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S-F-PRIME FILTER PROPERTY IN LATTICES

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Abstract. Let F be a fixed filter of a bounded distributive lattice $\mathcal{L}$ and S a join subset of $\mathcal{L}$. In this paper, we introduce the concept of S-F-prime filters as a new generalization of S-prime filters. We say that a proper filter P of $\mathcal{L}$ disjoint with S is an S-F-prime filter if there is an element $s \in S$ such that for all $a, b \in \mathcal{L}$ if $a \vee b \in P - (P \vee F)$, then $s \vee a \in P$ or $s \vee b \in P$. We extend the notion of S-F-prime property in commutative rings to S-F-prime property in lattices.

1. Introduction

All lattices considered in this paper are assumed to have a least element denoted by 0 and a greatest element denoted by 1, in other words they are bounded. As algebraic structures, lattices are undoubtedly a natural choice of generalizations of rings. In structure, lattices lie between semigroups and rings. The main aim of this article is that of extending some results obtained for ring theory to lattice theory. The main difficulty is figuring out what additional hypothesis the lattice or filter must satisfy to get similar results. Nevertheless, growing interest in developing the algebraic theory of lattices can be found in several papers and books (see for instance [7, 9, 11, 12, 13, 14, 15, 16]).

The notion of prime ideal plays a key role in the theory of commutative algebra, and it has been widely studied. See, for example, [1]. Recall from [1], a prime ideal P of R is a proper ideal having the property that $ab \in P$ implies either $a \in P$ or $b \in P$ for each $a, b \in R$. There are several ways to generalize the notion of a prime ideal. In 2003, Anderson and Smith in [3] defined weakly prime ideals which is a generalization of prime ideals (also see [10]). A proper ideal P of a ring R is said to be a weakly prime if $0 \neq xy \in P$ for each $x, y \in R$ implies either $x \in P$ or $y \in P$.

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Thus every prime ideal is weakly prime. In 2019, Hamed and Malek [17] introduced the notion of an S-prime ideal (also see [16, 18]), i.e. let $S \subseteq R$ be a multiplicative set and I an ideal of R disjoint from S. We say that I is S-prime if there exists an $s \in S$ such that for all $a, b \in R$ with $ab \in I$, we have $sa \in I$ or $sb \in I$. Almahdi et. al. [4] introduced the notion of a weakly S-prime ideal as follows: We say that I is a weakly S-prime ideal of R if there is an element $s \in S$ such that for all $x, y \in R$ if $0 \neq xy \in I$, then $sx \in I$ or $sy \in I$. Akray and Hussein generalized the concept of I-prime submodules in [6] (also see [5]). Let R be a commutative ring and I be a fixed ideal of R. Then a proper submodule P of an R-module M is called I-prime submodule of M if $rm \in P - IP$ for all $r \in R$ and $m \in M$ implies that either $m \in P$ or $r \in (P :_R M)$, and a proper ideal P of R is I-prime if for $a, b \in R$ with $ab \in P - IP$ implies either $a \in P$ or $b \in P$. So every weakly prime is I-prime.

Let F be a fixed filter of a bounded distributive lattice $\mathcal{L}$. Among many other results in this paper, the first, preliminaries section contains elementary observations needed later on. Section 3 is dedicated to the investigation of the some basic properties of F-prime filters. Following the concept of I-prime ideals, we define F-prime filters of $\mathcal{L}$. A proper filter P of $\mathcal{L}$ is F-prime if for $a, b \in \mathcal{L}$ with $a \vee b \in P - (F \vee P)$ implies either $a \in P$ or $b \in P$. In this section, we are interested in investigating F-prime filters to use other notions of I-prime ideals, and associate which exist in the literature as laid forth in [5, 6]. At first, we define the definition of F-prime filters (Definition 3.1) and we give an example (Example 3.2 (3)) of a F-prime filter of $\mathcal{L}$ that is not a weakly prime filter (so it is not prime). It is proved (Proposition 3.1)) that If $\mathcal{L}$ is a local lattice with unique maximal filter M, then every proper filter of $\mathcal{L}$ is a M-prime filter. It is shown that (Theorem 3.1) that if P is a F-prime filter of $\mathcal{L}$ that is not prime, then $P \subseteq F$. In the Corollary 3.2, we give a condition under which a F-prime filter of $\mathcal{L}$ is a prime filter. In the Theorem 3.2, We give three other characterizations of F-prime filters. In the rest of this section, we investigate the properties of F-prime filters similar to prime filters. In particular, we investigate the behavior of F-prime filters under homomorphism, in factor lattices and in cartesian products of lattices (see Theorem 3.3, Theorem 3.4, Theorem 3.5, Theorem 3.6 and Theorem 3.7).

We say that a subset $S \subseteq \mathcal{L}$ is *join subset* if $0 \in S$ and $s_1 \vee s_2 \in S$ for all $s_1, s_2 \in S$ (if P is a prime filter of $\mathcal{L}$, then $\mathcal{L} \setminus P$ is a join subset of $\mathcal{L}$). In Section 4, we give the definition of S-F-prime filter (Definition 4.1) and provide an example (Example 4.1 (6)) of an S-F-prime filter of $\mathcal{L}$ that is not an S-prime filter. In the Theorem 4.1, We give a characterization of S-F-prime filters. We provide some conditions under which an intersection of a family of S-F-prime filters of $\mathcal{L}$ is an S-F-prime filter (see Theorem 4.2). The rest of this section, we investigate the behavior of S-F-prime filters under homomorphism, in factor lattices, S-Noetherian lattices and in cartesian products of lattices (see Theorem 4.4, Theorem 4.5, Theorem 4.6, Theorem 4.7, Theorem 4.8, Theorem 4.9 and Theorem 4.10).

2. Preliminaries

A *poset* $(\mathcal{L}, \leqslant)$ is a *lattice* if $\sup\{a, b\} = a \vee b$ and $\inf\{a, b\} = a \wedge b$ exist for all $a, b \in \mathcal{L}$ (and call $\wedge$ the *meet* and $\vee$ the *join*). A lattice $\mathcal{L}$ is called a *distributive lattice* if $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all a, b, c in $\mathcal{L}$ (equivalently, $\mathcal{L}$ is distributive if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all a, b, c in $\mathcal{L}$). A non-empty subset F of a lattice $\mathcal{L}$ is called a *filter*, if for $a \in F$, $b \in \mathcal{L}$, $a \leqslant b$ implies $b \in F$, and $x \wedge y \in F$ for all $x, y \in F$ (so if $\mathcal{L}$ is a lattice with 0 and 1, then $1 \in F$, $\{1\}$ is a filter of $\pounds$ and $0 \in F$ if and only if $F = \mathcal{L}$). A proper filter P of $\mathcal{L}$ is called *prime* (resp. *weakly prime*) if $x \vee y \in P$ (resp. $1 \neq x \vee y \in P$), then $x \in P$ or $y \in P$. A proper filter F of $\mathcal{L}$ is said to be *maximal* if G is a filter in $\mathcal{L}$ with $F \subsetneqq G$, then $G = \mathcal{L}$. A lattice $\mathcal{L}$ is called *local* if it has exactly one maximal filter that contains all proper filters. Assume that P is a filter of $\mathcal{L}$ and let S be a join subset of $\mathcal{L}$ disjoint with S. We say that P is an *S-prime filter* of $\mathcal{L}$ if there is an element $s \in S$ such that for all $x, y \in \mathcal{L}$ if $x \vee y \in \mathcal{L}$, then $s \vee x \in$ or $s \vee y \in P$.

Let D be subset of a lattice $\mathcal{L}$. Then the filter generated by D, denoted by $T(D)$, is the intersection of all filters that is containing D. A filter F is called finitely generated if there is a finite subset D of F such that $F = T(D)$. If $x \in \mathcal{L}$, then a *complement* of x in $\mathcal{L}$ is an element $y \in \mathcal{L}$ such that $x \vee y = 1$ and $x \wedge y = 0$. The lattice $\mathcal{L}$ is *complemented* if every element of $\mathcal{L}$ has a complement in $\mathcal{L}$. If $\mathcal{L}$ and $\mathcal{L}'$ are lattices, then a *lattice homomorphism* $f : \mathcal{L} \to \mathcal{L}'$ is a map from $\mathcal{L}$ to $\mathcal{L}'$ satisfying $f(x \vee y) = f(x) \vee f(y)$ and $f(x \wedge y) = f(x) \wedge f(y)$ for $x, y \in \mathcal{L}$. For undefined notations or terminologies in lattice theory, we refer the reader to [7, 9]. First we need the following easy observations proved in [11, 12, 13, 14].

LEMMA 2.1. *Let $\mathcal{L}$ be a lattice.*

(1) A non-empty subset F of $\mathcal{L}$ is a filter of $\mathcal{L}$ if and only if $x \vee z \in F$ and $x \wedge y \in F$ for all $x, y \in F$, $z \in \mathcal{L}$. Moreover, since $x = x \vee (x \wedge y)$, $y = y \vee (x \wedge y)$ and F is a filter, $x \wedge y \in F$ gives $x, y \in F$ for all $x, y \in \mathcal{L}$.

(2) Let A be an arbitrary non-empty subset of $\mathcal{L}$. Then

$$T(A) = \{x \in \mathcal{L} : \ a_1 \wedge a_2 \wedge \cdots \wedge a_n \leqslant x \text{ for some } a_i \in A \ (1 \leqslant i \leqslant n)\}.$$

LEMMA 2.2. *Let F, G be filters of $\mathcal{L}$ and $z \in \mathcal{L}$. Then the following hold:*

(1) $F \vee G = \{a \vee b : a \in F, b \in G\}$ and $z \vee F = \{z \vee y : y \in F\}$ are filters of $\mathcal{L}$ with $F \vee G = F \cap G$.

(2) If $\mathcal{L}$ is distributive, then $F \wedge G = \{a \wedge b : a \in F, b \in G\}$ is a filter of $\mathcal{L}$ with $F, G \subseteq F \wedge G$

(3) If $\mathcal{L}$ is distributive, then $(G :_{\mathcal{L}} F) = \{x \in \mathcal{L} : x \vee F \subseteq G\}$ and $(F :_{\mathcal{L}} T(\{z\})) = (F :_{\mathcal{L}} z) = \{a \in \mathcal{L} : a \vee z \in F\}$ are filters of $\mathcal{L}$.

(4) If $v : \mathcal{L} \to \mathcal{L}'$ is a lattice homomorphism such that $v(1) = 1$, then $\mathrm{Ker}(v) = \{x \in \mathcal{L} : v(x) = 1\}$ *is a filter of $\mathcal{L}$.*

3. Characterization of F-prime filters

In this section, we collect some basic properties concerning F-prime filters. We remind the reader with the following definition.

Definition 3.1. Let F be a fixed filter of a lattice $\mathcal{L}$. A proper filter P of $\mathcal{L}$ is F-prime if for $a, b \in \mathcal{L}$ with $a \vee b \in P - (F \vee P)$ implies either $a \in P$ or $b \in P$.

Example 3.1. (1) If P is a proper filter of $\mathcal{L}$, then P is always P-prime since $P - (P \vee P) = P - P = \emptyset$.

(2) Let P and Q be filters of $\mathcal{L}$ with $P \subseteq Q$. If P is proper, then P is always Q-prime since $P - (P \vee Q) = P - P = \emptyset$. In particular, P is always $\mathcal{L}$-prime.

Example 3.2. (1) If $F = \{1\}$, then the F-prime and the weakly prime filters of $\mathcal{L}$ are the same.

(2) Suppose that P is a weakly prime (i.e. $\{1\}$-prime) filter of $\mathcal{L}$ and let F be a filter of $\mathcal{L}$. Let $a, b \in \mathcal{L}$ such that $a \vee b \in P - (P \vee F)$; so $a \vee b \neq 1$ since $1 \in P \vee F$. It follows that $a \in P$ or $b \in P$. Thus, every weakly prime filter (prime filter) is F-prime.

(3) Let $\mathcal{L} = \{0, a, b, c, 1\}$ be a lattice with the relations $0 \leqslant a \leqslant c \leqslant 1$, $0 \leqslant b \leqslant c \leqslant 1$, $a \vee b = c$ and $a \wedge b = 0$. An inspection will show that the nontrivial filters (i.e. different from $\mathcal{L}$ and $\{1\}$) of $\mathcal{L}$ are $F_1 = \{1, c\}$, $F_2 = \{1, c, a\}$ and $F_3 = \{1, c, b\}$. Then F_1 is an F_2-prime filter of $\mathcal{L}$ by Example 3.1 (2). Also, F_1 is not a weakly prime (prime) filter of $\mathcal{L}$ because $1 \neq a \vee b = c \in F_1$, $a \notin F_1$ and $b \notin F_1$. Thus an F-prime filter need not be a weakly prime filter (prime filter).

Example 3.3. Let F and G be filters of $\mathcal{L}$ such that $F \subseteq G$. If P is an F-prime filter of $\mathcal{L}$, then P is an G-prime filter (since $F \subseteq G$ gives $P - (P \vee G) \subseteq P - (P \vee F)$). However, the converse is not true in general. Indeed, assume that $\mathcal{L}$ is the lattice as in Example 3.2 (3) and let $F = \{1\} \subseteq G = F_2$. Then $P = F_1$ is an G-prime filter of $\mathcal{L}$ but not an F-prime filter of $\mathcal{L}$.

Proposition 3.1. *If $\mathcal{L}$ is a local lattice with unique maximal filter M, then every proper filter of $\mathcal{L}$ is an M-prime filter.*

Proof. Let P be a proper filter of $\mathcal{L}$. Then $P \subseteq M$ by [14, Lemma 2.1]. Now the assertion follows from Example 3.1 (2). □

An element $a \in \mathcal{L}$ is called *irreducible* if $a = x \vee y$, then either $x \in T(\{a\})$ or $y \in T(\{a\})$. Compare the next example with Theorem 2.11 (2) and Theorem 2.11 (3) in [5].

Example 3.4. The collection of ideals of $\mathbb{Z}$, the ring of integers, form a lattice under set inclusion which we shall denote by $\mathcal{L}$ with respect to the following definitions: $m\mathbb{Z} \vee n\mathbb{Z} = (m, n)\mathbb{Z}$ and $m\mathbb{Z} \wedge n\mathbb{Z} = [m, n]\mathbb{Z}$ for all ideals $m\mathbb{Z}$ and $n\mathbb{Z}$ of $\mathbb{Z}$, where (m, n) and $[m, n]$ are greatest common divisor and least common multiple of m, n, respectively. Note that $\mathcal{L}$ is a distributive complete lattice with least element the zero ideal and the greatest element $\mathbb{Z}$. By [11, Theorem 2.9 (ii)], $M = \mathcal{L} \setminus \{0\}$ is the only maximal filter of $\mathcal{L}$ and so $\mathcal{L}$ is a local lattice. It follows from Proposition 3.1 that every proper filter of $\mathcal{L}$ is a M-prime filter. Consider the M-prime filter $P = T(\{3\mathbb{Z}\}) = \{\mathbb{Z}, 3\mathbb{Z}\}$.

(1) Since $1 \neq 6\mathbb{Z} \vee 9\mathbb{Z} = 3\mathbb{Z} \in P$ with $6\mathbb{Z}, 9\mathbb{Z} \notin P$, we conclude that P is not a weakly prime ($\{1\}$-prime) filter. Moreover, $P - P \vee F = P - P = \emptyset$ gives P is F-prime for every filter $P \subseteq F$ of $\mathcal{L}$ but it is not $\{1\}$-prime.

(2) Since $6\mathbb{Z} \vee 9\mathbb{Z} = 3\mathbb{Z}$ with $6\mathbb{Z}, 9\mathbb{Z} \notin P$, we infer that $3\mathbb{Z}$ is not an irreducible element. Thus an M-prime filter need not be an irreducible element.

Henceforth we will assume that F is a fixed filter of $\mathcal{L}$.

THEOREM 3.1. *Let P be an F-prime filter of $\mathcal{L}$. If P is not prime, then $P \subseteq F$.*

PROOF. Suppose that $P \nsubseteq F$; we show that P is prime. Let $x, y \in \mathcal{L}$ such that $x \vee y \in P$. If $x \vee y \notin P \vee F$, then P is an F-prime gives $x \in P$ or $y \in P$. So we may assume that $x \vee y \in P \vee F$. By the hypothesis, there exists $z \in P$ such that $z \notin F$ which implies that $z \wedge (x \vee y) \in P - (P \vee F)$ by Lemma 2.1 (1). Then $(x \wedge z) \vee (y \wedge z) = z \wedge (x \vee y) \in P - (P \vee F)$ implies that $x \wedge z \in P$ or $y \wedge z \in P$; hence $x \in P$ or $y \in P$ by Lemma 2.1 (1), i.e. P is prime. □

COROLLARY 3.1. *Let P be an $\{1\}$-prime filter of $\mathcal{L}$. If P is not a prime filter, then $P = \{1\}$.*

PROOF. This is a direct consequence of Theorem 3.1. □

COROLLARY 3.2. *Let P be an F-prime filter of $\mathcal{L}$. If $P \nsubseteq F$, then P is a prime filter of $\mathcal{L}$.*

PROOF. This is a direct consequence of Theorem 3.1. □

We next give three other characterizations of F-prime filters. Compare the next theorem with Theorem 2.12 in [5].

THEOREM 3.2. *If P is a proper filter of $\mathcal{L}$, then the following statements are equivalent:*
(1) P is an F-prime filter of $\mathcal{L}$;
(2) For $a \in \mathcal{L} - P$, $(P :_{\mathcal{L}} a) = P \cup (P \vee F :_{\mathcal{L}} a)$;
(3) For $a \in \mathcal{L} - P$, $(P :_{\mathcal{L}} a) = P$ or $(P :_{\mathcal{L}} a) = (P \vee F :_{\mathcal{L}} a)$;
(4) For filters G and K of $\mathcal{L}$ with $G \vee K \subseteq P$ and $G \vee K \nsubseteq P \vee F$, either $G \subseteq P$ or $K \subseteq P$.

PROOF. (1) $\Rightarrow$ (2) Since the inclusion $P \cup (P \vee F :_{\mathcal{L}} a) \subseteq (P :_{\mathcal{L}} a)$ is clear, we will prove the reverse inclusion. Let $x \in (P :_{\mathcal{L}} a)$, where $a \in \mathcal{L} - P$. If $x \vee a \in P - (P \vee F)$, then $x \in P$, as P is an F-prime filter. If $x \vee a \in P \vee F$, then $x \in (P \vee F :_{\mathcal{L}} a)$, and so we have equality.

(2) $\Rightarrow$ (3) Since $(P :_{\mathcal{L}} a) \subseteq P \cup (P \vee F :_{\mathcal{L}} a)$ by (2), we conclude that either $(P :_{\mathcal{L}} a) \subseteq P$ or $(P :_{\mathcal{L}} a) \subseteq (P \vee F :_{\mathcal{L}} a)$ by [11, Remark 2.3 (i)], and so (3) holds.

(3) $\Rightarrow$ (4) On the contrary, assume that $G \nsubseteq P$ and $K \nsubseteq P$. It is enough to show that $G \vee K \subseteq P \vee F$. Let $x \in G$. If $x \notin P$, then $x \vee K \subseteq P$ implies that $K \subseteq (P :_{\mathcal{L}} x)$. Now, $K \nsubseteq P$ gives $K \subseteq (P \vee F :_{\mathcal{L}} x)$ by (3); hence $x \vee K \subseteq P \vee F$. So we may assume that $x \in P$. By assumption, there exists $y \in G$ such that $y \notin P$; thus $x \wedge y \notin P$ and $x \wedge y \in G$ by Lemma 2.1 (1). By an argument like that as above, $(x \wedge y) \vee G \subseteq P \vee F$. Let $k \in K$. Then $(x \wedge y) \vee k = (x \vee k) \wedge (y \vee k) \in P \vee F$ gives $x \vee k \in P \vee F$ by lemma 2.1 (1); so $x \vee K \subseteq P \vee F$. Therefore, $G \vee K \subseteq P \vee F$, as required.

(4) $\Rightarrow$ (1) Let $a, b \in \mathcal{L}$ such that $a \vee b \in P - (P \vee F)$. Set $G = T(\{a\})$ and $K = T(\{b\})$. Then $G \vee K \subseteq P$ and $G \vee K \nsubseteq P \vee F$ gives $a \in G \subseteq P$ or $b \in K \subseteq P$ by (4); i.e. (1) holds. □

We continue this section with the investigation of the stability of F-prime filters in various lattice-theoretic constructions.

THEOREM 3.3. *Assume that $v : \mathcal{L} \to \mathcal{L}'$ is a lattice homomorphism such that $v(1) = 1$ and let $\mathcal{L}$ be a complemented lattice. If v is an epimorphism and P is an F-prime filter of $\mathcal{L}$ with* $\mathrm{Ker}(v) \subseteq P$, *then $v(P)$ is an $v(F)$-prime filter of $\mathcal{L}'$.*

PROOF. Clearly, $v(P)$ and $v(F)$ are filters of $\mathcal{L}'$, as v is epimorphism. Let $x, y \in \mathcal{L}'$ such that $x \vee y \in v(P) - v(P) \vee v(F) = v(P) - v(P \vee F)$. Then there exist $a, b \in \mathcal{L}$ such that $x = v(a)$, $y = v(b)$ and $v(a \vee b) = x \vee y \in v(P) - v(P \vee F)$; so $a \vee b \notin P \vee F$ and $v(a \vee b) = v(p)$ for some $p \in P$. By the hypothesis, $p \vee p' = 1$ and $p \wedge p' = 0$ for some $p' \in \mathcal{L}$. Since $v(a \vee b \vee p') = v(a \vee b) \vee v(p') = v(1) = 1$, we conclude that $a \vee b \vee p' \in \mathrm{Ker}(v) \subseteq P$. As $a \vee b = (a \vee b) \vee (p \wedge p') = (a \vee b \vee p) \wedge (a \vee b \vee p') \in P - (P \vee F)$, we infer that $a \in P$ or $b \in P$ which implies that $x = v(a) \in v(P)$ or $y = v(b) \in v(P)$, i.e. the result holds. □

THEOREM 3.4. *Assume that $v : \mathcal{L} \to \mathcal{L}'$ is a lattice homomorphism such that $v(1) = 1$ and let $v(F)$ be a filter of $\mathcal{L}'$. If v is a monomorphism and P' is an $v(F)$-prime filter of $\mathcal{L}'$, then $P = v^{-1}(P')$ is an F-prime filter of $\mathcal{L}$.*

PROOF. Let $a, b \in \mathcal{L}$ such that $a \vee b \in P - (P \vee F)$. If $v(a \vee b) \in v(F)$, then $v(a \vee b) = v(f)$ for some $f \in F$; so $a \vee b = f \in P \vee F$ since v is injective, a contradiction. Thus, $v(a \vee b) \notin P' \vee v(F)$. Now, since $v(a \vee b) = v(a) \vee v(b) \in P' - P' \vee v(F)$ and P' is an $v(F)$-prime filter, we infer that $v(a) \in P'$ or $v(b) \in P'$. Hence, $a \in P$ or $b \in P$, and so $P = v^{-1}(P')$ is an F-prime filter of $\mathcal{L}$. □

In the following example, it is shown that the condition "$v(F)$ is a filter of $\mathcal{L}'$" in Theorem 3.4 cannot be omitted.

EXAMPLE 3.5. Assume that $\mathcal{L}$ is the lattice as in Example 3.2 (3). Then there is a mapping $v : \mathcal{L} \to \mathcal{L}$ given the formula $v(a) = v(c) = a$, $v(0) = 0 = v(b)$ and $v(1) = 1$, and it is clear that v is a lattice homomorphism and $v(F_2) = \{1, a\}$ is not a filter of $\mathcal{L}$.

COROLLARY 3.3. *If $\mathcal{L}$ is a sublattice of $\mathcal{L}'$ and P' is an F-prime filter of $\mathcal{L}'$, then $P' \cap \mathcal{L}$ is an F-prime filter of $\mathcal{L}$.*

PROOF. It suffices to apply Theorem 3.4 to the natural injection $\iota : \mathcal{L} \to \mathcal{L}'$ since $\iota^{-1}(P') = P' \cap \mathcal{L}$. □

Quotient lattices are determined by equivalence relations rather than by ideals as in the ring case. If F is a filter of a lattice $(\mathcal{L}, \leqslant)$, we define a relation on $\mathcal{L}$, given by $x \sim y$ if and only if there exist $a, b \in F$ satisfying $x \wedge a = y \wedge b$. Then $\sim$ is an equivalence relation on $\mathcal{L}$, and we denote the equivalence class of a by $a \wedge F$ and these collection of all equivalence classes by $\mathcal{L}/F$. We set up a partial order $\leqslant_Q$ on $\mathcal{L}/F$ as follows: for each $a \wedge F, b \wedge F \in \mathcal{L}/F$, we write $a \wedge F \leqslant_Q b \wedge F$ if and only if $a \leqslant b$. The following notation below will be used in this paper: It is straightforward to check that $(\mathcal{L}/F, \leqslant_Q)$ is a lattice with $(a \wedge F) \vee_Q (b \wedge F) = (a \vee b) \wedge F$ and $(a \wedge F) \wedge_Q (b \wedge F) = (a \wedge b) \wedge F$ for all elements $a \wedge F, b \wedge F \in \mathcal{L}/F$. Note that $f \wedge F = F$ if and only if $f \in F$ (see [12, Remark 4.2 and Lemma 4.3]).

Compare the next proposition with Proposition 2.2 in [5].

PROPOSITION 3.2. *If P is a proper filter of $\mathcal{L}$, then the following statements are equivalent:*
(1) P is an F-prime filter of $\mathcal{L}$;
(2) $P/(P \vee F)$ is a weakly prime filter of $\mathcal{L}/(P \vee F)$.

PROOF. (1) $\Rightarrow$ (2) Set $P \vee F = G$. Suppose that P is an F-prime filter of $\mathcal{L}$ and let $1 \wedge G \neq (x \wedge G) \vee_Q (y \wedge G) = (x \vee y) \wedge G \in P/G$ for $x \wedge G, y \wedge G \in \mathcal{L}/G$ (so $x \vee y \notin G$ and $x \vee y \in P$ by [12, Remark 4.2 and Lemma 4.3]). Then by the hypothesis, $x \vee y \in P - G$ gives $x \in P$ or $y \in P$; hence $x \wedge G \in P/G$ or $y \wedge G \in P/G$, i.e. (2) holds.

(2) $\Rightarrow$ (1) Assume that P/G is a weakly prime filter of $\mathcal{L}/G$ and let $x, y \in \mathcal{L}$ such that $x \vee y \in P - G$, where $G = P \vee F$. Then by assumption, $1 \wedge G \neq (x \vee y) \wedge G = (x \wedge G) \vee_Q (y \wedge G) \in P/G$ gives $x \wedge G \in P/G$ or $y \wedge G \in P/G$; hence $x \in P$ or $y \in P$. Thus, P is an F-prime filter of $\mathcal{L}$. □

LEMMA 3.1. *If G is a filter of $\mathcal{L}$, then $F_Q(G) = \{a \wedge G : a \in F\}$ is a filter of $\mathcal{L}/G$.*

PROOF. Let $a \wedge G, b \wedge G \in F_Q(G)$ and $x \wedge G \in \mathcal{L}/G$. Since $a \wedge b, a \vee x \in F$, we conclude that $(a \wedge G) \wedge_Q (b \wedge G) = (a \wedge b) \wedge G \in F_Q(G)$ and $(a \wedge G) \vee_Q (x \wedge G) = (a \vee x) \wedge G \in F_Q(G)$; so $F_Q(G)$ is a filter of $\mathcal{L}/G$ by Lemma 2.1 (1). □

Compare the next theorem with Proposition 2.14 (1) in [5].

THEOREM 3.5. *Assume G is a filter of a complemented lattice $\mathcal{L}$ and let P be a filter of $\mathcal{L}$ such that $G \subseteq P$. If P is an F-prime filter of $\mathcal{L}$, then P/G is an $F_Q(G)$-prime filter of $\mathcal{L}/G$.*

PROOF. Assume that $v : \mathcal{L} \to \mathcal{L}/G$ such that $v(a) = a \wedge G$ and let $x, y \in \mathcal{L}$. Then $v(x \vee y) = (x \vee y) \wedge G = (x \wedge G) \vee_Q (y \wedge G) = v(x) \vee_Q v(y)$. Similarly, $v(x \wedge y) = v(x) \wedge_Q v(y)$. So v is a lattice homomorphism from $\mathcal{L}$ onto $\mathcal{L}/G$, $v(1) = 1 \wedge G = 1_{\mathcal{L}/G}$ and $v(F) = F_Q$. Suppose that P is an F-prime filter of $\mathcal{L}$. Since $\mathrm{Ker}(v) = G \subseteq P$ and v is onto, we conclude that $v(P) = P/G$ (see [12, Lemma 3.4]) is an $F_Q(G)$-prime filter of $\mathcal{L}/G$ by Theorem 3.3. □

THEOREM 3.6. *Assume that P is a filter of a lattice $\mathcal{L}$ with $F \cap P = \{1\}$ and let G be a filter of $\mathcal{L}$ such that $G \subseteq P$. If P/G is an F_Q-prime filter of $\mathcal{L}/G$ and G is an F-prime filter of $\mathcal{L}$, then P is F-prime.*

PROOF. Let $a, b \in \mathcal{L}$ such that $a \vee b \in P - (P \vee F)$. Then $(a \wedge G) \vee_Q (b \wedge G) = (a \vee b) \wedge G \in P/G$. If $(a \vee b) \wedge G \notin F_Q$, then P/G is a F_Q-prime gives $a \wedge G \in P/G$ or $b \wedge G \in P/G$ which implies that $a \in P$ or $b \in P$ by [12, Lemma 4.3]. So we may assume that $(a \vee b) \wedge G \in F_Q$. It follows that $(a \vee b) \wedge G = f \wedge G$ for some $f \in F$; thus $(a \vee b) \wedge g_1 = f \wedge g_2$ for some $g_1, g_2 \in G$. Since $(a \vee b) \wedge g_1 \in P$ and $P \cap F = \{1\}$, we conclude that $f = 1$ by Lemma 2.1 (1); hence $(a \wedge g_1) \vee (b \wedge g_1) \in G - (G \vee F)$ (since $a \vee b \notin F$, we infer that $(a \vee b) \wedge g_1 \notin F$ by Lemma 2.1 (1)). Now, G is F-prime gives $a \wedge g_1 \in G$ or $b \wedge g_1 \in G$ which implies that $a \in G \subseteq P$ or $b \in G \subseteq P$, as needed. $\square$

Assume that $(\mathcal{L})_1, \leqslant_1), (\mathcal{L})_2, \leqslant_2)$ are lattices and let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$. We set up a partial order $\leqslant_c$ on $\mathcal{L}$ as follows: for each $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{L}$, we write $x \leqslant_c y$ if and only if $x_i \leqslant_i y_i$ for each $i \in \{1, 2\}$. The following notation below will be used in this paper: It is straightforward to check that $(\mathcal{L}, \leqslant_c)$ is a lattice with $x \vee_c y = (x_1 \vee y_1, x_2 \vee y_2)$ and $x \wedge_c y = (x_1 \wedge y_1, x_2 \wedge y_2)$. In this case, we say that $\mathcal{L}$ is a *decomposable lattice*.

Compare the next theorem with Theorem 2.15 in [5].

Theorem 3.7. *Let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ be a decomposable lattice and $F = F_1 \times F_2$, where F_i is a filter of $\mathcal{L}_i$, $i = 1, 2$. Then the F-prime filters of $\mathcal{L}$ have exactly one of the following three types:*
(1) $P_1 \times P_2$, where P_i is a proper filter of $\mathcal{L}_i$ with $P_i \subseteq F_i$, $i = 1, 2$;
(2) $P_1 \times \mathcal{L}_2$, where P_1 is an F_1-prime filter of $\mathcal{L}_1$ and $F_2 = \mathcal{L}_2$;
(3) $\mathcal{L}_1 \times P_2$, where P_2 is an F_2-prime filter of $\mathcal{L}_2$ and $F_1 = \mathcal{L}_1$.

Proof. First we discuss these filters and show that they are F-prime filters, then we show that there are no more F-prime filters. Since $P_1 \times P_2 - (P_1 \times P_2) \vee (F_1 \times F_2) = P_1 \times P_2 - (P_1 \vee F_1) \times (P_2 \vee F_2) = P_1 \times P_2 - P_1 \times P_2 = \emptyset$, we infer that $P_1 \times P_2$ is an F-prime. Suppose that P_1 is an F_1-prime filter of $\mathcal{L}_1$ and $F_2 = \mathcal{L}_2$. If $(a, b) \vee_c (c, d) = (a \vee c, b \vee d) \in P_1 \times \mathcal{L}_2 - (P_1 \times \mathcal{L}_2) \vee (F_1 \times \mathcal{L}_2) = P_1 \times \mathcal{L}_2 - (P_1 \vee F_1) \times \mathcal{L}_2 = (P_1 - P_1 \vee F_1) \times \mathcal{L}_2$ for some $(a, b), (c, d) \in \mathcal{L}$, then $a \vee c \in P_1 - (P_1 \vee F_1)$ gives $a \in P_1$ or $c \in P_1$ which implies that $(a, b) \in P_1 \times \mathcal{L}_2$ or $(c, d) \in P_1 \times \mathcal{L}_2$; so $P_1 \times \mathcal{L}_2$ is F-prime. Similarly, $\mathcal{L}_1 \times P_2$ is F-prime. Now, we show that there are no more F-prime filters. Suppose that $G_1 \times G_2$ is an F-prime filter of $\mathcal{L}$ and let $x, y \in \mathcal{L}_1$ such that $x \vee y \in G_1 - (G_1 \vee F_1)$. Then $(x, 1) \vee_c (y, 1) = (x \vee y, 1) \in G_1 \times G_2 - (G_1 \vee F_1) \times (G_2 \vee F_2)$ implies that $(x, 1) \in G_1 \times G_2$ or $(y, 1) \in G_1 \times G_2$ and so $x \in G_1$ or $y \in G_1$. Therefore, G_1 is F_1-prime. Similarly, G_2 is F_2-prime. If $G_1 \times G_2 = (G_1 \vee F_1) \times (G_2 \vee F_2)$, then $G_1 \subseteq F_1$ and $G_2 \subseteq F_2$ and so we are done. So we may assume that $G_1 \times G_2 \neq (G_1 \vee F_1) \times (G_2 \vee F_2)$, say $G_1 \neq G_1 \vee F_1$. Let $g_1 \in G_1 - (G_1 \vee F_1)$ (so $g_1 \notin F_1$) and $g_2 \in G_2$. Then $(g_1, 0) \vee_c (0, g_2) \in G_1 \times G_2 - (G_1 \vee F_1) \times (G_2 \vee F_2)$ gives $(g_1, 0) \in G_1 \times G_2$ or $(0, g_2) \in G_1 \times G_2$; hence $0 \in G_1$ or $0 \in G_2$ which implies that $G_1 = \mathcal{L}_1$ or $G_2 = \mathcal{L}_2$. Let $G_1 = \mathcal{L}_1$. Then $\mathcal{L}_1 \times G_2$ is F-prime, where G_2 is an F_2-prime filter of $\mathcal{L}_2$. $\square$

Corollary 3.4. *Let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ be a decomposable lattice. Then the weakly prime filters of $\mathcal{L}$ have exactly one of the following three types:*
(1) $\{1\} \times \{1\}$;
(2) $P_1 \times \{1\}$, where P_1 is a weakly prime filter of $\mathcal{L}_1$;
(3) $\{1\} \times P_2$, where P_2 is a weakly prime filter of $\mathcal{L}_2$.

PROOF. Take $F = \{1\} \times \{1\}$ in the Theorem 3.7. $\square$

4. Characterization of S-F-prime filters

We continue to use the notation already established, so F is a fixed filter of $\mathcal{L}$. In this section, we collect some basic properties concerning S-F-prime filters. We remind the reader the following definition.

DEFINITION 4.1. Let S be a join subset of $\mathcal{L}$. We say that a proper filter P of $\mathcal{L}$ with $P \cap S = \emptyset$ is an S-F-prime filter if there is an element $s \in S$ such that for all $a, b \in \mathcal{L}$ if $a \vee b \in P - (P \vee F)$, then $s \vee a \in P$ or $s \vee b \in P$.

EXAMPLE 4.1. (1) If $S = \{0\}$, then the F-prime and the S-F-prime filters of $\mathcal{L}$ are the same.

(2) If P is a F-prime filter of $\mathcal{L}$ disjoint with S, then P is an S-F-prime filter. Moreover, since every prime filter is F-prime, we infer that every prime filter of $\mathcal{L}$ disjoint with S is S-F-prime.

(3) If P is a proper filter of $\mathcal{L}$, then P is always S-P-prime since $P - (P \vee P) = P - P = \emptyset$.

(4) Let P and Q be filters of $\mathcal{L}$ with $P \subseteq Q$. If P is proper, then P is always S-Q-prime since $P - (P \vee Q) = P - P = \emptyset$. In particular, P is always S-$\mathcal{L}$-prime.

(5) If $\mathcal{L}$ is a local lattice with unique maximal filter M, then every proper filter of $\mathcal{L}$ is a S-M-prime filter by (4).

(6) It is clear that if P is an S-prime filter of $\mathcal{L}$, then P is an S-F-prime filter. However, the converse is not true in general. Indeed, let $D = \{a, b, c\}$. Then $\mathcal{L} = \{X : X \subseteq D\}$ forms a distributive lattice under set inclusion with greatest element D and least element $\emptyset$ (note that if $x, y \in \mathcal{L}$, then $x \vee y = x \cup y$ and $x \wedge y = x \cap y$). Set $P = \{D\}$, $F = \{\{a, b\}, D\}$ and $S = \{\{a\}, \emptyset\}$. Then S is a join subset of $\mathcal{L}$ disjoint with P and P is clearly an S-F-prime filter of $\mathcal{L}$. Since $\{a, b\} \vee \{c\} \in P$, $\{a\} \vee \{a, b\} \notin P$ and $\{a\} \vee \{c\} \notin P$, it follows that P is not a S-prime filter of $\mathcal{L}$. Thus an S-F-prime filter need not be an S-prime filter.

PROPOSITION 4.1. *Assume that P is a filter of $\mathcal{L}$ and let S be a join subset of $\mathcal{L}$ disjoint with P. The following hold:*

(1) Let Q be a filter of $\mathcal{L}$ such that $Q \cap S \neq \emptyset$. If P is an S-F-prime filter of $\mathcal{L}$, then $P \vee Q$ is an S-F-prime filter of $\mathcal{L}$;

(2) Let $\mathcal{L} \subseteq \mathcal{L}'$ an extension of lattices. If Q is an S-F-prime filter of $\mathcal{L}'$, then $Q \vee \mathcal{L}$ is an S-F-prime filter of $\mathcal{L}$.

PROOF. (1) Since $P \vee Q \subseteq P$, we conclude that $(P \vee Q) \cap S = \emptyset$. Suppose that $q \in S \cap Q$ and let $x, y \in \mathcal{L}$ such that $x \vee y \in (P \vee Q) - (P \vee Q \vee F)$ which implies that $x \vee y \in P - (P \vee F)$. Then there is an element $s \in S$ such that $s \vee x \in P$ or $s \vee y \in P$ which gives $(s \vee q) \vee x \in P \vee Q$ or $(s \vee q) \vee y \in P \vee Q$, where $s \vee q \in S$, i.e. (1) holds.

(2) Let $x, y \in \mathcal{L}$ such that $x \vee y \in (Q \vee \mathcal{L}) - (Q \vee \mathcal{L} \vee F)$; so $x \vee y \in Q - (Q \vee F)$. Then there is an element $s \in S$ such that $s \vee x \in Q$ or $s \vee y \in Q$ which implies that $s \vee x \in Q \vee \mathcal{L}$ or $s \vee y \in Q \vee \mathcal{L}$. This completes the proof. $\square$

Theorem 4.1. *Assume that P is a filter of $\mathcal{L}$ and let S be a join subset of $\mathcal{L}$ disjoint with P. The following assertions are equivalent:*

(1) P is an S-F-prime filter of $\mathcal{L}$;

(2) There exists $s \in S$ such that for all G, K two filters of $\mathcal{L}$, if $G \vee K \subseteq P$ and $G \vee K \nsubseteq P \vee F$, then $s \vee G \subseteq P$ or $s \vee K \subseteq P$.

Proof. (1) $\Rightarrow$ (2) By the hypothesis, there is an element $s \in S$ such that for all $x, y \in \mathcal{L}$, if $x \vee y \in P - (P \vee F)$, then $s \vee x \in P$ or $s \vee y \in P$. On the contrary, suppose that for all $t \in S$, there are G_t, K_t two filters of $\mathcal{L}$ with $G_t \vee K_t \subseteq P$ and $G_t \vee K_t \nsubseteq P \vee F$, but $t \vee G_t \nsubseteq P$ and $t \vee K_t \nsubseteq P$. Since $s \in S$, we conclude that there exist G_s, K_s two filters of $\mathcal{L}$ with $G_s \vee K_s \subseteq P$ and $G_s \vee K_s \nsubseteq P \vee F$, but $s \vee G_s \nsubseteq P$ and $s \vee K_s \nsubseteq P$. This implies that there exist $x_s, x'_s \in G_s$ and $y_s, y'_s \in K_s$ such that $s \vee x_s \notin P$, $s \vee y_s \notin P$ and $x'_s \vee y'_s \notin P \vee F$ (so $x'_s \vee y'_s \notin F$). It follows that $s \vee (x_s \wedge x'_s) = (s \vee x_s) \wedge (s \vee x'_s) \notin P$, $s \vee (y_s \wedge y'_s) = (s \vee y_s) \wedge (s \vee y'_s) \notin P$ and $(x_s \wedge x'_s) \vee (y_s \wedge y'_s) = (x_s \vee y_s) \wedge (x'_s \vee y_s) \wedge (x_s \vee y'_s) \wedge (x'_s \vee y'_s) \notin P \vee F$ by Lemma 2.1 (1). This shows that there exist $x_s \wedge x'_s \in G_s$ and $y_s \wedge y'_s \in K_s$ such that $s \vee (x_s \wedge x'_s) \notin P$ and $s \vee (y_s \wedge y'_s) \notin P$ which is a contradiction, as P is an S-F-prime filter, i.e. (2) holds.

(2) $\Rightarrow$ (1) Let $x, y \in \mathcal{L}$ such that $x \vee y \in P - (P \vee F)$. Set $G = T(\{x\})$ and $K = T(\{y\})$. Then $G \vee K \subseteq P$ and $G \vee K \nsubseteq P \vee F$ gives there exits $s \in S$ such that $s \vee x \in s \vee G \subseteq P$ or $s \vee y \in s \vee K \subseteq P$ by (2), i.e. (1) holds. □

Proposition 4.2. *Assume that P is a filter of $\mathcal{L}$ and let S be a join subset of $\mathcal{L}$ disjoint with P. Then P is an S-F-prime filter if and only if there exists $s \in S$ such that for all $G_1, \cdots, G_n$ filters of $\mathcal{L}$, if $G_1 \vee \cdots \vee G_n \subseteq P$ and $G_1 \vee \cdots \vee G_n \nsubseteq P \vee F$, then $s \vee G_i \subseteq P$ for some $i \in \{1, \cdots, n\}$.*

Proof. Let P be an S-F-prime filter of $\mathcal{L}$. Then there is an element $s \in S$ such that for all $x, y \in \mathcal{L}$, if $x \vee y \in P - (P \vee F)$, then $s \vee x \in P$ or $s \vee y \in P$. We use induction on n. We can take $n = 2$ as a base case by Theorem 4.1. Let $n \geqslant 3$, assume that the property holds up to the order $n-1$ and let $G_1, \cdots, G_n$ filters of $\pounds$ such that $G_1 \vee \cdots \vee G_n = (G_1 \vee \cdots \vee G_{n-1}) \vee G_n \subseteq P$ and $(G_1 \vee \cdots \vee G_{n-1}) \vee G_n \nsubseteq P \vee F$. Then by Theorem 4.1, $s \vee G_n \subseteq P$ or $(s \vee G_1) \vee G_2 \vee \cdots \vee G_{n-1} \subseteq P$. Since $(s \vee G_1) \vee G_2 \vee \cdots \vee G_{n-1} \nsubseteq P \vee F$, we infer from the induction hypothesis that $s \vee G_n \subseteq P$ or ($s \vee s \vee G_1 = s \vee G_1 \subseteq P$ or $s \vee G_i \subseteq P$ for some $i \in \{2, \cdots, n-1\}$). In the same way we prove that $s \vee G_i \subseteq P$ for some $i \in \{1, 2, \cdots, n\}$. The other side is clear. □

Corollary 4.1. *Let P be a proper filter of $\mathcal{L}$. Then P is an F-prime filter if and only if for all $G_1, \cdots, G_n$ filters of $\mathcal{L}$, if $G_1 \vee \cdots \vee G_n \subseteq P$ and $G_1 \vee \cdots \vee G_n \nsubseteq P \vee F$, then $G_i \subseteq P$ for some $i \in \{1, \cdots, n\}$.*

Proof. Take $S = \{0\}$ in Proposition 4.2. □

Corollary 4.2. *Assume that P is a filter of $\mathcal{L}$ and let S be a join subset of $\pounds$ disjoint with P. Then P is an S-F-prime filter if and only if there exists $s \in S$ such that for all $a_1, a_2, \cdots, a_n \in \mathcal{L}$, if $a_1 \vee \cdots \vee a_n \in P$ and $a_1 \vee \cdots \vee a_n \notin P \vee F$, then $s \vee a_i \in P$ for some $i \in \{1, \cdots, n\}$.*

PROOF. Assume that P is an S-F-prime filter of $\mathcal{L}$ and let $a_1, \cdots, a_n \in \mathcal{L}$ such that $a_1 \vee \cdots \vee a_n \in P$ and $a_1 \vee \cdots \vee a_n \notin P \vee F$. Therefore, $T(\{a_1\}) \vee \cdots \vee T(\{a_n\}) \subseteq P$ and $T(\{a_1\}) \vee \cdots \vee T(\{a_n\}) \nsubseteq P \vee F$. Then by Proposition 4.2, there exists $s \in S$ such that $s \vee a_i \in s \vee T(\{a_i\}) \subseteq P$ for some $i \in \{1, \cdots, n\}$. For the converse, take $n = 2$. □

Let S be a join subset of $\mathcal{L}$. We say that S is a strongly join subset if for each family $\{s_i\}_{i\in I}$ of elements of S we have $(\cap_{i\in I} T(\{s_i\})) \cap S \neq \emptyset$ [15, 16].

THEOREM 4.2. *Suppose that S is a strongly join subset of $\mathcal{L}$ and let $\{P_i\}_{i\in I}$ be a chain of S-F-prime filters of $\mathcal{L}$. Then $P = \cap_{i\in I} P_i$ is an S-F-prime filter of $\mathcal{L}$.*

PROOF. For each $i \in I$, there exists $s_i \in S$ such that for all $x, y \in \mathcal{L}$ with $x \vee y \in P_i - (P_i \vee F)$ (so $x \vee y \notin F$) we have $s_i \vee x \in P_i$ or $s_i \vee y \in P_i$. Consider $s \in (\cap_{i\in I} T(\{s_i\})) \cap S$, as $(\cap_{i\in I} T(\{s_i\})) \cap S \neq \emptyset$. Then for each $i \in I$, $s = s_i \vee a_i$, where $a_i \in \mathcal{L}$. Now, it suffices to show that for all $x, y \in \mathcal{L}$ such that $x \vee y \in P - (P \vee F)$ we have $s \vee x \in P$ or $s \vee y \in P$, i.e. P is S-F-prime. Let $a, b \in \mathcal{L}$ such that $a \vee b \in P - (P \vee F)$ and suppose that $s \vee a \notin P$. Then $s \vee a \notin P_j$ for some $j \in I$. Let $k \in I$. Then $P_k \subseteq P_j$ or $P_j \subseteq P_k$. We split the proof into two cases.

Case 1: $P_k \subseteq P_j$. Since $s \vee a \notin P_j$, we infer that $s \vee a = s_k \vee a_k \vee a \notin P_k$; so $s_k \vee a \notin P_k$. Clearly, $a \vee b \in P_j - (P_j \vee F)$. This shows that $s_k \vee b \in P_k$; hence $s_k \vee a_k \vee b = s \vee b \in P_k$. Thus, $s \vee b \in P$.

Case 2: $P_j \subseteq P_k$. Since $s \vee a = s_j \vee a_j \vee a \notin P_j$, we conclude that $s_j \vee a \notin P_j$; so $s_j \vee b \in P_j \subseteq P_k$ which gives $s \vee b = s_j \vee a_j \vee b \in P_k$, and so $s \vee b \in P$. □

THEOREM 4.3. *Assume that S is a join subset of $\mathcal{L}$ and let P be an S-F-prime filter of $\mathcal{L}$. If P is not S-prime, then $P \subseteq F$.*

PROOF. Suppose that $P \nsubseteq F$; we show that P is S-prime. Let $a, b \in \mathcal{L}$ such that $a \vee b \in P$. If $a \vee b \notin P \vee F$, Then P is an S-F-prime gives $s \vee a \in P$ or $s \vee b \in P$ for some $s \in S$. So we can assume that $a \vee b \in P \vee F$. By the hypothesis, there exists $p \in P$ such that $p \notin F$ which implies that $p \wedge (a \vee b) \in P - (P \vee F)$ by Lemma 2.1 (1). Since $(p \wedge a) \vee (p \wedge b) = p \wedge (a \vee b) \in P - (P \vee F)$, we conclude that there is an element $t \in S$ such that $t \vee (p \wedge a) = (t \vee p) \wedge (t \vee a) \in P$ or $t \vee (p \wedge b) = (t \vee p) \wedge (t \vee b) \in P$ which implies that $t \vee a \in P$ or $t \vee b \in P$ by Lemma 2.1 (1), i.e. P is S-prime. □

COROLLARY 4.3. *Let P be an S-$\{1\}$-prime filter of $\mathcal{L}$. If P is not an S-prime filter, then $P = \{1\}$.*

PROOF. This is a direct consequence of Theorem 4.3. □

COROLLARY 4.4. *Let P be an S-F-prime filter of $\mathcal{L}$. If $P \nsubseteq F$, then P is an S-prime filter of $\mathcal{L}$.*

PROOF. This is a direct consequence of Theorem 4.3. □

We continue this section with the investigation of the stability of S-F-prime filters in various lattice-theoretic constructions.

Theorem 4.4. *Let $v : \mathcal{L} \to \mathcal{L}'$ be a lattice homomorphism such that $v(1) = 1$ and $v(0) = 0$. Suppose that S is a join subset of a complemented lattice $\mathcal{L}$. If v is a epimorphism and P is a S-F-prime filter of $\mathcal{L}$ with* $\mathrm{Ker}(v) \subseteq P$*, then $v(P)$ is a $v(S)$-$v(F)$-prime filter of $\mathcal{L}'$.*

Proof. Clearly, $v(P), v(F)$ are filters of $\mathcal{L}'$ and $v(S)$ is a join subset of $\mathcal{L}'$. If $v(S) \cap v(P) \neq \emptyset$, then $v(s) = v(q)$ for some $s \in S$ and $q \in P$. By assumption, there exists $q' \in \mathcal{L}$ such that $q \vee q' = 1$ and $q \wedge q' = 0$. Therefore, $v(s \vee q') = v(q) \vee v(q') = v(1) = 1$ gives $s \vee q' \in \mathrm{Ker}(v) \subseteq P$. Since P is a filter, we conclude that $(s \vee q) \wedge (s \vee q') = s \in P \cap S$, a contradiction. Thus, $v(S) \cap v(P) = \emptyset$. Let $x, y \in \mathcal{L}'$ such that $x \vee y \in v(P) - v(P) \vee v(F) = v(P) - v(P \vee F)$. Then there exist $a, b \in \mathcal{L}$ such that $x = v(a)$, $y = v(b)$ and $v(a \vee b) = x \vee y \in v(P) - v(P \vee F)$; so $a \vee b \notin P \vee F$ and $v(a \vee b) = v(p)$ for some $p \in P$. By the hypothesis, $p \vee p' = 1$ and $p \wedge p' = 0$ for some $p' \in \mathcal{L}$. Since $v(a \vee b \vee p') = v(a \vee b) \vee v(p') = v(1) = 1$, we conclude that $a \vee b \vee p' \in \mathrm{Ker}(v) \subseteq P$. As $a \vee b = (a \vee b) \vee (p \wedge p') = (a \vee b \vee p) \wedge (a \vee b \vee p') \in P - (P \vee F)$, we infer that there exists $s \in S$ such that $s \vee a \in P$ or $s \vee b \in P$ which implies that $v(s) \vee x = v(s \vee a) \in v(P)$ or $v(s) \vee y = v(s \vee b) \in v(P)$, i.e. $v(P)$ is a $v(S)$-$v(F)$-prime filter of $\mathcal{L}'$. $\square$

Theorem 4.5. *Let $v : \mathcal{L} \to \mathcal{L}'$ be a lattice homomorphism such that $v(1) = 1$ and $v(0) = 0$. Suppose that S is a join subset of a lattice $\mathcal{L}$ and let $v(F)$ be a filter of $\mathcal{L}'$. If v is a monomorphism and P' is a $v(S)$-$v(F)$-prime filter of $\mathcal{L}'$, then $v^{-1}(P')$ is a S-F-prime filter of $\mathcal{L}$.*

Proof. Set $P = v^{-1}(P')$. It is clear that $P \cap S = \emptyset$. Let $a, b \in \mathcal{L}$ such that $a \vee b \in P - (P \vee F)$. If $v(a \vee b) \in v(F)$, then $v(a \vee b) = v(f)$ for some $f \in F$; so $a \vee b = f \in P \vee F$ since v is injective, a contradiction. Thus, $v(a \vee b) \notin P' \vee v(F)$. Now, since $v(a \vee b) = v(a) \vee v(b) \in P' - P' \vee v(F)$ and P' is a $v(S)$-$v(F)$-prime filter, we infer that there exists $s \in S$ such that $v(s) \vee v(a) = v(s \vee a) \in P'$ or $v(s) \vee v(b) = v(s \vee b) \in P'$. Hence, $s \vee a \in P$ or $s \vee b \in P$, and so $v^{-1}(P')$ is a S-F-prime filter of $\mathcal{L}$. $\square$

Corollary 4.5. *Let S be a join subset of $\mathcal{L}$. If $\mathcal{L}$ is a sublattice of $\mathcal{L}'$ and P' is a S-F-prime filter of $\mathcal{L}'$, then $P' \cap \mathcal{L}$ is a S-F-prime filter of $\mathcal{L}$.*

Proof. It suffices to apply Theorem 4.5 to the natural injection $\iota : \mathcal{L} \to \mathcal{L}'$ since $\iota^{-1}(P') = P' \cap \mathcal{L}$ and $\iota(S) = S$. $\square$

An element x of $\mathcal{L}$ is called identity join of a lattice $\mathcal{L}$, if there exists $1 \neq y \in \mathcal{L}$ such that $x \vee y = 1$. The set of all identity joins of a lattice $\mathcal{L}$ is denoted by $\mathrm{I}(\mathcal{L})$. Suppose that G is a filter of $\mathcal{L}$ and let S be a join subset of $\mathcal{L}$. An easy inspection will show that $S_Q(G) = \{s \wedge G : s \in S\}$ is a join subset of $\mathcal{L}/G$.

Proposition 4.3. *Assume that P is a filter of $\mathcal{L}$ and let S be a join subset of $\mathcal{L}$ disjoint with P such that $S_Q(F) \cap \mathrm{I}(\mathcal{L}/F) = \emptyset$. The following assertions are equivalent:*

(1) P is an S-F-prime filter of $\mathcal{L}$;

(2) $(P :_{\mathcal{L}} s)$ is an F-prime filter of $\mathcal{L}$ for some $s \in S$.

PROOF. (1) $\Rightarrow$ (2) Since P is an S-F-prime filter, we conclude that there is an element $s \in S$ such that for all $x, y \in \mathcal{L}$ with $x \vee y \in P - (P \vee F)$ we have $s \vee x \in P$ or $s \vee y \in P$. Now, we show that $(P :_{\mathcal{L}} s)$ is an F-prime filter of $\mathcal{L}$. Let $x, y \in \mathcal{L}$ such that $x \vee y \in (P :_{\mathcal{L}} s) - ((P :_{\mathcal{L}} s) \vee F)$. Then $x \vee y \notin F$ gives $(x \vee y) \wedge F \neq 1 \wedge F$. If $x \vee y \vee s \in F$, then $((x \vee y) \wedge F) \vee_Q (s \wedge F) = 1 \wedge F$ by [12, Remark 4.2] gives $s \wedge F \in S_Q(F) \cap \mathrm{I}(\mathcal{L}/F)$ which is impossible. So we may assume that $x \vee y \vee s \notin F$. Therefore, $x \vee y \vee s \in P - (P \vee F)$ gives $x \vee s \vee s = x \vee s \in P$ or $y \vee s \in P$ which means that $x \in (P :_{\mathcal{L}} s)$ or $y \in (P :_{\mathcal{L}} s)$. Thus $(P :_{\mathcal{L}} s)$ is an F-prime filter of $\mathcal{L}$.

(2) $\Rightarrow$ (1) Suppose that $(P :_{\mathcal{L}} s)$ is an F-prime filter of $\mathcal{L}$ for some $s \in S$ and let $a, b \in \mathcal{L}$ such that $a \vee b \in P - (P \vee F)$ (so $a \vee b \notin F$ and $a \vee b \vee s \in P$). Since $S_Q(F) \cap \mathrm{I}(\mathcal{L}/F) = \emptyset$, we conclude that $a \vee b \vee s \notin F$; so $a \vee b \in (P :_{\mathcal{L}} s) - ((P :_{\mathcal{L}} s) \vee F)$. Now, $(P :_{\mathcal{L}} s)$ is an F-prime gives $s \vee a \in P$ or $s \vee b \in P$, as required. □

In the following theorem, we give a condition under which the F-prime and the S-F-prime filters coincide.

THEOREM 4.6. *Assume that P is a filter of $\mathcal{L}$ and let S be a join subset of $\mathcal{L}$ disjoint with P such that $S_Q(F) \cap \mathrm{I}(\mathcal{L}/F) = \emptyset = S_Q(P) \cap \mathrm{I}(\mathcal{L}/P)$. Then P is F-prime if and only if P is S-F-prime.*

PROOF. One side is clear. To see the other side, it is enough to show that $P = (P :_{\mathcal{L}} s)$ for all $s \in S$ by Proposition 4.3. Since the inclusion $P \subseteq (P :_{\mathcal{L}} s)$ is clear, we will prove the reverse inclusion. Let $s \in S$ and $x \in (P :_{\mathcal{L}} s)$. Then $s \vee x \in P$ gives $(s \wedge P) \vee_Q (x \wedge P) = (s \vee x) \wedge P = 1 \wedge P$. Since $S_Q(P) \cap \mathrm{I}(\mathcal{L}/P) = \emptyset$, we conclude that $x \wedge P = 1 \wedge P$; so $x \in P$ by [12, Remark 4.2], and so we have equality. □

THEOREM 4.7. *Assume that G is a filter of $\mathcal{L}$ and let S be a join subset of $\mathcal{L}$. Let P be a proper filter of $\mathcal{L}$ containing G such that $(P/G) \cap S_Q(G) = \emptyset$. Then P is an S-F-prime filter of $\mathcal{L}$ if and only if P/G is an $S_Q(G)$-$F_Q(G)$-prime filter of $\mathcal{L}/G$.*

PROOF. Let P be an S-F-prime filter of $\mathcal{L}$. Then there exists $s \in S$ such that for all $x, y \in \mathcal{L}$, if $x \vee y \in P - (P \vee F)$, then $s \vee x \in \mathcal{L}$ or $s \vee y \in \mathcal{L}$. Let $a \wedge G, b \wedge G \in \mathcal{L}/G$ such that $(a \wedge G) \vee_Q (b \wedge G) = (a \vee b) \wedge G \in P/G - (P/G) \vee F_Q(G)$ (so $(a \vee b) \wedge G \notin F_Q(G)$ gives $a \vee b \notin F$) which implies that $a \vee b \in P - (P \vee F)$ by [12, Lemma 4.3]; hence $s \vee a \in P$ or $s \vee b \in P$. Therefore $(s \wedge G) \vee_Q (a \wedge G) \in P/G$ or $(s \wedge G) \vee_Q (b \wedge G) \in P/G$. Thus P/G is an $S_Q(G)$-$F_Q(G)$-prime filter of $\mathcal{L}/G$.

Conversely, if $P \cap S \neq \emptyset$, then $(P/G) \cap S_Q(G) \neq \emptyset$ which is a contradiction. Thus, $S \cap P = \emptyset$. Since P/G is an S_Q-$F_Q(G)$-prime filter of $\mathcal{L}/G$, we conclude that there exists $s \in S$ such that for all $x \wedge G, y \wedge G \in \mathcal{L}/G$ with $(x \wedge G) \vee_Q (y \wedge G) \in P/G - (P/G) \vee F_Q(G)$, we infer that $(s \wedge G) \vee_Q (x \wedge G) \in P/G$ or $(s \wedge G) \vee_Q (y \wedge G) \in P/G$. Now, let $a, b \in \mathcal{L}$ such that $a \vee b \in P - (P \vee F)$. Then $(a \wedge G) \vee_Q (b \wedge G) \in P/G - (P/G) \vee F_Q(G)$ gives $(s \wedge G) \vee_Q (a \wedge G) \in P/G$ or $(s \wedge G) \vee_Q (b \wedge G) \in P/G$; hence $s \vee a \in P$ or $s \vee b \in P$, i.e. the result holds. □

Theorem 4.8. *Let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ be a decomposable lattice, $P = P_1 \times P_2$, $F = F_1 \times F_2$ and $S = S_1 \times S_2$, where P_i, F_i are filters of $\mathcal{L}_i$ and S_i is a join subset of $\mathcal{L}_i$, $i = 1, 2$. Then the following hold:*

(1) If P_i is a proper filter of $\mathcal{L}_i$ with $P_i \subseteq F_i$, $i = 1, 2$, then P is an S-F-filter of $\mathcal{L}$;

(2) If P_1 is an S_1-F_1-prime filter of $\mathcal{L}_1$ and $F_2 = \mathcal{L}_2$, then $P_1 \times \mathcal{L}_2$ is an S-F-filter of $\mathcal{L}$;

(3) If P_2 is an S_2-F_2-prime filter of $\mathcal{L}_2$ and $F_1 = \mathcal{L}_1$, then $\mathcal{L}_1 \times P_2$ is an S-F-filter of $\mathcal{L}$.

Proof. (1) Since $P_1 \times P_2 - (P_1 \times P_2) \vee (F_1 \times F_2) =$

$$P_1 \times P_2 - (P_1 \vee F_1) \times (P_2 \vee F_2) = P_1 \times P_2 - P_1 \times P_2 = \emptyset,$$

we infer that $P = P_1 \times P_2$ is an S-F-prime.

(2) Suppose that P_1 is a S_1-F_1-prime filter of $\mathcal{L}_1$ and $F_2 = \mathcal{L}_2$. If

$$(a, b) \vee_c (c, d) = (a \vee c, b \vee d) \in P_1 \times \mathcal{L}_2 - (P_1 \times \mathcal{L}_2) \vee (F_1 \times \mathcal{L}_2) =$$

$P_1 \times \mathcal{L}_2 - (P_1 \vee F_1) \times \mathcal{L}_2 = (P_1 - P_1 \vee F_1) \times \mathcal{L}_2$ for some $(a, b), (c, d) \in \mathcal{L}$, then $a \vee c \in P_1 - (P_1 \vee F_1)$ gives there exists $s_1 \in S_1$ such that $s_1 \vee a \in P_1$ or $s_1 \vee c \in P_1$ which implies that $(s_1, 0) \vee_c (a, b) \in P_1 \times \mathcal{L}_2$ or $(s_1, 0) \vee_c (c, d) \in P_1 \times \mathcal{L}_2$, where $(s_1, 0) \in S$; so $P_1 \times \mathcal{L}_2$ is S-F-prime.

(3) The proof is similar to that in case (2) and we omit it. □

Theorem 4.9. *Let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ be a decomposable lattice, $P = P_1 \times P_2$, $F = F_1 \times F_2$ and $S = S_1 \times S_2$, where P_i, F_i are filters of $\mathcal{L}_i$ and S_i is a join subset of $\mathcal{L}_i$, $i = 1, 2$. If P is an S-F-prime filter of $\mathcal{L}$, then P_1 is an S_1-F_1-prime filter of $\mathcal{L}_1$ and $P_2 \cap S_2 \neq \emptyset$ or P_2 is an S_2-F_2-prime filter of $\mathcal{L}_2$ and $P_1 \cap S_1 \neq \emptyset$ or P_1 is an S_1-F_1-prime filter of $\mathcal{L}_1$ and P_2 is an S_2-F_2-prime filter of $\mathcal{L}_2$.*

Proof. Suppose that P is an S-F-prime filter of $\mathcal{L}$. Then we keep in mind that there exists a fixed $s = (s_1, s_2) \in S$ that satisfies the S-F-prime condition. Since $P \cap S = (P_1 \cap S_1) \times (P_2 \cap S_2) = \emptyset$, we have either $P_1 \cap S_1 = \emptyset$ or $P_2 \cap S_2 = \emptyset$. If $P_1 \cap S_1 \neq \emptyset$, we will show that P_2 is an S_2-F_2-prime filter of $\mathcal{L}_2$. Let $x \vee y \in P_2 - (P_2 \vee F_2)$ for some $x, y \in \mathcal{L}_2$ (so $x \vee y \notin F_2$). Then $(1, x) \vee_c (1, y) = (1, x \vee y) \in P - (P \vee F)$ gives $s \vee_c (1, x) = (1, s_2 \vee x) \in P$ or $s \vee_c (1, y) = (1, s_2 \vee y \in P$. This shows that $s_2 \vee x \in P_2$ or $s_2 \vee y \in P_2$. Hence, P_2 is an S_2-F_2-prime filter of $\mathcal{L}_2$. Similarly, if $P_2 \cap S_2 \neq \emptyset$, then P_1 is an S_1-F_1-prime filter of $\mathcal{L}_1$. Now assume that $P_1 \cap S_1 = \emptyset = P_2 \cap S_2$. We will show that P_1 is an S_1-F_1-prime filter of $\mathcal{L}_1$ and P_2 is an S_2-F_2-prime filter of $\mathcal{L}_2$. Suppose that P_1 is not an S_1-F_1-prime filter of $\mathcal{L}_1$. Then there exist $a, b \in \mathcal{L}_1$ such that $a \vee b \in P_1 - (P_1 \vee F_1)$ (so $a \vee b \notin F_1$) but $s_1 \vee a \notin P_1$ and $s_1 \vee b \notin P_1$. Then $(a, 0) \vee_c (b, 1) = (a \vee b, 1) \in P - (P \vee F)$ gives $s \vee_c (a, 0) = (s_1 \vee a, s_2) \in P$ or $s \vee_c (b, 1) = (s_1 \vee b, 1) \in P$; so $s_1 \vee a \in P_1$ or $s_1 \vee b \in P_1$ which is a contradiction. Therefor, P_1 is an S_1-F_1-prime filter of $\mathcal{L}_1$. Similarly, P_2 is an S_2-F_2-prime filter of $\mathcal{L}_2$. □

Let S be a join subset of $\mathcal{L}$. We say that a filter G of $\mathcal{L}$ is S-finite if $s \vee G \subseteq K \subseteq G$ for some finitely generated filter K of $\mathcal{L}$ and some $s \in S$. We say that $\mathcal{L}$

is S-Noetherian if each filter of $\mathcal{L}$ is S-finite [2, 15, 16]. The proof of the following lemma can be found in [16, Lemma 2.20], but we give the details for convenience.

LEMMA 4.1. *Suppose that S is a join subset of $\mathcal{L}$ and let P be a filter of $\mathcal{L}$ which is maximal among all non-S-finite filters of $\mathcal{L}$. Then P is a F-prime filter of $\mathcal{L}$.*

PROOF. Clearly, $\mathcal{L}$ is S-finite. Since every prime filter is F-prime by Example 3.2 (2), it suffices to show that P is prime. If P is not prime, let $a, b \notin P$ with $a \vee b \in P$. Since $P \subsetneqq P \wedge T(\{a\})$ and $P \subsetneqq (P :_{\mathcal{L}} a)$, we conclude that there exist $s, t \in S$, $p_1, \cdots, p_n \in P$, $b_1, \cdots, b_n \in \mathcal{L}$ and $c_1, \cdots, c_k \in (P :_{\mathcal{L}} a)$ such that $s \vee (P \wedge T(\{a\}) \subseteq T(B)$ and $t \vee (P :_{\mathcal{L}} a) \subseteq T(C)$, where $B = \{p_1 \wedge (a \vee b_1), \cdots, p_n \wedge (a \vee b_n)\}$ and $C = \{c_1, \cdots, c_k\}$. Now, let $x \in P$. Then $s \vee x \in s \vee P \subseteq s \vee (P \wedge T(\{a\}))$ gives $s \vee x = \wedge_{i=1}^{n}(s \vee x \vee p_i) \wedge (\wedge_{i=1}^{n}(s \vee x \vee a \vee b_i))$; so $y = \wedge_{i=1}^{n}(s \vee x \vee b_i) \in (P :_{\mathcal{L}} a)$. It follows that $t \vee y = \wedge_{i=1}^{k}(t \vee y \vee c_i) \in (P :_{\mathcal{L}} a)$. Therefore, $s \vee x \vee t = \wedge_{i=1}^{n}(s \vee x \vee p_i \vee t) \wedge (\wedge_{i=1}^{k}(a \vee c_i \vee t \vee y))$. Hence, $(s \vee t) \vee P \subseteq T(D) \subseteq P$, where $D = \{p_1 \vee t, \cdots, p_n \vee t, a \vee c_1, \cdots, a \vee c_k\} \subseteq P$; so P is S-finite, a contradiction. Thus, P is a prime filter of $\mathcal{L}$. $\square$

Let $\mathbb{F}(\mathcal{L})$ be the set of all filters of $\mathcal{L}$.

PROPOSITION 4.4. *Let S be a join subset of $\mathcal{L}$. Then $\mathcal{L}$ is S-Noetherian if and only if every F-prime filter of $\mathcal{L}$ (disjoint from S) is S-finite.*

PROOF. One side is clear. To see the other side, assume that every F-prime filter P of $\mathcal{L}$ with $P \cap S = \emptyset$ is S-finite. On the contrary, suppose That $\mathcal{L}$ is not S-Noetherian. Then the set $\Omega = \{G \in \mathbb{F}(\mathcal{L}) : G \text{ is non-S-finite}\}$ is not empty. Moreover, $(\Omega, \subseteq)$ is a partial order. It is easy to see that Ω is closed under taking unions of chains and so Ω has at least one maximal element by Zorn's Lemma, say P. Then Lemma 4.1 shows that P is an F-prime filter. If $P \cap S \neq \emptyset$, then $s \vee P \subseteq T(\{s\}) \subseteq P$ for every $s \in P \cap S$ gives P is S-finite, a contradiction. Thus $P \cap S = \emptyset$. Now, by the hypothesis, P is S-finite which is impossible since $P \in \Omega$. Thus $\mathcal{L}$ is S-Noetherian. $\square$

We obtain the following S-version of Cohen's Theorem [8].

THEOREM 4.10. *Let S be a join subset of $\mathcal{L}$. The following assertions are equivalent:*

(1) $\mathcal{L}$ is S-Noetherian;

(2) Every S-F-prime filter of $\mathcal{L}$ is S-finite;

(3) Every F-prime filter of $\mathcal{L}$ is S-finite.

PROOF. (1) $\Rightarrow$ (2) This is clear.

(2) $\Rightarrow$ (3) Let P be an F-prime filter of $\mathcal{L}$. If $P \cap S \neq \emptyset$, then $s \vee P \subseteq T(\{s\}) \subseteq P$ for every $s \in P \cap S$ gives P is S-finite. If $P \cap S = \emptyset$, then P is an S-F-filter of $\mathcal{L}$ by Example 4.1 (2); so by (2), P is S-finite.

(3) $\Rightarrow$ (1) Follows from Proposition 4.4. $\square$

References

[1] M. F. Atiyah and I. G. MacDonald, Introduction to Commutative Algebra, Addison-Wesley, Reading, MA, 1969.

[2] D. D. Anderson and T. Dumitrescu, *S-Noetherian rings*, Comm. Algebra **30** (2002), 4407-4416.

[3] D. D. Anderson and E. Smith, *Weakly prime ideals*, Houston J. Math. **29 (4)** (2003), 831-840.

[4] F. A. A. Almahdi, E. M. Bouba, and M. Tamekkante, *On weakly S-prime ideals of commutative rings*, An. St. Univ. Ovidius Constanta **29 (2)** (2021), 173-186.

[5] I. Akray, *I-prime ideals*, J. Algebra Relat. Topics **4 (2)** (2016), 41-47.

[6] I. Akray and H. S. Hussein, *I-prime submodules*, Acta Math. Acad. Paedagog. Nyházi. **33** (2017), 165-173.

[7] G. Birkhoff, *Lattice theory*, Amer. Math. Soc., 1973.

[8] I. S. Cohen, *Commutative rings with restricted minimum condition*, Duke Math. J. **17** (1950), 27-42.

[9] G. Călugăreanu, *Lattice Concepts of Module Theory*, Kluwer Academic Publishers, 2000.

[10] S. Ebrahimi Atani and F. Farzalipour, *On weakly primary ideals*, Georgian Math. J. **12** (2005), 423-429.

[11] S. Ebrahimi Atani and M. Sedghi Shanbeh Bazari, *On* 2*-absorbing filters of lattices*, Discuss. Math. Gen. Algebra Appl. **36** (2016), 157-168.

[12] S. Ebrahimi Atani, *G-Supplemented property in the lattices*, Mathematica Bohemica **147 (4)** (2022), 525-545.

[13] S. Ebrahimi Atani, *Note on weakly* 1*-absorbing prime filters*, Bull. Int. Math. Virtual Inst. **13 (3)** (2023), 465-478.

[14] S. Ebrahimi Atani, *On Baer filters of bounded distributive lattices*, Quasigroups and Related Systems **32** (2024), 1-20.

[15] S. Ebrahimi Atani, *On S-2-absorbing filters of lattices*, Bull. Int. Math. Virtual Inst. **14 (1)** (2024), 115-128.

[16] S. Ebrahimi Atani, *S-prime property in lattices*, Mathematica, to appear.

[17] A. Hamed and A. Malek, *S-prime ideals of a commutative ring*, Beitr. Algebra Geom., (2019).

[18] E. S. Sevim, T. Arabaci, Ü. Tekir, and S. Koc, *On S-prime submodules*, Turk. J. Math. **43 (2)** (2019), 1036-1046.

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