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INTEGRAL REPRESENTATIONS OF THE PELL AND PELL-LUCAS NUMBERS

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ABSTRACT. We present integral representations of Pell and Pell-Lucas numbers for the first time in this paper. In this note, we first give new integral representations of the Pell numbers P_{kn} and the Pell-Lucas numbers Q_{kn} and then using integral representations of the Pell numbers P_{kn} and the Pell-Lucas numbers Q_{kn}, we give integral representations of the Pell numbers P_{kn+r} and the Pell-Lucas numbers Q_{kn+r}, where $n \in \mathbb{Z}_{\geqslant 0} = \{0, 1, 2, \ldots\}$ is a non-negative integer, $k \in \mathbb{Z}_{>0} = \{1, 2, 3, \ldots\}$ is an arbitrary but fixed positive integer, while $r \in \mathbb{Z}_{\geqslant 0}$ is an arbitrary but fixed non-negative integer.

1. Introduction and a simple review of recent developments

The integral representations of the special numbers that are obtained from different counting sequences are used as a tool in a large number of studies. This fact indicates the importance of obtaining the integral representations of different special numbers.

There are many papers devoted to the study of the integral representations of some special numbers. These integral representations of special numbers have attracted much attention. The integral representations of special numbers found in the literature are proved using standard or advanced mathematical techniques from the integral calculus.

We will now give a brief overview of the most recent developments of the integral representations related to the special numbers that are obtained from different counting sequences.

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Most of the works in the literature deal with the integral representations related to the Catalan numbers.

Recall from [**12**] and [**15**] that the Catalan numbers C_n are defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad n = 0, 1, 2, \ldots .$$

Dana-Picard [**3**] showed that a Catalan number can be defined in many different ways by the properties of a combinatorial system. Also, in that paper, Dana-Picard presented the integral representations for these Catalan numbers.

Dana-Picard and Zeitoun [**4**] computed closed forms for two multiparameter families of definite integrals. They obtained combinatorial formulas.

Dana-Picard [**5**] derived a combinatorial identity and obtained two integral representations of the Catalan numbers.

Dana-Picard [**6**] obtained integral identities and new integral representations of the Catalan numbers by searching for closed forms of an integral depending on a parameter.

Using the Wallis formula and a non-straightforward recurrence formula, Dana-Picard and Zeitoun [**7**] gave a sequence of improper integrals for which a closed formula can be computed. This gives a new integral representation for the Catalan numbers.

Penson and Sixdeniers [**17**] established an integral representation for the Catalan numbers by means of the Mellin transform.

Recall from [**1**] that the Fibonacci numbers $F_n, \quad n = 0, 1, 2, \ldots,$ are defined by $F_0 = 0, F_1 = 1$ and

$$F_{n+2} = F_{n+1} + F_n, \quad n = 0, 1, 2, \ldots$$

and Lucas numbers $L_n, \quad n = 0, 1, 2, \ldots,$ are defined by $L_0 = 2, L_1 = 1$ and

$$L_{n+2} = L_{n+1} + L_n, \quad n = 0, 1, 2, \ldots .$$

Glasser and Zhou [**11**] introduced an integral representation for the Fibonacci numbers. Stewart [**18**] gave integral representations for the Fibonacci and Lucas numbers.

Recall from [**14**] that the Motzkin numbers $M_n, \quad n = 0, 1, 2, \ldots,$ are defined by

$$M_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k, \quad n = 0, 1, 2, \ldots .$$

Mccalla and Nkwanta [**13**] derived integral representations of Motzkin numbers.

Recall from [**16**] that the Pell numbers $P_n, \quad n = 0, 1, 2, \ldots,$ are defined by $P_0 = 0, P_1 = 1$ and

$$P_{n+2} = 2P_{n+1} + P_n, \quad n = 0, 1, 2, \ldots$$

and the formula of the general term is given by

$$P_n = \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^n - (1-\sqrt{2})^n\right]. \tag{1.1}$$

Recall from [**16**] that the Pell-Lucas numbers $Q_n, \quad n = 0, 1, 2, \ldots$, are defined by $Q_0 = 2, Q_1 = 2$ and

$$Q_{n+2} = 2Q_{n+1} + Q_n, \quad n = 0, 1, 2, \ldots$$

and the formula of the general term is given by

$$Q_n = (1 + \sqrt{2})^n + (1 - \sqrt{2})^n. \tag{1.2}$$

We refer the reader to Koshy's (2014) book, which provides an interesting historical overview of the origins of Pell and Pell-Lucas numbers, for any remaining undefined terms related to Pell and Pell-Lucas numbers.

The Pell numbers P_n and the Pell-Lucas numbers Q_n are frequently employed in the practical applications and scientific investigation fields. For information on new developments of these kinds of numbers, please refer to [**2**], [**8**], [**9**], [**10**] and closely related references therein.

The purpose of this note is to present a new integral representation of Pell numbers P_{kn} and Pell-Lucas numbers Q_{kn}, followed by the construction of integral representations of Pell numbers P_{kn+r} and Pell-Lucas numbers Q_{kn+r} based on integral representations of Pell numbers P_{kn} and Pell-Lucas numbers Q_{kn}, where $n \in \mathbb{Z}_{\geqslant 0} = \{0, 1, 2, \ldots\}$ is a non-negative integer, $k \in \mathbb{Z}_{>0} = \{1, 2, 3, \ldots\}$ is an arbitrary but fixed positive integer, while $r \in \mathbb{Z}_{\geqslant 0}$ is an arbitrary but fixed non-negative integer.

The following section presents several facts concerning Pell and Pell-Lucas numbers.

2. Preliminaries

We will now examine some of the key facts concerning Pell and Pell-Lucas numbers in this section.

Let $\alpha = 1 + \sqrt{2}$. From (1.1), it follows that

$$P_n = \frac{1}{2\sqrt{2}} \left(\alpha^n - \frac{(-1)^n}{\alpha^n} \right), \tag{2.1}$$

called Binet's formula for the Pell numbers and from (1.2), it follows that

$$Q_n = \alpha^n + \frac{(-1)^n}{\alpha^n}, \tag{2.2}$$

called Binet's formula for the Pell-Lucas numbers.

There are the following relations between these two types of numbers and α.

(1) From (2.1) and (2.2), we obtain that the connection among the Pell numbers, the Pell-Lucas numbers, and α is for $n \in \mathbb{Z}_{\geqslant 0}$

$$\alpha^n = \frac{Q_n + 2\sqrt{2} P_n}{2}. \tag{2.3}$$

(2) For the connection between the Pell numbers and the Pell-Lucas numbers, straightforward computation yields from (2.1) and (2.2) for $n \in \mathbb{Z}_{\geqslant 0}$

$$\begin{aligned} Q_n^2 - 8P_n^2 &= \left(\alpha^n + \frac{(-1)^n}{\alpha^n}\right)^2 - 8\left(\frac{1}{2\sqrt{2}}\left(\alpha^n - \frac{(-1)^n}{\alpha^n}\right)\right)^2 \\ &= \left(\alpha^{2n} + \frac{(-1)^{2n}}{\alpha^{2n}} + 2\alpha^n \frac{(-1)^n}{\alpha^n}\right) \\ &\quad - 8\left(\frac{1}{8}\left(\alpha^{2n} + \frac{(-1)^{2n}}{\alpha^{2n}} - 2\alpha^n \frac{(-1)^n}{\alpha^n}\right)\right) \\ &= 4(-1)^n. \end{aligned} \tag{2.4}$$

(3) For $m, r \in \mathbb{Z}_{\geqslant 0}$, from (2.1) and (2.2) we establish the Pell index addition formulae by direct calculation

$$\begin{aligned} P_rQ_m + Q_rP_m &= \frac{1}{2\sqrt{2}}\left(\alpha^r - \frac{(-1)^r}{\alpha^r}\right)\left(\alpha^m + \frac{(-1)^m}{\alpha^m}\right) \\ &\quad + \frac{1}{2\sqrt{2}}\left(\alpha^r + \frac{(-1)^r}{\alpha^r}\right)\left(\alpha^m - \frac{(-1)^m}{\alpha^m}\right) \\ &= \frac{1}{2\sqrt{2}}\left(\alpha^{r+m} + \alpha^r\frac{(-1)^m}{\alpha^m} - \alpha^m\frac{(-1)^r}{\alpha^r} - \frac{(-1)^{r+m}}{\alpha^{r+m}}\right) \\ &\quad + \frac{1}{2\sqrt{2}}\left(\alpha^{r+m} - \alpha^r\frac{(-1)^m}{\alpha^m} + \alpha^m\frac{(-1)^r}{\alpha^r} - \frac{(-1)^{r+m}}{\alpha^{r+m}}\right) \\ &= 2\frac{1}{2\sqrt{2}}\left(\alpha^{r+m} - \frac{(-1)^{r+m}}{\alpha^{r+m}}\right) \\ &= 2P_{m+r}. \end{aligned} \tag{2.5}$$

(4) For $m, r \in \mathbb{Z}_{\geqslant 0}$, from (2.1) and (2.2) we establish the Pell-Lucas index addition formulae by direct calculation

$$\begin{aligned} Q_mQ_r + 8P_mP_r &= \left(\alpha^m + \frac{(-1)^m}{\alpha^m}\right)\left(\alpha^r + \frac{(-1)^r}{\alpha^r}\right) \\ &\quad + 8\left(\frac{1}{2\sqrt{2}}\right)^2\left(\alpha^m - \frac{(-1)^m}{\alpha^m}\right)\left(\alpha^r - \frac{(-1)^r}{\alpha^r}\right) \\ &= \alpha^{m+r} + (-1)^r\alpha^{m-r} + (-1)^m\alpha^{r-m} + (-1)^{m+r}\alpha^{-m-r} \\ &\quad + \alpha^{m+r} - (-1)^r\alpha^{m-r} - (-1)^m\alpha^{r-m} + (-1)^{m+r}\alpha^{-m-r} \\ &= 2\left(\alpha^{m+r} + \frac{(-1)^{m+r}}{\alpha^{m+r}}\right) \\ &= 2Q_{m+r}. \end{aligned} \tag{2.6}$$

3. Integral representations for the Pell Numbers P_{kn} and the Pell-Lucas Numbers Q_{kn}

Our purpose in this section is to present integral representations for the Pell numbers P_{kn} and for the Pell-Lucas numbers Q_{kn}, where $n \in \mathbb{Z}_{\geqslant 0} = \{0, 1, 2, \dots\}$ is a non-negative integer and $k \in \mathbb{Z}_{>0} = \{1, 2, 3, \ldots\}$ is an arbitrary but fixed positive integer.

The following theorem gives an integral representation of the Pell numbers.

THEOREM 3.1. *We have an integral representation of the Pell numbers P_{kn} by the integral*

$$P_{kn} = \frac{nP_k}{2^n} \int_{-1}^{1} \left(Q_k + 2\sqrt{2}P_k x\right)^{n-1} dx \tag{3.1}$$

for $n \in \mathbb{Z}_{\geqslant 0}$ and arbitrary but fixed $k \in \mathbb{Z}_{>0}$.

PROOF. Denote the integral to be found by I. We make the substitution

$$u = g(x) = Q_k + 2\sqrt{2}P_k x$$

because its differential is $du = 2\sqrt{2}P_k dx$, which, apart from the factor $2\sqrt{2}P_k$, occurs in the integral. Then, we obtain $dx = \frac{1}{2\sqrt{2}P_k} du$. Before substituting, determine the new upper and lower limits of integration. When $x = -1$, the new lower limit is $u = g(-1)$ and when $x = 1$, the new upper limit is $u = g(1)$. Now, we can substitute to obtain

$$\begin{aligned}
\frac{nP_k}{2^n} I &= \frac{nP_k}{2^n} \int_{-1}^{1} \left(Q_k + 2\sqrt{2}P_k x\right)^{n-1} dx \\
&= \frac{nP_k}{2^n} \frac{1}{2\sqrt{2}P_k} \int_{g(-1)}^{g(1)} u^{n-1} du \\
&= \frac{1}{2\sqrt{2}} \frac{n}{2^n} \frac{1}{n} \left[u^n\right]_{g(-1)}^{g(1)} \\
&= \frac{1}{2\sqrt{2}} \frac{1}{2^n} \left[\left(Q_k + 2\sqrt{2}P_k x\right)^n\right]_{-1}^{1} \\
&= \frac{1}{2\sqrt{2}} \left[\left(\frac{Q_k + 2\sqrt{2}P_k x}{2}\right)^n\right]_{-1}^{1} \\
&= \frac{1}{2\sqrt{2}} \left[\left(\frac{Q_k + 2\sqrt{2}P_k}{2}\right)^n - \left(\frac{Q_k - 2\sqrt{2}P_k}{2}\right)^n\right]. \qquad (3.2)
\end{aligned}$$

From (2.3) and (2.4), direct calculation gives

$$\begin{aligned}\frac{1}{\alpha^n} &= \frac{2}{Q_n+2\sqrt{2}P_n}\\ &= \frac{2\left(Q_n-2\sqrt{2}P_n\right)}{\left(Q_n+2\sqrt{2}P_n\right)\left(Q_n-2\sqrt{2}P_n\right)}\\ &= \frac{2\left(Q_n-2\sqrt{2}P_n\right)}{Q_n^2-8P_n^2}\\ &= \frac{2}{4\left(-1\right)^n}\left(Q_n-2\sqrt{2}P_n\right)\\ &= \frac{\left(-1\right)^n}{2}\left(Q_n-2\sqrt{2}P_n\right).\end{aligned}$$

Hence, we have

$$\frac{\left(-1\right)^n}{\alpha^n} = \frac{Q_n-2\sqrt{2}P_n}{2}. \tag{3.3}$$

From (3.2) and (3.3), it follows that

$$\begin{aligned}\frac{nP_k}{2^n}I &= \frac{1}{2\sqrt{2}}\left[\left(\alpha^k\right)^n-\left(\frac{\left(-1\right)^k}{\alpha^k}\right)^n\right]\\ &= \frac{1}{2\sqrt{2}}\left[\alpha^{kn}-\frac{\left(-1\right)^{kn}}{\alpha^{kn}}\right]\\ &= P_{kn}.\end{aligned}$$

Thus, the proof of Theorem 3.1 is completed. □

Corollary 3.1. *We have an integral representation of the Pell numbers P_n by the integral*

$$P_n = \frac{n}{2}\int_{-1}^{1}\left(1+x\sqrt{2}\right)^{n-1}dx$$

for $n \in \mathbb{Z}_{\geqslant 0}$.

PROOF. If we write $k = 1$ in (3.1), then we obtain the integral representations of Pell numbers P_n as follows:

$$P_n = \frac{nP_1}{2^n} \int_{-1}^{1} \left(Q_1 + x2\sqrt{2}P_1\right)^{n-1} dx$$
$$= \frac{n}{2^n} \int_{-1}^{1} \left(2 + x2\sqrt{2}\right)^{n-1} dx$$
$$= \frac{n}{2} \int_{-1}^{1} \left(1 + x\sqrt{2}\right)^{n-1} dx.$$

Thus, the proof of Corollary 3.1 is completed. □

The following corollary gives an integral representation of the Pell numbers with even integer index.

COROLLARY 3.2. *We have an integral representation of the Pell numbers* P_{2n} *by the integral*

$$P_{2n} = \frac{n}{2^{n-1}} \int_{-1}^{1} \left(6 + x4\sqrt{2}\right)^{n-1} dx. \tag{3.4}$$

for $n \in \mathbb{Z}_{\geqslant 0}$.

PROOF. If we set $k = 2$ in (3.1), then we get an integral representation of the Pell numbers with even integer index by

$$P_{2n} = \frac{nP_2}{2^n} \int_{-1}^{1} \left(Q_2 + x2\sqrt{2}P_2\right)^{n-1} dx$$
$$= \frac{n}{2^{n-1}} \int_{-1}^{1} \left(6 + x4\sqrt{2}\right)^{n-1} dx.$$

The proof of Corollary 3.2 is complete. □

The following corollary gives an integral representation of the Pell numbers with odd integer index..

COROLLARY 3.3. *We have an integral representation of the Pell numbers* P_{2n+1} *by the integral*

$$P_{2n+1} = \frac{1}{2^{n+1}} \int_{-1}^{1} \left(4n + 6 + (n+1)\, x4\sqrt{2}\right) \left(6 + x4\sqrt{2}\right)^{n-1} dx \tag{3.5}$$

for $n \in \mathbb{Z}_{\geqslant 0}$.

PROOF. We first recall the obvious identity $P_{2n+2} = 2P_{2n+1} + P_{2n}$. Then, from this identity, it follows that

$$P_{2n+1} = \frac{1}{2}\left(P_{2n+2} - P_{2n}\right). \tag{3.6}$$

Using a reindexing of $n \mapsto n+1$ in (3.4), from (3.4) and (3.6) straightforward computation yields

$$\begin{aligned} P_{2n+1} &= \frac{1}{2}\left(P_{2n+2} - P_{2n}\right) \\ &= \frac{1}{2}\left(\frac{n+1}{2^n}\int\limits_{-1}^{1}\left(6 + x4\sqrt{2}\right)^n dx - \frac{n}{2^{n-1}}\int\limits_{-1}^{1}\left(6 + x4\sqrt{2}\right)^{n-1} dx\right) \\ &= \frac{1}{2^{n+1}}\int\limits_{-1}^{1}\left(4n + 6 + (n+1)\,x4\sqrt{2}\right)\left(6 + x4\sqrt{2}\right)^{n-1} dx. \end{aligned}$$

The proof of Corollary 3.3 is complete. □

The following corollary gives a thinly disguised form of Binet's formula for P_{kn}.

COROLLARY 3.4. *The Pell numbers P_{kn} can be represented by*

$$P_{kn} = \frac{n}{2\sqrt{2}}\int\limits_{\frac{1}{(-\alpha)^k}}^{\alpha^k} t^{n-1} dt$$

for $n \in \mathbb{Z}_{\geqslant 0}$ and arbitrary but fixed $k \in \mathbb{Z}_{>0}$.

PROOF. Using the substitution $t = \frac{1}{2}\left(Q_k + 2\sqrt{2}P_k x\right)$ in (3.1), we have $dt = \frac{2\sqrt{2}P_k}{2}dx$ and $dx = \frac{2}{2\sqrt{2}P_k}dt$. To find the new limits of integration (3.1) we note that when $x = -1$,

$$t = \frac{1}{2}\left(Q_k - 2\sqrt{2}P_k\right) = \frac{1}{(-\alpha)^k}$$

and when $x = 1$,

$$t = \frac{1}{2}\left(Q_k + 2\sqrt{2}P_k\right) = \alpha^k.$$

Therefore, from (3.1) we obtain

$$\begin{aligned}
P_{kn} &= \frac{nP_k}{2^n}\int_{-1}^{1}\left(Q_k+2\sqrt{2}P_kx\right)^{n-1}dx \\
&= \frac{nP_k}{2^n}\int_{\frac{1}{(-\alpha)^k}}^{\alpha^k}(2t)^{n-1}\frac{2}{2\sqrt{2}P_k}dt \\
&= \frac{nP_k}{2^n}2^{n-1}\frac{2}{2\sqrt{2}P_k}\int_{\frac{1}{(-\alpha)^k}}^{\alpha^k}(t)^{n-1}dt \\
&= \frac{n}{2\sqrt{2}}\int_{\frac{1}{(-\alpha)^k}}^{\alpha^k}t^{n-1}dt.
\end{aligned}$$

The proof of Corollary 3.4 is complete. □

The following theorem gives an integral representation of the Pell-Lucas numbers.

THEOREM 3.2. *We have an integral representation of the Pell-Lucas numbers Q_{kn} by the integral*

$$Q_{kn} = \frac{1}{2^n}\int_{-1}^{1}\left(Q_k+P_k(n+1)x2\sqrt{2}\right)\left(Q_k+P_kx2\sqrt{2}\right)^{n-1}dx \tag{3.7}$$

for $n \in \mathbb{Z}_{\geqslant 0}$ and arbitrary but fixed $k \in \mathbb{Z}_{>0}$.

PROOF. Let

$$J = \int\left(Q_k+P_k(n+1)x2\sqrt{2}\right)\left(Q_k+P_kx2\sqrt{2}\right)^{n-1}dx.$$

To evaluate this integral we use the integration by parts. Let

$$u = Q_k+P_k(n+1)x2\sqrt{2}$$

and

$$dv = \left(Q_k+P_kx2\sqrt{2}\right)^{n-1}dx.$$

Then,

$$du = P_k(n+1)2\sqrt{2}dx$$

and

$$v = \int\left(Q_k+P_kx2\sqrt{2}\right)^{n-1}dx.$$

To evaluate the integral that we have obtained, $v=\int\left(Q_k+P_kx2\sqrt{2}\right)^{n-1}dx$, if we let $t=Q_k+P_kx2\sqrt{2}$, then $dt=2\sqrt{2}P_kdx$, so $dx=\frac{1}{2\sqrt{2}P_k}dt$. Therefore,

$$\begin{aligned} v &= \int \frac{1}{2\sqrt{2}P_k}t^{n-1}dt \\ &= \frac{1}{n2\sqrt{2}P_k}t^n \\ &= \frac{1}{n2\sqrt{2}P_k}\left(Q_k+P_kx2\sqrt{2}\right)^n. \end{aligned}$$

If we let

$$I=\frac{1}{2^n}\int_{-1}^{1}\left(Q_k+P_k(n+1)x2\sqrt{2}\right)\left(Q_k+P_kx2\sqrt{2}\right)^{n-1}dx,$$

then we obtain

$$\begin{aligned} I &= \frac{1}{2^n}\left\{[uv]_{-1}^{1}-\int_{-1}^{1}vdu\right\} \\ &= \frac{1}{2^n}\left\{\frac{1}{n2\sqrt{2}P_k}\left[\left(Q_k+P_k(n+1)x2\sqrt{2}\right)\left(Q_k+P_kx2\sqrt{2}\right)^n\right]_{-1}^{1}\right. \\ &\quad \left.-P_k(n+1)2\sqrt{2}\frac{1}{n2\sqrt{2}P_k}\int_{-1}^{1}\left(Q_k+P_kx2\sqrt{2}\right)^n dx\right\} \\ &= \frac{1}{nP_k2\sqrt{2}}\left(\frac{Q_k+P_k2\sqrt{2}}{2}\right)^n\left(Q_k+P_k(n+1)2\sqrt{2}\right) \\ &\quad -\frac{1}{nP_k2\sqrt{2}}\left(\frac{Q_k-P_k2\sqrt{2}}{2}\right)^n\left(Q_k-P_k(n+1)2\sqrt{2}\right) \\ &\quad -\frac{n+1}{n2^n}\int_{-1}^{1}\left(Q_k+P_kx2\sqrt{2}\right)^n dx. \end{aligned} \tag{3.8}$$

From (3.1), we have that

$$P_{k(n+1)}\frac{2^{n+1}}{(n+1)\,P_k}=\int_{-1}^{1}\left(Q_k+2\sqrt{2}P_kx\right)^n dx. \tag{3.9}$$

Hence, from (3.8) and (3.9) we get

$$\begin{aligned} I &= \frac{1}{nP_k 2\sqrt{2}} \left(\frac{Q_k + P_k 2\sqrt{2}}{2} \right)^n \left(Q_k + P_k (n+1) 2\sqrt{2} \right) \\ &\quad - \frac{1}{nP_k 2\sqrt{2}} \left(\frac{Q_k - P_k 2\sqrt{2}}{2} \right)^n \left(Q_k - P_k (n+1) 2\sqrt{2} \right) \\ &\quad - \frac{n+1}{n2^n} \frac{2^{n+1}}{(n+1)\, P_k} P_{kn+k}. \end{aligned}$$

By (2.3), (2.5) and (3.3), it follows that

$$\begin{aligned} I &= \frac{1}{nP_k 2\sqrt{2}} \alpha^{kn} \left(Q_k + P_k (n+1) 2\sqrt{2} \right) \\ &\quad - \frac{1}{nP_k 2\sqrt{2}} \frac{(-1)^{kn}}{\alpha^{kn}} \left(Q_k - P_k (n+1) 2\sqrt{2} \right) \\ &\quad - \frac{2}{nP_k} P_{kn+k} \\ &= \frac{1}{nP_k} \left[\frac{1}{2\sqrt{2}} \left(\alpha^{kn} - \frac{(-1)^{kn}}{\alpha^{kn}} \right) Q_k \right. \\ &\quad + \left(\alpha^{kn} + \frac{(-1)^{kn}}{\alpha^{kn}} \right) (n+1) P_k \\ &\quad \left. -2P_{kn+k} \right] \\ &= \frac{1}{nP_k} \left[Q_k P_{kn} + (n+1) P_k Q_{kn} - 2P_{kn+k} \right] \\ &= \frac{1}{nP_k} \left[nP_k Q_{kn} + Q_k P_{kn} + P_k Q_{kn} - 2P_{kn+k} \right]. \end{aligned} \tag{3.10}$$

Here, if we remember the formula given in (2.5) and substitute k for r and kn for m in this formula, we see that the following equation is satisfied:

$$Q_k P_{kn} + P_k Q_{kn} - 2P_{kn+k} = 0.$$

Hence, (3.10) gives us the result $I = Q_{kn}$. Consequently, the proof of the theorem by obtaining the result given in (3.7) is completed. □

Corollary 3.5. *We have an integral representation of the Pell-Lucas numbers* Q_n *by the integral*

$$Q_n = \int_{-1}^{1} \left(1 + (n+1) x\sqrt{2} \right) \left(1 + x\sqrt{2} \right)^{n-1} dx$$

for $n \in \mathbb{Z}_{\geqslant 0}$.

Proof. As a result of writing $k = 1$ at (3.7), we obtain integral representations for Pell-Lucas numbers Q_n in the following way:

$$\begin{aligned} Q_n &= \frac{1}{2^n}\int_{-1}^{1}\left(Q_1 + P_1(n+1)x2\sqrt{2}\right)\left(Q_1 + P_1x2\sqrt{2}\right)^{n-1}dx \\ &= \frac{1}{2^n}\int_{-1}^{1}\left(2 + 2(n+1)x\sqrt{2}\right)\left(2 + 2x\sqrt{2}\right)^{n-1}dx \\ &= \int_{-1}^{1}\left(1 + (n+1)x\sqrt{2}\right)\left(1 + x\sqrt{2}\right)^{n-1}dx \end{aligned}$$

The proof of Corollary 3.5 is completed. □

The following corollary gives an integral representation of the Pell-Lucas numbers with even integer index.

Corollary 3.6. *We have an integral representation of the Pell-Lucas numbers* Q_{2n} *by the integral*

$$Q_{2n} = \frac{1}{2^n}\int_{-1}^{1}\left(6 + 4(n+1)x\sqrt{2}\right)\left(6 + 4x\sqrt{2}\right)^{n-1}dx \tag{3.11}$$

for $n \in \mathbb{Z}_{\geqslant 0}$.

Proof. If we set $k = 2$ in (3.7), then we get an integral representation of the even Pell-Lucas numbers by

$$\begin{aligned} Q_{2n} &= \frac{1}{2^n}\int_{-1}^{1}\left(Q_2 + P_2(n+1)x2\sqrt{2}\right)\left(Q_2 + P_2x2\sqrt{2}\right)^{n-1}dx \\ &= \frac{1}{2^n}\int_{-1}^{1}\left(6 + 4(n+1)x\sqrt{2}\right)\left(6 + 4x\sqrt{2}\right)^{n-1}dx. \end{aligned}$$

The proof of Corollary 3.6 is complete. □

The following corollary gives an integral representation of the Pell-Luas numbers with even integer index.

Corollary 3.7. *We have an integral representation of the Pell-Lucas numbers* Q_{2n+1} *by the integral*

$$Q_{2n+1} = \frac{1}{2^n}\int_{-1}^{1}\left(6 + 8n + 4(n+1)x\sqrt{2}\right)\left(6 + 4x\sqrt{2}\right)^{n-1}dx \tag{3.12}$$

for $n \in \mathbb{Z}_{\geqslant 0}$.

PROOF. Recalling that the identity in (80) takes the form of

$$2Q_{m+r} = Q_m Q_r + 8P_m P_r,$$

by substituting $2n$ for m and 1 for r, we get the following identity

$$\begin{aligned} 2Q_{2n+1} &= Q_{2n}Q_1 + 8P_{2n}P_1 \\ &= 2Q_{2n} + 8P_{2n}. \end{aligned} \tag{3.13}$$

Substituting the integral representations obtained for Q_{2n} and P_{2n} in 3.13 results in the following integral representation for Q_{2n+1}:

$$\begin{aligned} Q_{2n+1} &= Q_{2n} + 4P_{2n} \\ &= \frac{1}{2^n}\int_{-1}^{1}\left(6+4(n+1)x\sqrt{2}\right)\left(6+4x\sqrt{2}\right)^{n-1}dx \\ &\quad + 4\frac{n}{2^{n-1}}\int_{-1}^{1}\left(6+x4\sqrt{2}\right)^{n-1}dx \\ &= \frac{1}{2^n}\int_{-1}^{1}\left(6+8n+4(n+1)x\sqrt{2}\right)\left(6+4x\sqrt{2}\right)^{n-1}dx. \end{aligned}$$

Thus, the proof of Corollary 3.7 is completed. □

The following corollary gives a thinly disguised form of Binet's formula for Q_{kn}.

COROLLARY 3.8. *The Pell-Lucas numbers Q_{kn} can be represented by*

$$Q_{kn} = n\int_{\frac{1}{(-\alpha)^k}}^{\alpha^k} t^{n-1}dt + \frac{2}{(-\alpha)^k}.$$

for $n \in \mathbb{Z}_{\geqslant 0}$ and arbitrary but fixed $k \in \mathbb{Z}_{>0}$.

PROOF. The proof of Corollary 3.8 can be done similarly to the proof of Corollary 3.4. □

4. Integral representations for the Pell Numbers P_{kn+r} and the Pell-Lucas Numbers Q_{kn+r}

This section presents the integral representations of Pell numbers P_{kn+r} and Pell-Lucas numbers Q_{kn+r}, derived from the integral representations of Pell numbers P_{kn} and Pell-Lucas numbers Q_{kn}, where $n \in \mathbb{Z}_{\geqslant 0} = \{0, 1, 2, \ldots\}$ is a non-negative integer, $k \in \mathbb{Z}_{>0} = \{1, 2, 3, \ldots\}$ is an arbitrary but fixed positive integer, while $r \in \mathbb{Z}_{\geqslant 0}$ is an arbitrary but fixed non-negative integer.

THEOREM 4.1. *For $n \in \mathbb{Z}_{\geqslant 0}$ and arbitrary but fixed $k \in \mathbb{Z}_{>0}$ and $r \in \mathbb{Z}_{\geqslant 0}$, the Pell numbers P_{kn+r} can be represented by the integral*

$$P_{kn+r} = \frac{1}{2^{n+1}} \int_{-1}^{1} \left[\left(nP_k Q_r + P_r Q_k + P_k P_r (n+1) x 2\sqrt{2} \right) \times \left(Q_k + 2\sqrt{2} P_k x \right)^{n-1} \right] dx. \tag{4.1}$$

PROOF. The Pell index addition formula

$$P_r Q_m + Q_r P_m = 2P_{m+r}$$

given by (2.5) with m replaced with kn produces

$$2P_{kn+r} = P_{kn} Q_r + P_r Q_{kn}.$$

When the integral representations of P_{kn} and Q_{kn} given by (3.1) and (3.7) respectively are substituted into the given index addition formula, the result follows immediately:

$$\begin{aligned} 2P_{kn+r} &= Q_r P_{kn} + P_r Q_{kn} \\ &= Q_r \frac{nP_k}{2^n} \int_{-1}^{1} \left(Q_k + 2\sqrt{2} P_k x \right)^{n-1} dx \\ &\quad + P_r \frac{1}{2^n} \int_{-1}^{1} \left(Q_k + P_k (n+1) x 2\sqrt{2} \right) \left(Q_k + P_k x 2\sqrt{2} \right)^{n-1} dx \\ &= \frac{1}{2^n} \int_{-1}^{1} \left(nP_k Q_r + P_r Q_k + P_k P_r (n+1) x 2\sqrt{2} \right) \left(Q_k + 2\sqrt{2} P_k x \right)^{n-1} dx, \end{aligned}$$

and completes the proof. □

REMARK 4.1. It is possible to obtain integral representations for P_n, P_{2n}, and P_{2n+1} by substituting $(1,0)$, $(2,0)$, and $(2,1)$ for (k,r) in the integral representation at (4.1) given by Theorem 4.1.

THEOREM 4.2. *For $n \in \mathbb{Z}_{\geqslant 0}$ and arbitrary but fixed $k \in \mathbb{Z}_{>0}$ and $r \in \mathbb{Z}_{\geqslant 0}$, the Pell-Lucas numbers Q_{kn+r} can be represented by the integral*

$$Q_{kn+r} = \frac{1}{2^{n+1}} \int_{-1}^{1} \left[\left(8nP_k P_r + Q_k Q_r + P_k Q_r (n+1) x 2\sqrt{2} \right) \times \left(Q_k + P_k x 2\sqrt{2} \right)^{n-1} \right] dx. \tag{4.2}$$

PROOF. The Pell-Lucas index addition formula

$$Q_m Q_r + 8P_m P_r = 2Q_{m+r}$$

given by (1.6) with m replaced with kn produces

$$Q_{kn}Q_r + 8P_{kn}P_r = 2Q_{kn+r}.$$

When the integral representations of P_{kn} and Q_{kn} given by (3.1) and (3.7) respectively are substituted into the given index addition formula, the result follows immediately:

$$\begin{aligned}
2Q_{kn+r} &= 8P_rP_{kn} + Q_rQ_{kn} \\
&= 8P_r\frac{nP_k}{2^n}\int_{-1}^{1}\left(Q_k + 2\sqrt{2}P_kx\right)^{n-1}dx \\
&\quad + Q_r\frac{1}{2^n}\int_{-1}^{1}\left(Q_k + P_k(n+1)x2\sqrt{2}\right)\left(Q_k + P_kx2\sqrt{2}\right)^{n-1}dx \\
&= \frac{1}{2^n}\int_{-1}^{1}\left(8nP_kP_r + Q_kQ_r + P_kQ_r(n+1)x2\sqrt{2}\right)\left(Q_k + P_kx2\sqrt{2}\right)^{n-1}dx,
\end{aligned}$$

and completes the proof. □

Remark 4.2. Q_n, Q_{2n} and Q_{2n+1}, respectively, have integral representations when we substitute $(1,0)$, $(2,0)$, and $(2,1)$ for (k,r) at (4.2) given by Theorem 4.2.

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