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HOMOLOGY SEQUENCES AND THEOREMS IN PERSISTENCE SETUP - A SURVEY

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Abstract. Persistent homology is one of the main tools of topological data analysis. It measures the topological properties of shapes through filtrations. In this paper, we review the concepts of persistence and we demonstrate the process of obtaining the persistent analogs of computational tools of homology. In particular, the Mayer-Vietoris sequence, the long exact sequence and the excision theorem. We provide new examples to emphasise the difficulties in the process, and the creativity exhibited by the researchers to overcome these difficulties. And we give an alternative proof for the excision theorem.

1. Introduction

Topological Data Analysis (TDA) is a branch of mathematics which makes use of topological methods and invariants to analyse shapes of datasets. One of the core tools used in this research area is persistent homology. It is an algebraic tool which measures topological properties of shapes. The concept is first introduced by Edelsbrunner, Letscher and Zomorodian in 2002 [**7**]. The authors point out all possible difficulties of extracting topological information from shapes and reducing it to actual topological features by removing the noise, and they demonstrate how to overcome these difficulties. In 2005, new computation methods and new algorithms which are suitable for arbitrary coefficients arise [**15**]. In 2005, barcodes, as a tool for visualising persistence, appears [**2**]. In 2008, Ghrist gives a detailed survey of barcodes [**8**]. More tools for visualisation of persistent homology get introduced in the following years such as persistence diagrams [**6**] and persistence landscape [**1**].

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At the same time, researchers work on applications of this new concept. Sensor network coverage [**3**], breast cancer identification [**10**], machine learning [**12**], image processing [**5**] are among examples. Even though persistent homology is a relatively young research area, it takes place in many applications due to the already existing strengths of homology: computability and reliability.

A natural question is, how far can homological properties be extended to persistence setup? In 2011, Di Fabio and Landi show that a Mayer-Vietoris sequence can be constructed in persistence homology but the sequence is not exact [**4**]. Even without the exactness, this sequence is useful in applications as the authors demonstrate. In 2018, Varli, Yılmaz and Pamuk show that a long sequence is also constructable at the cost of losing the exactness [**14**]. But the authors also show that trying a new approach with a different algebraic structure provides the exactness of both sequences. In 2019, Palser gives an excision theorem for persistence and shows that it holds for both persistent homology groups and modules [**11**].

This paper is a survey of the process of creating persistent homological properties based on homological properties. The paper is organized as follows. In Section 2, we start with a brief review of homology. Then we give the Mayer-Vietoris sequence, long exact sequence of a pair, and the excision theorem of homology which are discussed in persistence setup in the further sections. In Section 3, we follow up with a review of the persistent homology. We mention the central objects, filtrations, in persistence and we compute the persistent homology groups of these objects. In Section 4, we review the attempts of carrying homological properties given in Section 2 to the persistent homology. We provide examples demonstrating the difficulties that came out of these attempts for each of the homology sequences mentioned above. Then we see how these difficulties are overcome by making use of another algebraic structure, persistence modules, and trying a different approach to these sequences. We give an alternative proof for Palser's excision theorem in persistent homology and end the section with an example case where we verify the theorem.

2. Homology

2.1. Homology groups. We start with a general definition of a chain complex and its homology.

DEFINITION 2.1. A sequence $\boldsymbol{C}$ consisting of abelian groups C_n and homomorphisms $\partial_n : C_n \to C_{n-1}$, $n \in \mathbb{Z}$,

$$\boldsymbol{C} : \quad \cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots$$

such that $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{Z}$ is called a *chain complex.*

For all $n \in \mathbb{Z}$, the abelian group C_n is called the nth chain group. The elements of $\ker(\partial_n)$ are called *n-cycles* and this subgroup is denoted by $Z_n(\boldsymbol{C})$, and the elements of $\mathrm{im}(\partial_{q+1})$ are called *n-boundaries* and this subgroup is denoted by $B_n(\boldsymbol{C})$.

DEFINITION 2.2. Let $\boldsymbol{C}$ be a chain complex. The quotient group

$$H_n(\boldsymbol{C}) = Z_n(\boldsymbol{C})/B_n(\boldsymbol{C}) = \ker(\partial_n)/\mathrm{im}(\partial_{n+1})$$

is called the nth homology group of $\boldsymbol{C}$.

There are ways of deriving the abelian groups C_n of the chain complex $\boldsymbol{C}$ from a topological space X, for $n \geqslant 0$. In that case, we denote the chain complex obtained from X by $\boldsymbol{C}(X)$, and it takes the form,

$$\cdots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

where $C_{-1}(X)$ is defined to be trivial. The nth homology group of X is denoted by $H_n(X)$. One of the most common examples of this construction is *singular homology*. We refer the reader to [**9**] for details of this homology theory.

Another homology theory is based on *simplicial complexes*, and it is called *simplicial homology*. A simplicial complex is a finite set consisting of *simplicies*, which are convex sets spanned by affine independent points in $\mathbb{R}^n$, such that either two simplicies intersect in a face belonging to both simplicies, or they are disjoint. For computational purposes, simplicial homology is commonly used in persistent homology. Thus, we briefly mention simplicial homology at this point.

2.2. Simplicial homology. In this homology theory, the nth chain groups of the chain complex are the free abelian groups generated by n-simplicies in an oriented simplicial complex $\mathcal{K}$, and denoted by $C_n(\mathcal{K})$. A typical element of $C_n(\mathcal{K})$ is of the form

$$\sum_q c_q \sigma_q,$$

where $c_q \in \mathbb{Z}$, and $\sigma_q \in \mathcal{K}$ is an n-simplex.

DEFINITION 2.3. [**13**] The boundary homomorphism $\partial_q : C_q(\mathcal{K}) \to C_{q-1}(\mathcal{K})$ is defined for any q-simplex $\sigma = [p_0, ..., p_q]$ in $C_q(\mathcal{K})$ as

$$\partial_q \sigma = \sum_{i=0}^{q} (-1)^i [p_0, ..., \hat{p_i}, ..., p_q],$$

where $\hat{p_i}$ means removing the vertex p_i.

The boundary homomorphism has the following important property. The boundary of a boundary is zero.

THEOREM 2.1. *[**13**]* $\partial_{q-1}\partial_q = 0$ *for* $q \geqslant 1$.

Then, from an n-dimensional simplicial complex $\mathcal{K}$, the following chain complex is obtained,

$$\cdots \to 0 \xrightarrow{\partial_{n+1}} C_n(\mathcal{K}) \xrightarrow{\partial_n} C_{n-1}(\mathcal{K}) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1(\mathcal{K}) \xrightarrow{\partial_1} C_0(\mathcal{K}) \xrightarrow{\partial_0} 0.$$

The nth homology group of a simplicial complex $\mathcal{K}$ is

$$H_n(\mathcal{K}) = Z_n(\mathcal{K})/B_n(\mathcal{K}) = \ker(\partial_q)/\mathrm{im}(\partial_{q+1}).$$

Throughout the paper, $<\sigma_1, \sigma_2, ..., \sigma_n>$ denotes the free abelian group generated by simplicies $\sigma_1, \sigma_2, ..., \sigma_n$. A generator $z + B_n(\mathcal{K})$ of $H_n(\mathcal{K})$ is denoted by $\overline{z}$. And the group of coefficients taken into account in homology computations is $\mathbb{Z}$.

Simplicial complexes can be used to represent topological spaces. The term that describes the representation is *triangulation*. Not all topological spaces are triangulable. But for triangulable ones, the simplicial homology of a simplicial complex representing a topological space X agrees with the singular homology of X.

Homology groups carry significant information about the topological space they are derived from. In particular, the nth homology group reveals information about the number of connected components of the space when $n = 0$, and the number of n-dimensional holes in the space when $n > 0$. Throughout the paper, we make use of this geometric interpretation of homology groups when its needed to reduce the amount of computations. We label the connected components in a topological space (simplicial complex) and use these labels as representers of generators in 0th homology group. Similarly, we use labels for n-dimensional holes in the space (or complex) and these labels represent the generators of the nth homology group.

2.3. Relative homology. For a subspace A of a topological space X, we form a quotient group $C_n(X)/C_n(A)$, and denote it by $C_n(X, A)$. This is the nth relative chain group of the pair (X, A). The boundary maps between these relative chain groups are induced by the ones between the chain groups of X. We have the following chain complex

$$\cdots \to C_n(X, A) \xrightarrow{\partial_n} C_{n-1}(X, A) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1(X, A) \xrightarrow{\partial_1} C_0(X, A) \xrightarrow{\partial_0} 0.$$

Definition 2.4. [**9**] Let A be a subspace of a topological space X. The *nth relative homology group* of the pair (X, A) is defined as

$$H_n(X, A) = \ker(\partial_n)/\mathrm{im}(\partial_{n+1}).$$

By forming these quotient groups, we ignore the topological data contained in the subspace A. By choosing A properly, we can reduce the amount of work needed to obtain the desired information about the space.

Next, we give three tools for homology computations consisting of two exact sequences and a theorem.

2.4. Mayer-Vietoris sequence.

Definition 2.5. [**9**] Let A and B be subspaces of a topological space X such that $X = A^o \cup B^o$. Then, there is an exact sequence of homology groups

$$\cdots \to H_n(A \cap B) \xrightarrow{(i_{1_*}, i_{2_*})} H_n(A) \oplus H_n(B) \xrightarrow{j_{1_*} - j_{2_*}} H_n(X) \xrightarrow{d} H_{n-1}(A \cap B) \to \cdots,$$

where $i_{1_*}, i_{2_*}, j_{1_*}, j_{2_*}$ are all induced by inclusion maps and d is the connecting homomorphism.

One can simplify homology computations of X by choosing A and B such that the homology groups of A, B or $A \cap B$ are trivial or easily computable, and using the Mayer-Vietoris sequence.

2.5. Long exact sequence of a pair. [**9**] Another useful tool is the long exact sequence for the homology groups. Let A be a subspace of a topological space X. Then we have the exact sequence

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{p_*} H_n(X,A) \xrightarrow{d} H_{n-1}(A) \to \cdots,$$

where i_* and p_* are induced by the inclusion map and the quotient map respectively, and d is the connecting homomorphism.

2.6. The excision theorem. This theorem relates the relative homology of a space to the relative homology of its subspaces.

Theorem 2.2. *[**9**] Let A and B be subspaces of a topological space X such that $X = A^o \cup B^o$. Then we have,*

$$H_n(B, A \cap B) \cong H_n(X,A),$$

and the isomorphism is induced by the inclusion map.

3. Persistent homology

The idea of persistence is basically keeping track of topological features in a filtration of a topological space (or a simplicial complex) and determine the ones that persist for a relatively long time. In this paper, the feature we are interested in is homology. As we move forward in a filtration, homology classes are *born*. Some of these classes may *die* soon and some may *persist* on staying alive for longer, or maybe forever. We call the ones that die soon *noise*, and we focus on the persisting ones.

3.1. Persistent homology for point cloud data (PCD). A data set consisting of a cloud of points is called a PCD [**8**] (Figure 1). One of the uses of persistent homology is obtaining a meaningful information about the space from which a given PCD is sampled. To be able to do that, we first need a structure on the data. Points in a PCD can be considered as 0-simplices so that a PCD can be

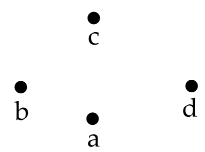

Figure 1. A PCD consisting of four points.

thought as a simplicial complex. Then, one can compute homology of this complex but it would not reveal any information about the shape of the object from which the PCD is sampled. Thus, this method is improved in the following way. For each point in a PCD, we draw balls with equal radius d centering on the point. Then we form simplices from the points which have intersecting balls. As d gets larger, we obtain a nested sequence of simplicial complexes, thus a filtered complex [**8**] of the given PCD. Figure 2 demonstrates an example of this process. Let the simplicial

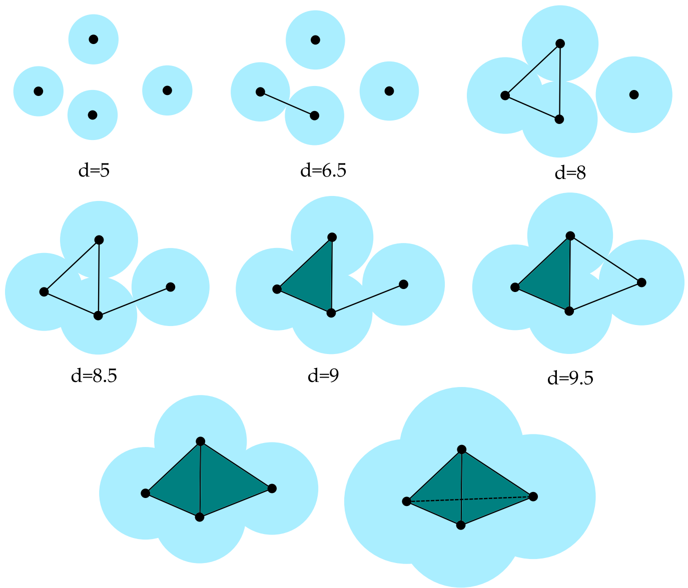

FIGURE 2. A filtration obtained from a PCD. We are interested in d values where new simplices are born.

complexes obtained in Figure 2 be $\mathcal{K}_0, ..., \mathcal{K}_7$ with respect to the order in the figure. Clearly we have

$$\mathcal{K}_0 \subset \mathcal{K}_1 \subset ... \subset \mathcal{K}_7.$$

$\mathcal{K}_0$ has four connected components. Thus $H_0(\mathcal{K}_0) \cong \mathbb{Z}^4$. For $\mathcal{K}_1$, $H_0(\mathcal{K}_1) \cong \mathbb{Z}^3$ as it has only three connected components. Although it is visible, we give an algebraic explanation of what happens as we move from $\mathcal{K}_0$ to $\mathcal{K}_1$. In $\mathcal{K}_1$, the boundary of $[a, b]$ is $[b] - [a]$. Thus, $[a]$ and $[b]$ fall in the same equivalence class in $H_1(\mathcal{K}_1)$. So, we say that a homology class dies as we pass from $\mathcal{K}_0$ to $\mathcal{K}_1$. For $\mathcal{K}_2$, we have $H_1(\mathcal{K}_2) \cong \mathbb{Z}$ as it contains a 1-dimensional hole. However, in $\mathcal{K}_4$, this hole is a boundary of a 2-simplex. Thus, a homology class was born in $\mathcal{K}_2$ but then it dies in $\mathcal{K}_4$.

DEFINITION 3.1. **(Persistent homology group)** The *nth persistent homology group* of the classes that are born before the ith complex and still alive in the jth complex is defined as [**15**]

$$H_n^{i,j}(\mathcal{K}) = \mathrm{im}\{H_n(\mathcal{K}_i) \to H_n(\mathcal{K}_j)\}.$$

EXAMPLE 3.1. We demonstrate a few computations for the filtration in Figure 2.

$$H_0^{1,2}(\mathcal{K}) = \mathrm{im}\{H_0(\mathcal{K}_1) \to H_0(\mathcal{K}_2)\}.$$

For $\mathcal{K}_1$ we have

$$\cdots \to 0 \to C_1(\mathcal{K}_1) \xrightarrow{\partial_1} C_0(\mathcal{K}_1) \xrightarrow{\partial_0} 0,$$

and $C_0(\mathcal{K}_1) = <[a],[b],[c],[d]>, C_1(\mathcal{K}_1) = <[a,b]>$. Thus we have

$$\begin{aligned} Z_0(\mathcal{K}_1) &= \ker(\partial_0) = <[a],[b],[c],[d]>, \\ B_0(\mathcal{K}_1) &= \operatorname{im}(\partial_1) = <\partial_1[a,b]> = <[b]-[a]>, \end{aligned}$$

and

$$H_0(\mathcal{K}_1) = <[a],[b],[c],[d]>/<[b]-[a]>.$$

Here we see that $[b]-[a] \in <[b]-[a]>$, so $[b] + <[b]-[a]> = [a] + <[b]-[a]>$ and finally $\overline{[b]} = \overline{[a]}$. We represent this class by $\overline{[a]}$. Then

$$H_0(\mathcal{K}_1) = <[a]+B_0(\mathcal{K}_1), [c]+B_0(\mathcal{K}_1), [d]+B_0(\mathcal{K}_1)>.$$

For $\mathcal{K}_2$ we have

$$\cdots \to 0 \to C_1(\mathcal{K}_2) \xrightarrow{\partial_1} C_0(\mathcal{K}_2) \xrightarrow{\partial_0} 0,$$

and $C_0(\mathcal{K}_2) = <[a],[b],[c],[d]>, C_1(\mathcal{K}_2) = <[a,b],[a,c],[b,c]>$. Thus we have

$$\begin{aligned} Z_0(\mathcal{K}_2) &= \ker(\partial_0) = <a,b,c,d>, \\ B_0(\mathcal{K}_2) = \operatorname{im}(\partial_1) &= <\partial_1[a,b], \partial_1[a,c], \partial_1[b,c])>, \\ &= <[b]-[a],[c]-[a],[c]-[b]>, \\ &= <[b]-[a],[c]-[a]> \quad \text{(since } [c]-[b] \text{ is spanned by the other two)}, \end{aligned}$$

and

$$H_0(\mathcal{K}_2) = <[a],[b],[c],[d]>/<[b]-[a],[c]-[a]>.$$

Here we see that $[c]-[a] \in <[c]-[a]>$, so $[c] + <[c]-[a]> = [a] + <[c]-[a]>$ and finally $\overline{[c]} = \overline{[a]}$ and similarly $\overline{[b]} = \overline{[a]}$. We keep using $\overline{[a]}$ to represent this class. Then

$$H_0(\mathcal{K}_2) = <[a]+B_0(\mathcal{K}_2), [d]+B_0(\mathcal{K}_2)>.$$

Since the map between $H_0(\mathcal{K}_1)$ and $H_0(\mathcal{K}_2)$ is induced by the composition of inclusion maps, we know that for any $z + B_0(\mathcal{K}_1)$, $i_*(z + B_0(\mathcal{K}_1)) = z + B_0(\mathcal{K}_2)$. Thus we obtain

$$\operatorname{im}\{H_0(\mathcal{K}_1) \to H_0(\mathcal{K}_2)\} = <[a]+B_0(\mathcal{K}_2), [c]+B_0(\mathcal{K}_2), [d]+B_0(\mathcal{K}_2)>.$$

We know that $[c]-[a] \in B_0(\mathcal{K}_2)$, so

$$H_0^{1,2}(\mathcal{K}) = <[a]+B_0(\mathcal{K}_2), [d]+B_0(\mathcal{K}_2)>.$$

Since there are no new points added to the space as the filtration grows, the persistent homology group $H_0^{1,2}(\mathcal{K})$ and the 0th homology group of $\mathcal{K}_2$ happened to be the same in this case.

In the figure we see that both $\mathcal{K}_2$ and $\mathcal{K}_5$ have 1-dimensional holes. The one in $\mathcal{K}_2$ dies in $\mathcal{K}_4$ and the one in $\mathcal{K}_5$ actually was born as we move to $\mathcal{K}_5$. Indeed,

$$\begin{aligned} H_1(\mathcal{K}_2) &= <[a,b]+[b,c]-[a,c]+B_1(\mathcal{K}_2)>, \\ H_1(\mathcal{K}_5) &= <[a,c]+[c,d]-[a,d]+B_1(\mathcal{K}_5)>, \\ \operatorname{im}\{H_1(\mathcal{K}_2) \to H_1(\mathcal{K}_5)\} &= <[a,b]+[b,c]-[a,c]+B_1(\mathcal{K}_5)>. \end{aligned}$$

$B_1(\mathcal{K}_5)=<\partial_2[a,b,c]>=<[a,b]+[b,c]-[a,c]>$, and it means $<[a,b]+[b,c]-[a,c]+B_1(\mathcal{K}_5)>=<0+B_1(\mathcal{K}_5)>$. Thus,

$$H_1^{2,5}(\mathcal{K}) = 0.$$

It means that there are no first homology classes that survive in the filtration from $\mathcal{K}_2$ to $\mathcal{K}_5$.

3.2. Persistent homology for real-valued functions. Another way of forming a filtration is to consider a real-valued function f and the part of the domain space that takes values less than or equal to some real number a. As we increase a, we have a growing space as desired. A similar way is to consider the part of the domain that takes values greater than or equal to a.

Definition 3.2. **(Sublevel-superlevel sets)** [**11**] Let X be a topological space and $f : X \to \mathbb{R}$ be a real-valued function. The sublevel and superlevel sets of f with respect to a real number a_k are defined as follows:

$$\begin{aligned} X_k &= \{x \in X : f(x) \leqslant a_k\}, \\ X^k &= \{x \in X : f(x) \geqslant a_k\}, \end{aligned}$$

respectively.

For $a_0 < a_1 < ... < a_n$, we have,

$$\begin{aligned} &X_0 \subset X_1 \subset \cdots \subset X_n, \\ &X^n \subset X^{n-1} \subset \cdots \subset X^0. \end{aligned}$$

In both cases, we obtain a filtration of the space.

Example 3.2. Let X be the green curve in Figure 3 and consider $f : X \to \mathbb{R}$ as the height function. Then the sublevel sets of X with respect to the values a_0, a_1, a_2, a_3 form a filtration,

$$X_0 \subset X_1 \subset X_2 \subset X_3$$

of X as in Figure 3. X_0 has only one connected component, so $H_0(X_0)$ has one generator. Let us denote its homology class by a. X_1 has two connected components, which means that we have a new 0th homology class. We call it b. As we move to X_2, these homology classes are still alive. However, in X_3, there is only one connected component. One of the homology classes dies since it falls in the same class with the other one. We use the older representer a for this class. Observe that the homology only changes when the horizontal line passes through a critical point. Since there are no critical points between a_1 and a_2, $H_0(X_1) \cong H_0(X_2)$.

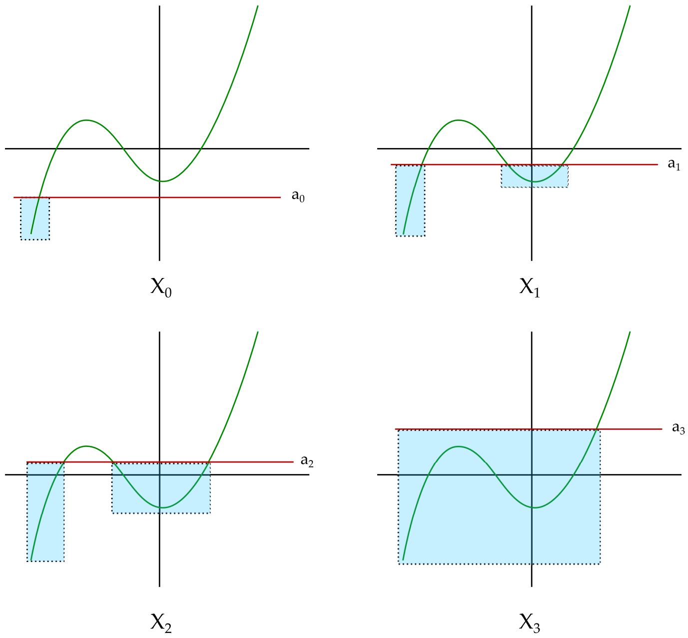

FIGURE 3. Sublevel sets of X by f.

Homology is a homotopy invariant. Two homotopy equivalent spaces have isomorphic homology groups [**9**]. However, this property does not hold in persistence setup. We demonstrate this with an example. In Figure 5, we have a rotated

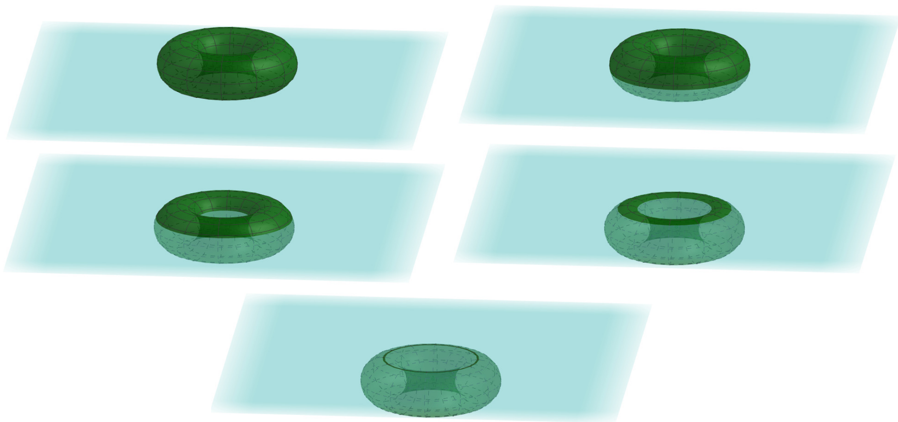

FIGURE 4. Sublevel set filtration of a horizontal torus by the height function. It consists of parts of the torus below a plane $z = z_0$.

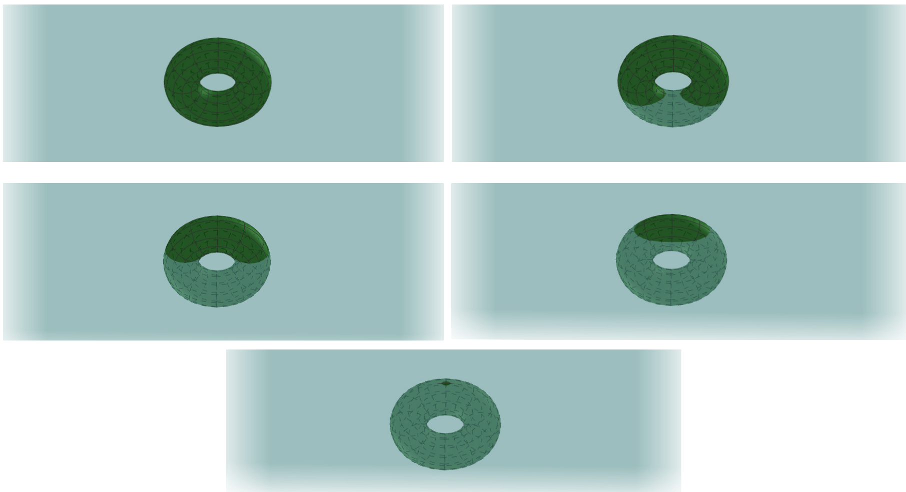

FIGURE 5. Sublevel set filtration of a vertical torus by the height function.

version of the torus from Figure 4. In both cases, the torus is placed on the xy-plane and the filtration starts with the plane $z = 0$. Let the sublevel sets of the spaces in figures be denoted by X_i and $X_i^{'}$, respectively, for $i = 0, 1, 2, 3$. Even in the first step when the plane touches the torus, the first figure has a circle as the first sublevel set X_0 in the filtration and $H_1(X_0) \cong \mathbb{Z}$. In the second figure, we have a single point as $X_0^{'}$ when the plane touches the torus. So $H_1(X_0^{'}) = 0$. Thus, the two homotopic spaces in the example will have different persistence homology groups.

3.3. Barcodes. We compute the persistent homology for the filtrations of a space, but obtaining an information about the space from the computation results requires a little bit more work. We collect all information about the birth and death steps of the homology classes and place them on a diagram. The resulting diagram is called a *barcode* [**2**]. We form the barcode of the filtration given in Figure 2 as follows.

The blue bars represent the lifetime of the homology classes. The red lines are to separate the first and 2nd homology classes. Short blue bars are called noise and ignored. Long blue bars are considered to obtain information about the features of the space.

4. Homology sequences and theorems in persistence setup

In this section, we see how the Mayer-Vietoris sequence, the long exact sequence of a pair, and the excision theorem of standard homology transfers to the persistence setup. To accomplish this, it needs a change of the algebraic point of view to the problem. Thus, the following structures are introduced.

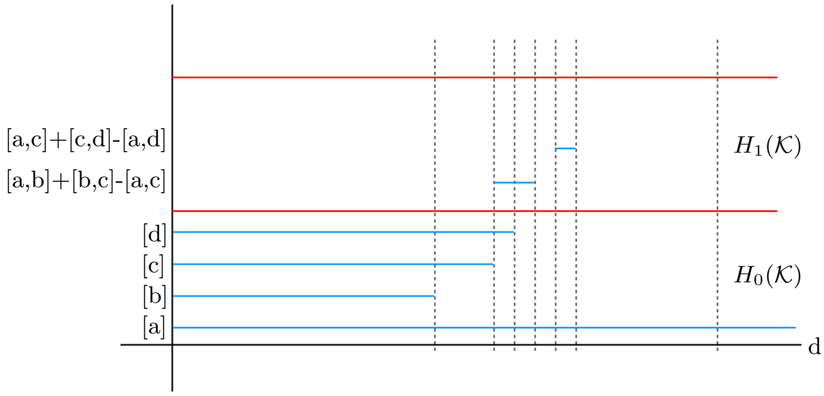

FIGURE 6. Barcode of the filtration in Figure 2.

DEFINITION 4.1. **(Persistence module)** [**15**] Let R be a ring, and let I be a partially ordered set. A *persistent module* $\mathcal{P}$ consists of P_i $(i \in I)$, which are modules over R, and maps $\psi_{i,k} : P_i \to P_k$ where $i \leqslant k$, satisfying the following conditions.

i) $\psi_{i,i}$ is the identity map on P_i.
ii) For any pair $\psi_{j,k} : P_j \to P_k$ and $\psi_{i,j} : P_i \to P_j$, the composition $\psi_{j,k} \circ \psi_{i,j}$ is equal to $\psi_{i,k} : P_i \to P_k$.

EXAMPLE 4.1. Consider a filtration of a topological space X,

$$X_0 \xrightarrow{i_{0,1}} X_1 \xrightarrow{i_{1,2}} \cdots \xrightarrow{i_{n-1,n}} X_n.$$

where $i_{k,k+1}$ is the inclusion map for $k = 0, 1, ..., n-1$. There is an identity map for each X_i. And the composition of inclusion maps $i_{j+1,j+2} \circ i_{j,j+1} = i_{j,j+2}$. We apply the functor H_k to this sequence. The identity maps are mapped to the identity maps, $H_k(i_{X_j}) = i_{H_k(X_j)}$. And,

$$\begin{aligned} i_{j+1,j+2} \circ i_{j,j+1} &= i_{j,j+2} \\ \Rightarrow H_k(i_{j+1,j+2} \circ i_{j,j+1}) &= H_k(i_{j,j+2}) \\ \Rightarrow H_k(i_{j+1,j+2}) \circ H_k(i_{j,j+1}) &= H_k(i_{j,j+2}) \\ \Rightarrow i_{*_{j+1,j+2}} \circ i_{*_{j,j+1}} &= i_{*_{j,j+2}}. \end{aligned}$$

Then, the sequence

$$H_k(X_0) \xrightarrow{i_{*_{0,1}}} H_k(X_1) \xrightarrow{i_{*_{1,2}}} \cdots \xrightarrow{i_{*_{n-1,n}}} H_k(X)$$

forms a persistence module.

Definition 4.2. Suppose we have two persistence modules $\mathcal{P}$ and $\mathcal{Q}$ over the same index set I,

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P_i & \xrightarrow{\psi_{i,j}} & P_j & \xrightarrow{\psi_{j,k}} & P_k & \longrightarrow & \cdots \\ & & \downarrow f_i & & \downarrow f_j & & \downarrow f_k & & \\ \cdots & \longrightarrow & Q_i & \xrightarrow{\phi_{i,j}} & Q_j & \xrightarrow{\phi_{j,k}} & Q_k & \longrightarrow & \cdots . \end{array}$$

If we have a family of linear maps $f_n : P_n \to Q_n$ such that the diagram above commutes, then this family is called a *morphism* between $\mathcal{P}$ and $\mathcal{Q}$, and it is denoted by $f : \mathcal{P} \to \mathcal{Q}$. If, in addition, each f_n is an isomorphism, then f is an *isomorphism of persistence modules* $\mathcal{P}$ and $\mathcal{Q}$.

Let $\mathbb{F}$ be a field and $\mathbb{F}[t]$ be the graded polynomial ring with the standard grading. We consider the direct sum of all $H_k(X_i)$. By defining the operation

$$t(z_0, z_1, z_2, ...) = \left(0, i_{*_{0,1}}(z_0), i_{*_{1,2}}(z_1), i_{*_{2,3}}(z_2)...\right),$$

we obtain an $\mathbb{F}[t]$-module structure of the persistence module and denote it by

$$\mathcal{H}_k(X) = \bigoplus_i H_k(X_i). \text{ [15]}$$

We bring the kth homology groups of all subspaces in the filtration together and form a graded module structure. Each time we apply t, we move one step upward in the grading. It means a one step shift of the birth time and the death time of a homology class.

4.1. Mayer-Vietoris sequence. To build a structure for Mayer-Vietoris sequence in persistent homology, we begin with some observations. Let X be a topological space and $A, B \subset X$ such that $X = A^o \cup B^o$. We consider filtrations of A and B such that,

$$X_j = A_j \cup B_j,$$
$$(A \cap B)_j = A_j \cap B_j.$$

For a step j in the filtration, we consider the nth homology group of X_j, A_j, B_j and $(A \cap B)_j$. We have the Mayer-Vietoris sequence

$$\cdots \to H_n((A \cap B)_j) \to H_n(A_j) \oplus H_n(B_j) \to H_n(X_j) \to H_{n-1}((A \cap B)_j) \to \cdots .$$

Also, consider the Mayer-Vietoris sequence at step k where $j < k$,

$$\cdots \to H_n((A \cap B)_k) \to H_n(A_k) \oplus H_n(B_k) \to H_n(X_k) \to H_{n-1}((A \cap B)_k) \to \cdots .$$

We connect these sequences vertically by the maps induced by the inclusion maps in the filtrations.

$$\begin{array}{ccccccccc} \cdots \to & H_n((A \cap B)_j) & \xrightarrow{\alpha_j} & H_n(A_j) \oplus H_n(B_j) & \xrightarrow{\beta_j} & H_n(X_j) & \xrightarrow{d_j} & H_{n-1}((A \cap B)_j) & \to \cdots \\ & \downarrow i^1_{*_n} & & \downarrow i^2_{*_n} & & \downarrow i^3_{*_n} & & \downarrow i^1_{*_{n-1}} & \\ \cdots \to & H_n((A \cap B)_k) & \xrightarrow{\alpha_k} & H_n(A_k) \oplus H_n(B_k) & \xrightarrow{\beta_k} & H_n(X_k) & \xrightarrow{d_k} & H_{n-1}((A \cap B)_k) & \to \cdots . \end{array}$$

Consider, say, $\operatorname{im}(i^3_{*_n}) = \operatorname{im}\{H_n(X_j) \to H_n(X_k)\}$. This is exactly the definition of the persistent homology group $H_n^{j,k}(X)$. The same is true for each column above. If we restrict α_k, β_k and d_k to the images of the vertical maps, then we have the following sequence of persistent homology groups [**4**],

$$\cdots \to H_n^{j,k}(A \cap B) \xrightarrow{\alpha} H_n^{j,k}(A) \oplus H_n^{j,k}(B) \xrightarrow{\beta} H_n^{j,k}(X) \xrightarrow{d} H_{n-1}^{j,k}(A \cap B) \to \cdots .$$

This sequence is a chain complex, but it is not exact in general. We demonstrate it with an example.

Example 4.2. Consider Figure 5, where we form the sublevel sets of the torus. Let X be the torus and pick A and B as in Figure 7 such that $X = A^o \cup B^o$ holds.

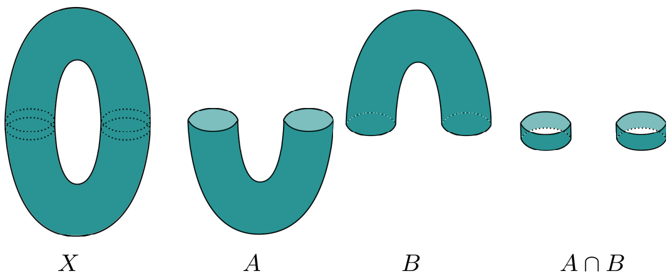

Figure 7. A topological space X, and its subspaces A, B and $A \cap B$.

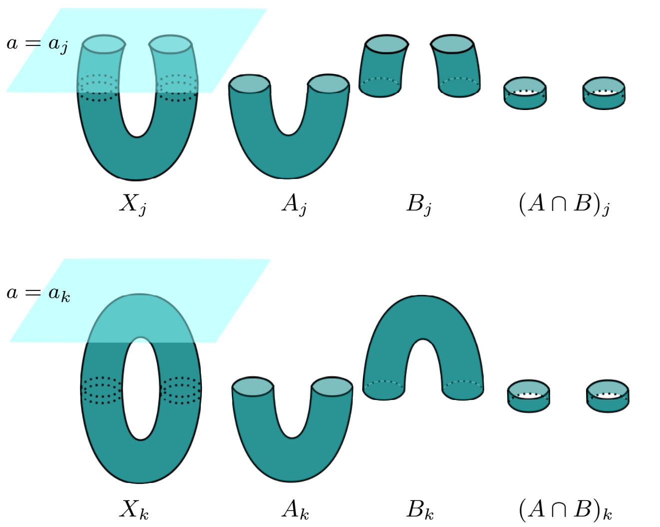

Figure 8. Sublevel sets of X, A, B and $A \cap B$ for a_j and a_k.

For $a = a_j$ and $a = a_k$, form a two step sublevel set filtration of spaces X, A, B, and $A \cap B$ as in Figure 8. In the a_j sublevel sets, $A_j \cap B_j$ has two 1-dimensional holes. So $H_1((A\cap B)_j)$ has two generators. Call them a and b. X_j does not have any enclosed voids, which means $H_2(X_j)$ is trivial. A_j has one 1-dimensional hole, thus one generator for $H_1(A_j)$ since a and b are the same classes at this step. We use a to represent this homology class. As B_j and $(A\cap B)_j$ are homeomorphic, $H_1(B_j) = <a, b>$. $H_1((A\cap B)_k)$ and $H_1(A_k)$ are the same as $H_1((A\cap B)_j)$ and $H_1(A_j)$ as the spaces remain the same in this step. $H_1(B_k)$ is $<a>$ as a and b are the same classes. And $H_2(X_k)$ has one generator since it has an enclosed void. Call this generator c. With these observations, we form the persistent homology groups by considering the homology classes that were alive at step j and remain alive at step k. Then we have the sequence

$$\cdots \to H_2^{j,k}(X) \xrightarrow{d} H_1^{j,k}(A\cap B) \xrightarrow{\alpha} H_1^{j,k}(A)\oplus H_1^{j,k}(B) \xrightarrow{\beta} H_1^{j,k}(X) \to \cdots,$$

which also could be expressed as

$$\cdots \to 0 \xrightarrow{d} <a,b> \xrightarrow{\alpha} <a> \oplus <a> \xrightarrow{\beta} <a> \to \cdots.$$

In this sequence im(d)=0. But since $\alpha(b) = 0$, we see that $b \in \ker(\alpha)$. It means im$(d) \neq$ker(α). Thus, the sequence is not exact.

To overcome this problem and obtain an exact sequence, instead of considering persistent homology groups, we consider persistent homology modules.

$$\cdots \to \mathcal{H}_k(A\cap B) \xrightarrow{\alpha} \mathcal{H}_k(A)\oplus\mathcal{H}_k(B) \xrightarrow{\beta} \mathcal{H}_k(X) \xrightarrow{d} \mathcal{H}_{k-1}(A\cap B) \to \cdots.$$

Here we define

$$\begin{aligned} \alpha &= (\alpha_0, \alpha_1, ..., \alpha_n),\\ \beta &= (\beta_0, \beta_1, ..., \beta_n),\\ \text{and}\ \ d &= (d_0, d_1, ..., d_n). \end{aligned}$$

where α_i, β_i and d_i are the maps of the Mayer-Vietoris sequence at step i. Since the Mayer-Vietoris sequence is exact for each step i, we have

$$\begin{aligned} \text{im}(\alpha_i) &= \ker(\beta_i),\\ \text{im}(\beta_i) &= \ker(d_i),\\ \text{and}\ \ \text{im}(d_i) &= \ker(\alpha_i). \end{aligned}$$

For each $i \in \{0, ..., n\}$, the equalities above hold. We combine these with how we define α, β and d and obtain,

$$\begin{aligned} \text{im}(\alpha) &= \ker(\beta),\\ \text{im}(\beta) &= \ker(d),\\ \text{and}\ \ \text{im}(d) &= \ker(\alpha), \end{aligned}$$

which means that the Mayer-Vietoris sequence of persistent homology modules is exact [**14**]. We show how this approach solves the problem that we face in the above example. Consider the whole filtration of the torus as in Figure 9. a_j and a_k in our example correspond to a_2 and a_4 here, respectively.

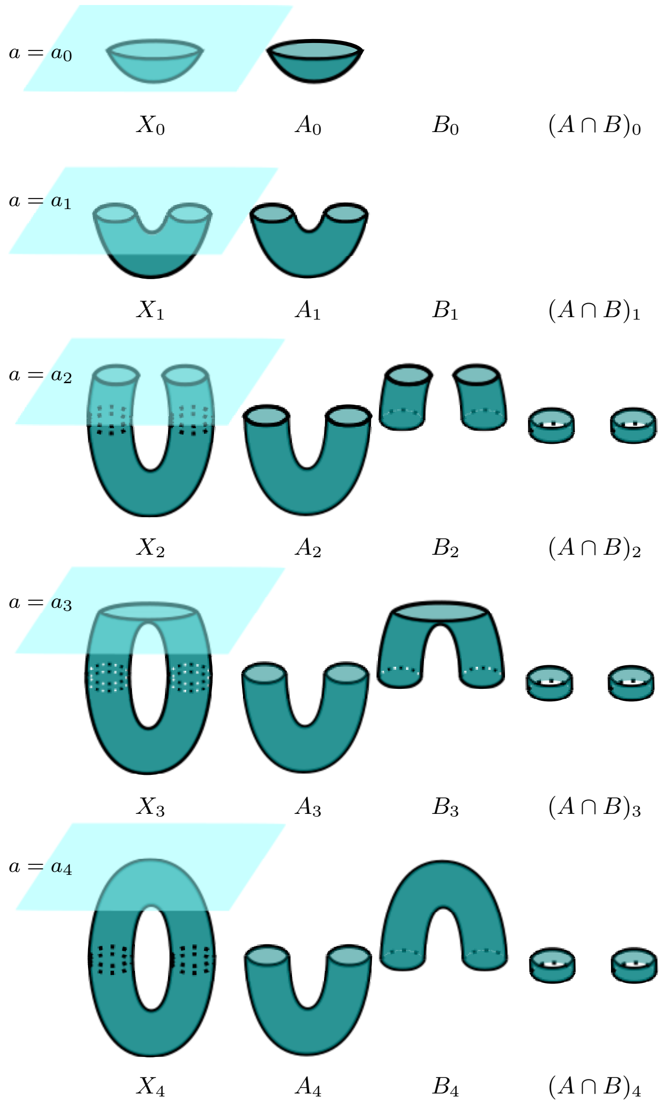

FIGURE 9. A filtration of X, A, B and $A \cap B$.

First, we give an example of an element of $\mathcal{H}_1(B)$. Consider the first homology groups of all B_i in the filtration in a direct sum. Then, we have examples of elements such as,

$$(0, 0, a, 0, 0) \in \mathcal{H}_1(B) \text{ or } (0, 0, b, 0, 0) \in \mathcal{H}_1(B).$$

The problematic homology class in the example was the class b and it occurs in step $j = 2$ and it is an element of $\ker(\alpha)$. In the persistent module, the corresponding element is $(0, 0, b, 0, 0)$. Considering the definition of the map α in module setup, we see that,

$$\alpha((0,0,b,0,0)) = ((0,0,0,0,0),(0,0,b,0,0)) \in \mathcal{H}_1(A) \oplus \mathcal{H}_1(B),$$

which means this element is not in $\ker(\alpha)$ anymore. If we apply t two times, then we have,

$$t^2(0,0,b,0,0) = (0,0,0,0,b),$$

so, this element is in $\ker(\alpha)$ as

$$\alpha((0,0,0,0,b)) = ((0,0,0,0,0),(0,0,0,0,0)),$$

and we are at step $k = 4$. We see that X has an enclosed void at step 4. So, $H_2(X)$ has c as a generator. In the module setup, we have an element

$$(0,0,0,0,c).$$

We take the image of this element under the map d and obtain,

$$d((0,0,0,0,c)) = (0,0,0,0,b),$$

because the Mayer-Vietoris sequence in the last step gives us $d_4(c) = b$. Thus, the exactness of the sequence is preserved.

4.2. The long exact sequence. We follow a similar procedure to form a long exact sequence of persistent homology. Consider a topological space X, a subspace $A \subset X$, the pair (X, A), and the filtrations of these spaces. For the steps j and k of a filtration where $j < k$, we have the long exact sequences and we have the maps induced by inclusion maps to connect these sequences,

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H_n(A_j) & \xrightarrow{\gamma_j} & H_n(X_j) & \xrightarrow{\delta_j} & H_n(X_j, A_j) & \xrightarrow{d_j} & H_{n-1}(A_j) & \longrightarrow \cdots \\
& \Big\downarrow {\scriptstyle i^1_{*n}} & & \Big\downarrow {\scriptstyle i^2_{*n}} & & \Big\downarrow {\scriptstyle i^3_{*n}} & & \Big\downarrow {\scriptstyle i^1_{*n-1}} & \\
\cdots \longrightarrow & H_n(A_k) & \xrightarrow{\gamma_k} & H_n(X_k) & \xrightarrow{\delta_k} & H_n(X_k, A_k) & \xrightarrow{d_k} & H_{n-1}(A_k) & \longrightarrow \cdots .
\end{array}$$

We consider the images of vertical maps and obtain the persistent homology groups of the spaces. Defining the maps γ, δ and d as restrictions of γ_k, δ_k and d_k to the images of the vertical maps, we have the following sequence [**14**]

$$\cdots \to H_n^{j,k}(A) \xrightarrow{\gamma} H_n^{j,k}(X) \xrightarrow{\delta} H_n^{j,k}(X, A) \xrightarrow{d} H_{n-1}^{j,k}(A) \to \cdots .$$

This sequence is a chain complex. However, for the same reason as the Mayer-Vietoris sequence of the persistent homology groups, it is not exact. At step k we have the exactness of the kth horizontal sequence, but it is not always the case when the maps γ, δ and d of step k are restricted to the images of the vertical maps.

Example 4.3. Consider the case given in Figure 10. We compute the homology groups as,

$$\begin{array}{lll}
H_1(X_j/A_j) = 0, & H_0(A_j) = <a,b>, & H_0(X_j) = <a,b>, \\
H_1(X_k/A_k) = <U,V>, & H_0(A_k) = <a,b>, & H_0(X_k) = <a>.
\end{array}$$

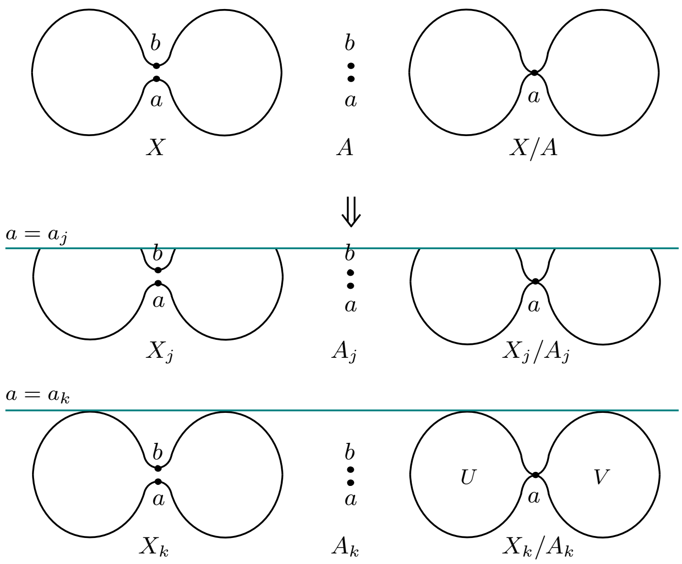

FIGURE 10. Sublevel sets at a_j, a_k of X, A and X/A.

We form the persistent homology groups as follows :

$$\cdots \to H_1^{j,k}(X,A) \xrightarrow{\gamma} H_0^{j,k}(A) \xrightarrow{\delta} H_0^{j,k}(X) \xrightarrow{d} \cdots ,$$

which also could be expressed in the following form since in this case we have $H_1(X,A) \cong H_1(X/A)$:

$$\cdots \to 0 \xrightarrow{\gamma} <a,b> \xrightarrow{\delta} <a> \xrightarrow{d} \cdots .$$

We see that $\ker(\delta) = <b> \neq 0 = \mathrm{im}(\gamma)$, and therefore the sequence is not exact. We change the approach as in the Mayer-Vietoris sequence case, and use persistent homology modules. Consider the sequence

$$\cdots \to \mathcal{H}_n(A) \xrightarrow{\gamma} \mathcal{H}_n(X) \xrightarrow{\delta} \mathcal{H}_n(X,A) \xrightarrow{d} \mathcal{H}_{n-1}(A) \to \cdots ,$$

where

$$\begin{aligned} \gamma &= (\gamma_0, \gamma_1, ..., \gamma_n), \\ \delta &= (\delta_0, \delta_1, ..., \delta_n), \\ \text{and} \quad d &= (d_0, d_1, ..., d_n). \end{aligned}$$

The long sequence at each step i is exact, which means

$$\begin{aligned} \mathrm{im}(\gamma_i) &= \ker(\delta_i), \\ \mathrm{im}(\delta_i) &= \ker(d_i), \\ \text{and} \quad \mathrm{im}(d_i) &= \ker(\gamma_i). \end{aligned}$$

The way we defined γ, δ and d gives us

$$\begin{aligned} \operatorname{im}(\gamma) &= \ker(\delta), \\ \operatorname{im}(\delta) &= \ker(d), \\ \text{and} \quad \operatorname{im}(d) &= \ker(\gamma). \end{aligned}$$

Therefore, the long sequence of persistent homology modules is exact [**14**]. Instead of the restrictions of the maps, we consider the maps themselves in module setup. Then, the way we define the maps between the modules lets us make use of the exactness of the sequences at each step. This solves the problems we face in the group setup.

4.3. The excision theorem. For a relatively complicated space X, instead of trying to compute the homology classes directly, we can choose suitable subspaces $A, B \subset X$ such that $X = A^o \cup B^o$ and apply excision to extract information we seek. To carry this idea to the persistence setup, we first need to make sure that at every step j in the filtration, we have the property $X_j = A_j^o \cup B_j^o$, where interiors are with respect to X_j. Consider the subspace topology on each subspace in the filtration. Recall that by taking any open set in X and intersect it with X_j, we obtain an open set in the subspace topology on X_j. By defining $A_j = A \cap X_j$ and $B_j = A \cap X_j$, we derive filtrations of A and B. Since we assume $X = A^o \cup B^o$, for all three fitrations we see that [**11**],

$$\begin{aligned} X_j = X_j \cap X &= X_j \cap (A^o \cup B^o), \\ &= (X_j \cap A^o) \cup (X_j \cap B^o), \\ &= (X_j \cap A^o)^o \cup (X_j \cap B^o)^o \quad \text{(since these sets are open in } X_j), \\ &\subseteq (X_j \cap A)^o \cup (X_j \cap B)^o, \\ &= A_j^o \cup B_j^o. \end{aligned}$$

Thus, for every step in the filtration, we have the excision theorem. Consider steps j and k such that $j < k$. Then we have,

$$\begin{array}{ccc} H_n(B_j, B_j \cap A_j) & \xrightarrow{f_j} & H_n(X_j, A_j) \\ \big\downarrow i_*^1 & & \big\downarrow i_*^2 \\ H_n(B_k, B_k \cap A_k) & \xrightarrow{f_k} & H_n(X_k, A_k), \end{array}$$

where f_j and f_k are isomorphisms. Since all of the maps in the diagram are induced by inclusion maps, the diagram is commutative.

We express the diagram as in Figure 11. From the commutativity of the diagram, we have

$$\operatorname{im}(i_*^2 \circ f_j) = \operatorname{im}(f_k \circ i_*^1).$$

Since f_j is an isomorphism,

$$\operatorname{im}(i_*^2 \circ f_j) = \operatorname{im}(i_*^2).$$

Moreover,

$$\operatorname{im}(f_k \circ i_*^1) = \operatorname{im}(f_k|_{\operatorname{im}(i_*^1)}).$$

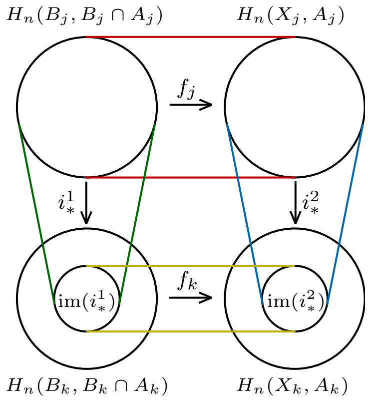

FIGURE 11. A visual representation of the square diagram.

Bringing these observations together, we have,

$$\mathrm{im}(i_*^2) = \mathrm{im}(f_k|_{\mathrm{im}(i_*^1)}).$$

Since f_k is an isomorphism, the restriction $f_k|_{\mathrm{im}(i_*^1)}$ is also an isomorphism between $\mathrm{im}(i_*^1)$ and $\mathrm{im}(i_*^2)$. In our case, from the definition of persistent homology groups we have

$$\mathrm{im}(i_*^1) = H_n^{j,k}(B, B \cap A),$$
$$\mathrm{im}(i_*^2) = H_n^{j,k}(X, A).$$

Then, we see that the excision theorem holds for persistent homology groups [**11**] as,

$$H_n^{j,k}(B, B \cap A) \cong H_n^{j,k}(X, A).$$

In the case of persistent homology modules, since f_i is an isomorphism at each step i, $f = (f_0, f_1, ..., f_n)$ is also an isomorphism between the persistent homology modules. That is

$$\mathcal{H}_n(B, B \cap A) \cong \mathcal{H}_n(X, A),$$

which means that the excision theorem also holds for persistent homology modules. [**11**].

EXAMPLE 4.4. Consider Figure 12, where we have subspaces A and B of a topological space X such that $X = A^o \cup B^o$. The sublevel set filtrations of these spaces are also given in the figure. We form quotient spaces as in Figure 13, and determine their first homology groups.

We see in Figure 13 that both X_0/A_0 and $B_0/(B_0 \cap A_0)$ have no 1-dimensional holes, which means that their first homology groups are trivial. All the other spaces in Figure 13 have three 1-dimensional holes, each. Thus, their first homology groups have three generators, each. We represent these classes by a, b and c. Bringing these

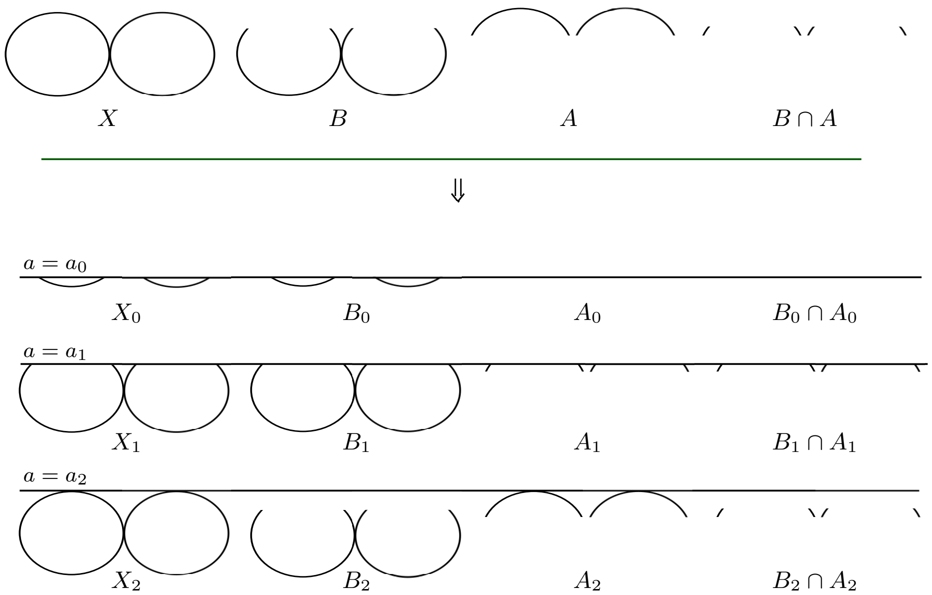

FIGURE 12. A sublevel set filtration of a topological space X, and its subspaces A, B and $B \cap A$.

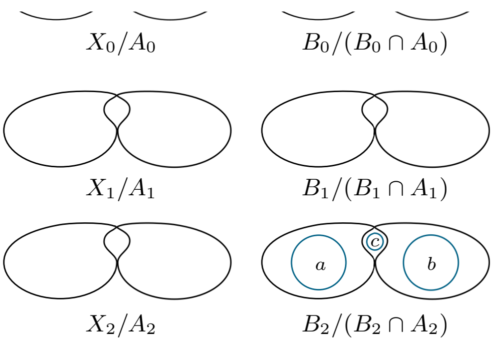

FIGURE 13. Quotient spaces obtained from the filtration in Figure 12.

together, we obtain the relative homology groups,

$$
\begin{aligned}
&H_1(X_0, A_0) = 0, && H_1(B_0, (B_0 \cap A_0)) = 0, \\
&H_1(X_1, A_1) = <a, b, c>, && H_1(B_1, (B_1 \cap A_1)) = <a, b, c>, \\
&H_1(X_2, A_2) = <a, b, c>, && H_1(B_2, (B_2 \cap A_2)) = <a, b, c>.
\end{aligned}
$$

We compute the persistent homology groups,

$$H_1^{0,1}(X,A)=0, \qquad H_1^{0,1}(B,B\cap A)=0,$$
$$H_1^{0,2}(X,A)=0, \qquad H_1^{0,2}(B,B\cap A)=0,$$
$$H_1^{1,2}(X,A)= <a,b,c>, \qquad H_1^{1,2}(B,B\cap A)= <a,b,c>.$$

We observe that,

$$H_1^{0,1}(X,A)\cong H_1^{0,1}(B,B\cap A),$$
$$H_1^{0,2}(X,A)\cong H_1^{0,2}(B,B\cap A),$$
$$H_1^{1,2}(X,A)\cong H_1^{1,2}(B,B\cap A).$$

We verify in the example that the excision theorem holds for each first persistent homology group.

5. Conclusion

Data analysis is a part of many research areas. As the datasets get more complex day by day, we need better and faster computation methods to be able to keep analysing the data effectively. Any improvements to the methods being used not only extends the theory but also increase the efficiency in the applications. Thus, carrying computation methods of usual homology to persistent homology plays a key role in the improvement of data analysis via persistent homology.

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