $|S - \{u\}| + m(G - (S - \{u\})) = |S| - 1 + m(G - S)$. Therefore, S is not an I-set. □

Corollary 4.1. *If S is an I-set of G, then every vertex of S is adjacent to at least one maximum order component of $G - S$ or if $N(u) \cap (V - S)$ intersects components the $T_1, T_2, \cdots, T_k$ in $G - S$, then $\sum\limits_{i=1}^{k} |T_i| < m(G - S)$.*

Proposition 4.8. *Let G be a simple graph. Let S be an I-set of G. Then $b_I(G) = 1$ if and only if there exists an I-set S of G such that either $G - S$ has a unique maximum order component, say T and $\kappa'(T) = 1$ (or) there exists $u \in S$ such that u is adjacent to exactly one vertex of exactly one maximum order component of $G - S$ and if u is adjacent to two or more non maximum order* *components, then the sum of the order of such components should not exceed the cardinality of a maximum order component of $G-S$.*

PROOF. Let S be an I-set of G. Let $b_I(G)=1$. Then there exists $uv \in E(G)$ such that $I(G-uv) < I(G)$.
Case(i):

Let $uv \in< G-S >$. If uv belongs to either a non maximum order component or a maximum order component, with another maximum order component existing in $G-S$ (or) it belongs to a unique maximum order component of $G-S$ but not a cut edge of that component, then
$I(G-uv)=I(G)$, a contradiction. Therefore, $G-S$ contains exactly one maximum order component and uv is a cut edge of that component. That is, $G-S$ has a unique maximum order component T and $\kappa'(T)=1$.
Case(ii):

$u \in S$ and $v \in V(G-S)$. If v belongs to a non maximum order component say T, then as u is adjacent to a maximum order component of $G-S$, we get that $I(G-uv)=I(G)$, a contradiction. Therefore, v belongs to a maximum order component say T. If u is adjacent to more than one vertex of T (or) u is adjacent to more non maximum order components with sum of their orders greater than the cardinality of a maximum order component of $G-S$, then $I(G-uv)=I(G)$, a contradiction. Therefore, v belongs to a maximum order component say T and u is adjacent only to v in T and it is not adjacent to any maximum order component of $G-S$.
The converse is obvious. □

PROPOSITION 4.9. *Let S be an non I-set of G such that $|S|+m(G-S)=I(G)+k$, $k \geqslant 1$. Then $b_I(G)=1$ if and only if either $G-S$ has a maximum order component T with a cut edge e such that the cardinality of every component of $T-e$ is less than or equal to $|T|-(k+1)$ (or) if $G-S$ has a unique maximum order component T with a cut edge e such that the cardinality of the maximum order component of $T-e$ is $|T|-l$ where $l=k+1-r$, $r \geqslant 1$, then there exists a set S_1 of vertices $u_1,u_2,\cdots,u_{l+1}$ such that the maximum order component of $G-(S-\{u_1,u_2,\cdots,u_{l+1}\})$ is $|T|-l+1$.*

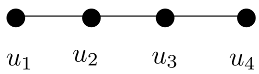
PROOF. Let S be a non I-set of G such that $|S|+m(G-S)=I(G)+k, k \geqslant 1$. Let $b_I(G)=1$. Then there exists $uv \in E(G)$ such that $I(G-uv) < I(G)$.
Case(i):

Let $uv \in< G-S >$. If uv belongs to either a non maximum order component or a maximum order component, with another maximum order component existing in $G-S$ (or) it belongs to a unique maximum order component of $G-S$ but not a cut edge of that component, then $I(G-uv)=I(G)$, a contradiction. Therefore, $G-S$ contains exactly one maximum order component and uv is a cut edge of that component. If the removal of uv from T results in components of order less than or equal to $|T|-(k+1)$, then $I(G-uv) < I(G)$, a contradiction. Therefore, every component of $T-uv$ has order less than or equal to $|T|-(k+1)$.
Case(ii):

$u \in S$ and $v \in V(G-S)$. In this case, $|S| + m(G - s)$ can be reduced by at most one by the removal of uv by shifting u from S to $V - S$. Hence the resulting set say S_1 satisfies $|S_1| + m(G - S_1) \geqslant I(G)$, a contradiction. Suppose the removal of e from T results in a component of T results in a component of T of maximum order $|T| - l$ where $l = k + 1 - r$, $r \geqslant 1$. Then $|S| + m(G - S - e) \geqslant I(G)$. Hence, we require a set S_1 of $l + 1$ vertices to be removed from S such that the maximum order component of $G - (S - S_1)$ is less than or equal to $|T| - l + 1$, in which case $|S - S_1| + m(G - (S - S_1)) < I(G)$, a contradiction. Hence the result. The converse is obvious. □

ILLUSTRATION 4.1.

P_4 :



$S = \{u_1, u_4\}$ be non I-set of P_4. $|S| + m(G - S) = 4 > I(P_4) = 3$. Remove the edge u_2u_3. Then $|S| + m(G - S - u_2u_3) = 3 = I(G)$. Let $S_1 = S - \{u_1, u_4\}$. Then $|S_1| + m((G - u_2u_3) - S_1) = 0 + 2 = 2 < I(P_4)$.

COROLLARY 4.2. $b_I(G) = 1$ *if and only if either there exists an I-set S satisfying the condition of the proposition 4.8 or there exists a maximum cardinality I-set satisfying the conditions of proposition 4.9.*

OBSERVATION 4.2. If S_1 is a non I-set of G with $|S_1|$ less than the order of a maximum order I-set of G say S, then there exists a subset S_2 of $V(G)$ such that $|S_2| = |S|$ and $|S_2| + m(G - S_2) \geqslant I(T)$.

PROPOSITION 4.10. *Let G be a connected graph. Then $b_I(G) \leqslant I'(G) - 1$.*

PROOF. Since $I'(G) \geqslant \Delta(G) + 1$ and $b_I(G) \leqslant \Delta(G)$,
we have $b_I(G) \leqslant \Delta(G) \leqslant I'(G) - 1$. The bound is sharp as seen in P_3. □

PROPOSITION 4.11.

Let G and H be two graphs. Then $b_I(G + H) \leqslant max\{b_I(G), b_I(H)\}$.

PROOF. Clearly, $I(G + H) = min\{I(G) + |V(H)|, I(H) + |V(G)|\}$.
Case(i): $I(G) + |V(H)| \leqslant I(H) + |V(G)|$. Then $I(G + H) = I(G) + |V(H)|$. Let E_1 be a set of edges of cardinality $b_I(G)$ whose removal from G reduces $I(G)$. Then $I(G + H - E_1) \leqslant I(G - E_1) + |V(H)| < I(G) + |V(H)|$. Therefore, $b_I(G + H) \leqslant |E_1| = b_I(G)$.
Similarly, we can prove that if $I(H) + |V(G)| \leqslant I(G) + |V(H)|$, then $b_I(G + H) \leqslant b_I(H)$. Therefore, $b_I(G + H) \leqslant max\{b_I(G), b_I(H)\}$. □

REMARK 4.7. The bound is sharp as seen in $P_3 + K_4$. $I(P_3) = 2, I(K_4) = 4$ and $b_I(P_3) = 2$; $b_I(K_4) = 1$. $b_I(P_3 + K_4) = max\{b_I(P_3), b_I(K_4)\} = 2$.

PROPOSITION 4.12. *Let T be any tree. If T is I-excellent, then $b_I(T) = 1$.*

PROOF. Let T be an I-excellent graph. Let u be a pendent vertex of T and let v be its support.

Clearly, $T - uv = (T - \{u\}) \cup \{u\}$. $I(T - \{u\}) < I(T)$, since T is I-excellent and hence $u \in T$. $I(T - uv) = I(T - \{u\})$. Therefore, $I(T - uv) < I(T)$.

Hence $b_I(T) = 1$. □

REMARK 4.8. The converse is not true. That is, there exists non I-excellent trees with $b_I(T) = 1$. For example, P_7 is not I-excellent, but $b_I(P_7) = 1$.

PROPOSITION 4.13. *$b_I(K_2 \times P_n) = 2$, for every $n, n \geqslant 2$.*

PROOF. Since $b_I(G) \leqslant \Delta(G)$, $b_I(K_2 \times P_n) \leqslant 3$.
If $n = \lceil \frac{k+2}{2} \rceil \lceil \frac{k+3}{2} \rceil - 1$, $k = 1, 2, \cdots$, then $b_I(P_n) = 2$ and the removal of those two edges from P_n which reduces $I(P_n)$ also reduces $I(K_2 \times P_n)$. Therefore, $b_I(K_2 \times P_n) \leqslant 2$.
If $n \neq \lceil \frac{k+2}{2} \rceil \lceil \frac{k+3}{2} \rceil - 1$, $k = 1, 2, \cdots$, then $b_I(P_n) = 1$ and the removal of one edge from each of the two P_n-layers reduces $I(K_2 \times P_n)$. The removal of any single edge from $K_2 \times P_n$ does not reduce its integrity. Therefore, $b_I(K_2 \times P_n) \geqslant 2$. Hence $b_I(K_2 \times P_n) = 2$. □

PROPOSITION 4.14. *$b_I(K_2 \times C_n) = 2$, for every n, where $n \geqslant 3$.*

PROOF. Since $b_I(G) \leqslant \Delta(G), b_I(K_2 \times C_n) \leqslant 3$. Proceeding as in the proposition 4.13, we get that $b_I(K_2 \times C_n) = 2$, for every $n, n \geqslant 3$. □

THEOREM 4.3. *Let G be any connected graph. If G is I-excellent , then $b_I(G) \leqslant \delta(G)$.*

PROOF. Let G be an I-excellent graph. Let u be a vertex of degree $\delta(G)$. Since G is I-excellent, u belongs to an I-set of G. Remove all the edges incident at u. Let H be the resulting graph. Then $I(H) < I(G)$.
Therefore, $b_I(G) \leqslant \delta(G)$. □

REMARK 4.9. The bound is sharp as seen in P_4.

PROPOSITION 4.15. *Let G be a connected graph. Let S be an I-set of G such that among all the I-sets of G, S contains a vertex u such that $|N(u) \cap (G - S)|$ is minimum. Then $b_I(G) \leqslant |N(u) \cap (G - S)|$.*

PROOF. Let u satisfy the hypothesis. Remove all the edges from u to $G - S$. Let H be the resulting graph.
$I(H) \leqslant I(H - (S - \{u\})) + m(H - \{u\}) = I(G - S) - 1 + m(G - S) = I(G) - 1$.
Therefore, $b_I(G) \leqslant |N(u) \cap (G - S)|$. □

REMARK 4.10. The bound is sharp as seen in P_5 and $D_{r,s}$.

REMARK 4.11. Let G be a connected graph. Let S be an I-set of G such that among all I-sets of G, S contains a vertex u such that number of edges from u to the maximum components of $G-S$ and to the components of $G-S$ with cardinality of $m(G-S)-1$ is minimum.
Then $b_I(G) \leqslant t$ where t is the number of edges from u to the maximum components of $G-S$ and to the components of $G-S$ with cardinality of $m(G-S)-1$ is minimum.

PROOF. The proof follows from the fact that the removal of such edges will not affect the cardinality of the maximum order component of $G-(S-\{u\})$. $\square$

PROPOSITION 4.16. *Let G be a connected graph and S be an I-set of G of maximum cardinality . Then $b_I(G) \leqslant n - I(G) + m(G-S)$.*

PROOF. Let S be an I-set of G of maximum cardinality. Then $m(G-S) = I(G) - |S|$ is minimum. Let $u \in S$. The maximum number of edges from u to $V-S$ is $|V-S|$. The removal of these edges from u will result in a graph with less integrity than $I(G)$. Therefore, $b_I(G) \leqslant |V-S| = n - |S| = n - I(G) + m(G-S)$. $\square$

PROPOSITION 4.17. *Let G and $\overline{G}$ be connected graph. Then $b_I(G) + b_I(\overline{G}) \leqslant |E(G)| + \Delta(G) - \delta(G)$ and hence for a regular graph, $b_I(G) + b_I(\overline{G}) \leqslant |E(G)|$.*

PROOF. $b_I(G) \leqslant \Delta(G)$ and $b_I(\overline{G}) \leqslant \Delta(\overline{G})$.
Therefore, $b_I(\overline{G}) \leqslant \Delta(\overline{G}) = n - \delta(G) - 1$.
Thus, $b_I(G) + b_I(\overline{G}) \leqslant \Delta(G) + n - \delta(G) - 1 \leqslant |E(G)| + \Delta(G) - \delta(G)$. $\square$

PROPOSITION 4.18. *Let G^+ be the corona of G.
Then $I(G^+) \leqslant I(G) + m(G-S)$ if $G \neq K_n$ where S is an I-set of maximum cardinality in G. The bound is sharp as seen in P_n.*

PROOF. Let S be an I-set of G of maximum cardinality. For any I-set of T of G,
$I(G^+) \leqslant |T| + 2m(G-T) = I(G) + m(G-T)$.
$I(G) + m(G-S) = \min\limits_{T \subset V(G)} \{I(G) + m(G-T)\}$, since S is an I-set of G of maximum cardinality. Therefore, $I(G^+) \leqslant I(G) + m(G-S)$. $\square$

REMARK 4.12. $I(K_n^+) = n+1$.

COROLLARY 4.3. *$b_I(G^+) \leqslant b_I(G)$ since for any removal of $b_I(G)$ edges in G, $I(G)$ becomes reduced and hence $I(G^+)$ also is reduced.*

PROPOSITION 4.19. *Let $\overline{C_n}$ be the complement of the Cycle C_n, $n \geqslant 5$. Then $I(\overline{C_n}) = n-1$.*

PROOF. Since $C_n (n \geqslant 5)$ has girth at least 5, $I(\overline{C_n}) = n-1$ (since in theorem 2.1(c) of [**3**], it is proved that $I(G) = n-1$ if and only if $\overline{G}$ has girth at least 5). $\square$

COROLLARY 4.4. *For $n \geqslant 5$, $b_I(\overline{C_n}) = 1$.*

PROOF. Let $V(C_n) = \{u_1, u_2, \cdots, u_n\}$. Then $S = \{u_1, u_2, \cdots, u_{n-2}\}$ is an I-set of $\overline{C_n}$ and $I(\overline{C_n}) = n-1$ and $b_I(\overline{C_n}) = 1$, since the removal of the edge in the maximum order component of $\overline{C_n} - S$ reduces the integrity. $\square$

PROPOSITION 4.20. *Let $\overline{P_n}$ be the complement of P_n. Then $I(\overline{P_n}) = n-1$.*

PROOF. Since P_n $(n \geqslant 4)$ does not contain $2K_2$ as an induced subgraph, by theorem 2.7(b) of [**3**], it is proved that,
$I(\overline{P_n}) = \alpha_0(\overline{P_n}) + 1 = n-2+1 = n-1$. $\square$

PROPOSITION 4.21. *If $I(G) = n - 1$, then $b_I(G) \leqslant 2$.*

PROOF. Let $I(G) = n - 1$. By theorem 2.7(b) of [**3**], $2K_2$ as not an induced subgraph of G. Hence, $I(G) = \alpha_0(G)+1$. Therefore, $\alpha_0(G) = n-2$ and $\beta_0(G) = 2$. Let S be a minimum vertex cover of G. Then $|V(G) - S| = 2$. Let $V(G - S) = \{v_1, v_2\}$. Since S is a dominating set, there exists v_1, v_2 such that v_1 is adjacent to u_1 and v_2 is adjacent to u_2. If $u_1 \neq u_2$, then remove the edge u_1v_1 from G. Let V_1 be the resulting graph.
Then $|S \cup \{u_1\}| + m(G - (S \cup \{u_1\})) = |S| - 1 + 1 = |S| = n - 2$. Therefore, $I(G) \leqslant n - 2$. Hence $b_I(T) = 1$. If $u_1 = u_2$, then remove the edges u_1v_1 and u_2v_2, then the resulting graph G_1 with $I(G_1) \leqslant n - 2$.
Therefore, $b_I(G) = 2$. □

PROPOSITION 4.22. *If $I(G) = \alpha_0(G) + 1$, then $b_I(G) \leqslant \beta_0(G)$.*

PROOF. Let S be a minimum vertex cover of G. Then $m(G-S) = 1$. Therefore, $|S| + m(G - S) = \alpha_0(G) + 1 = I(G)$. Therefore, S is an I-set of G. Let u be any vertex in S. Then all the edges from u to $V - S$ are removed. Let G_1 be the resulting graph. Then the number of edges removed is less than or equal to $\beta_0(G)$.
$|S - \{u\}| + m(G - (S \cup \{u\})) = |S| - 1 + 1 = |S| = \alpha_0(G) < I(G)$. Therefore, $b_I(G) \leqslant \beta_0(G)$. □

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