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HERMITE-HADAMARD-TYPE INEQUALITIES FOR EXPONENTIAL TYPE HARMONICALLY $M(\alpha, S)$-CONVEX FUNCTIONS

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Abstract. In this paper, we introduce a new form of convex mapping known as exponential type harmonically $m(\alpha, s)$-convex function. The generalized convex function for the product of functions of exponential type Hermite-Hadamard inequalities are as well studied. Additionally, we investigate different cases of double and triple integrals to obtain some new products of this function.

1. Introduction

The theory of convex functions has become highly important in different areas of mathematics and their use in optimization and modern analysis can't be underrated. Their applications generalized and extended in different directions by several authors because of their adaptivity. For proper details, (see [1-20]). The generalization of classical convexity is known to have gotten so many applications out of which the Hermite-Hadamard inequality is mostly used. For example, Bital and Khan [8] introduced some new Hermite-Hadamard dual inequalities, Tariq et al. [24] introduced Hermite-Hadamard-type inequalities for product of functions by using convex functions, Noor et al. [19] introduced (α, m, h)-convex function and also found few fundamental inequalities for the class of twice-differentiable functions, Toader [23] introduced m-convex function and established some results

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on Hermite-Hadamard type inequalities. For further study on this, please refer to the stated articles ([7-23]), [16, 26].

Motivated by previous studies in literature, researchers were moved to establish another class of convexity know as exponential convexity which has been developed in several ways. For example, Dragomir and Gomm [10] introduced the concept of exponential type convexity, Awan et. al [2] introduced Hemite-Hadamard inequalities for exponential convex functions, Kadakal and Iscan [16] introduced exponential type convexity and some related inequalities. For proper study on exponential convexity, see [4] and its references. Propelled by the work of Tariq et al., we introduce exponential-type harmonically $m(\alpha, s)$- convex functions and establish Hermite-Hadamard-type inequalities for functions of this class. Some special results are studied by introducing different cases of double and triple integrals on this functions. Our results generalize different classes of convexity functions, as exponential type convex function, harmonically p-function, harmonically (α, s)- convex function, exponential type harmonically (α, s)-convex function, (α, m, h)-convex function and (α, m)-convex functions.

2. Preliminaries

Some well known definitions which are related to our work are stated out in this section.

Definition 2.1. ([2]) Let $f : I \to (0, \infty)$ be a mapping, then f is m-convex if, for every $x, y \in I, m \in [0, 1]$ and $\lambda \in [0, 1]$

$$f(\lambda x + m(1-\lambda)y) \leqslant \lambda f(x) + m(1-\lambda)f(y). \tag{2.1}$$

Definition 2.2. ([18]) Let $f : I \to (0, \infty)$ be a mapping, then f is (α, m)-convex if, for every $x, y \in I, (\alpha, m) \in [0, 1]^2$ and $\lambda \in [0, 1]$

$$f(\lambda x + m(1-\lambda)y) \leqslant \lambda^\alpha f(x) + (1-\lambda^\alpha)f(y). \tag{2.2}$$

Definition 2.3. ([19]) Let $f : I \to (0, \infty)$ be a mapping, then f is (α, m, h)-convex if, for all $x, y \in I, \lambda \in [0, 1], (\alpha, m) \in [0, 1]^2$ and $h : [0, 1] \longrightarrow [0, 1]$,

$$f(\lambda x + m(1-\lambda)y) \leqslant h(\lambda^\alpha f(x)) + mh(1-\lambda^\alpha)f(y). \tag{2.3}$$

Definition 2.4. [16]A nonnegative function $f : I \longrightarrow \mathbb{R}$ is said to be exponential type convex, if for all $x, y \in I$ and $\lambda \in [0, 1]$

$$f(\lambda y + (1-\lambda)x) \leqslant (e^{1-\lambda} - 1)f(x) + (e^\lambda - 1)f(y). \tag{2.4}$$

Definition 2.5. ([14]) Let $f : I \subset [0, \infty) \longrightarrow \mathbb{R}$ be a mapping and I be the real interval, then

$$f\left(\frac{xy}{\lambda x + (1-\lambda)y}\right) = \lambda f(y) + (1-\lambda)f(x), \quad \forall \ x, y \in I \ \ and \ \ \lambda \in [0, 1] \tag{2.5}$$

is called harmonically convex function.

Definition 2.6. ([1]) A nonnegative function $f : I \longrightarrow \mathbb{R}$ is said to be exponential type harmonically $(\alpha, s)_h$-convex function if $\forall \ x, y \in I, \ \ h(.) > 0, \lambda \in (0, 1]$

and $(\alpha, s) \in (0,1)$

$$f(\frac{xy}{\lambda y+(1-\lambda)x}) = f(\frac{\lambda}{x}+\frac{1-\lambda}{y})^{-1} \leqslant (e^{h(1-\lambda^{\alpha})^{s}}-1)f(x)+(e^{h(\lambda^{\alpha})^{s}}-1)f(y). \tag{2.6}$$

THEOREM 2.1. *([20]). Let f and g be nonnegative convex functions on $[\phi_1, \phi_2]$, suppose they are real valued. Then, the following inequalities hold*

$$\frac{1}{\phi_2-\phi_1}\int_{\phi_1}^{\phi_2} f(x)g(x)dx \leqslant \frac{1}{3}M(\phi_1,\phi_2)+\frac{1}{6}N(\phi_1,\phi_2), \tag{2.7}$$

$$2f(\frac{\phi_1+\phi_2}{2})g(\frac{\phi_1+\phi_2}{2}) \leqslant (\frac{1}{\phi_2-\phi_1})\int_{\phi_1}^{\phi_2} f(x)g(x)dx+\frac{1}{6}M(\phi_1,\phi_2)+\frac{1}{3}N(\phi_1,\phi_2), \tag{2.8}$$

where
$M(\phi_1,\phi_2)=f(\phi_1)g(\phi_1)+f(\phi_2)g(\phi_2)$, $N(\phi_1,\phi_2)=f(\phi_1)g(\phi_2)+f(\phi_2)g(\phi_1)$.

THEOREM 2.2. *([25]). Suppose f and g are two functions, if*

$$(f(x)-f(y))(g(x)-g(y)) \geqslant 0$$

holds for all $x, y \in \mathbb{R}$, then f and g are said to be similarly ordered functions.

3. Main results

We define the following:

DEFINITION 3.1. A nonnegative function $f: I \longrightarrow \mathbb{R}$ is said to be exponential type harmonically $m(\alpha, s)$-convex, if for all $x, y \in I$, $\lambda \in (0,1)$ and $m(\alpha, s) \in (0,1]^2$

$$f(\frac{xy}{\lambda y+m(1-\lambda)x}) = f((\frac{\lambda}{x}+\frac{m(1-\lambda)}{y})^{-1}) \leqslant (e^{mh(1-\lambda^{\alpha})^{s}}-1)f(x)+(e^{h(\lambda^{\alpha})^{s}}-1)f(y). \tag{3.1}$$

REMARK 3.1. Some notable results of exponential type harmonically $m(\alpha, s)$-convex functions:

(1) If $(e^{mh(1-\lambda^{\alpha})^{s}}-1)=1$, $(e^{h(\lambda^{\alpha})^{s}}-1)=1$ and m = 1 in inequality (3.1), we recover harmonically p-function [15].

(2) On the condition $f(\frac{xy}{\lambda y+(1-\lambda)x}) \leqslant f(\lambda x+(1-\lambda)y)$, $h=I$ giving

that $\lambda > \lambda^2$, $\lambda^2 \leqslant \lambda$ which implies $h(\lambda)=\lambda$, $\alpha=m=s=1$ in inequality (3.1), then it becomes exponential type-convex function [16].

(3) If $e^{h(a)} = h(a) + 1$, which implies $e^{h(\lambda^\alpha)^s} - 1 = h(\lambda^\alpha)^s$ and m = 1 in inequality (3.1), then it gives the definition of harmonically (α, s)-convex function [22].

(4) If $m = 1$ in inequality (3.1), we recover exponential type harmonically $(\alpha, s)_h$-convex function in [1].

(5) If $f(\frac{xy}{\lambda y + (1-\lambda)x}) \leqslant f(\lambda x + (1-\lambda)y)$ and $e^{h(\cdot)} = h(\cdot) + 1 \Rightarrow e^{h(\cdot)} - 1 = h(\cdot)$

which implies $e^{h(\lambda^\alpha)^s} - 1 = h(\lambda^\alpha)^s$ and $e^{mh(1-\lambda^\alpha)^s} - 1 = mh(1-\lambda^\alpha)^s$,

with $s = 1$ in (3.1), we recover (α, m, h)-convex function in [19].

(6) In addition to the stated conditions in (5), if $h(\lambda) = \lambda$, $h(1-\lambda) = 1 - \lambda$ in (3.1), we recover (α, m)-convex function in [25].

LEMMA 3.1. *For all $x, y \in I$, $\lambda \in [0, 1]$ and $e^{h(.)} = h(.) + 1$, the following hold*

$$f(\frac{xy}{\lambda y + (1-\lambda)x}) \leqslant e^{h(\lambda)x + h(1-\lambda)y} - 1. \tag{3.2}$$

PROOF. We start by using left hand side of (11) to have

$$\begin{aligned} f(\frac{xy}{\lambda y + (1-\lambda)x}) &\leqslant (\frac{\lambda y + (1-\lambda)x}{xy})^{-1} \\ &\leqslant \frac{1}{\frac{\lambda y + (1-\lambda)x}{xy}} \\ &\leqslant \frac{1}{\lambda\frac{1}{x} + (1-\lambda)\frac{1}{y}} \end{aligned}$$

take $X = 1/x$ and $Y = 1/y$, it becomes

$$\begin{aligned} f(\frac{1}{\lambda X + (1-\lambda)Y}) &\leqslant (\lambda x + (1-\lambda)y) \\ &\leqslant e^{h(\lambda)x + h(1-\lambda)y} - 1 \end{aligned} \tag{3.3}$$

that is,

$$f(\frac{xy}{\lambda y + (1-\lambda)x}) \leqslant e^{h(\lambda)x + (1-\lambda)y} - 1. \tag{3.4}$$

□

THEOREM 3.1. *Let f and g be two exponential type harmonically $m(\alpha, s)$-convex functions on $[\phi_1, \phi_2]$, f and g being a nonnegative real-valued functions with*

$fg \in [\phi_1, \phi_2]$, where $\phi_1, b \in I$ and $\phi_1 < \phi_2$, $e^{h(\lambda^\alpha)^s} - 1 \geqslant 0$, $(e^a - 1)(e^b - 1) \leqslant e^{ab} - 1$. Then, the following inequalities hold.

$$\frac{1}{m\phi_1 - \phi_2} \int_{\phi_2}^{m\phi_1} f(x)g(x)dx \leqslant q(\phi_1\phi_2)p + q(\phi_1\phi_2)N + q(\phi_1, \phi_2)R, \tag{3.5}$$

where

$$\begin{aligned} q(\phi_1, \phi_2) &= f(\phi_1)g(\phi_1) \\ q(\phi_2, \phi_2) &= f(\phi_2)g(\phi_2) \\ q(\phi_1, \phi_2) &= f(\phi_1)g(\phi_2) + f(\phi_2)g(\phi_1), \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} p &= \int_0^1 (e^{[h(\lambda^\alpha)^s]^2} - 1)d\lambda \\ N &= \int_0^1 (e^{m^2[h(1-\lambda^\alpha)^s]^2} - 1)d\lambda \\ R &= \int_0^1 (e^{mh(\lambda^\alpha)^s h(1-\lambda^\alpha)^s} - 1)d\lambda. \end{aligned} \tag{3.7}$$

Proof. Since f and g are exponential type harmonically $m(\alpha, s)$-convex functions on $[\phi_1, \phi_2]$. It follows that,

$$\begin{aligned} & f(\frac{\phi_1\phi_2}{\lambda\phi_2 + m(1-\lambda)\phi_1})g(\frac{\phi_1\phi_2}{\lambda\phi_2 + m(1-\lambda)\phi_1}) \\ & \leqslant [(e^{h(\lambda^\alpha)^s} - 1)f(\phi_1) + (e^{mh(1-\lambda^\alpha)^s} - 1)f(\phi_2)] \\ & \times[(e^{h(\lambda^\alpha)^s} - 1)g(\phi_1) + (e^{mh(1-\lambda^\alpha)^s} - 1)g(\phi_2)] \\ & \leqslant [(e^{h(\lambda^\alpha)^s} - 1)(e^{h(\lambda^\alpha)^s} - 1)](f(\phi_1)g(\phi_1)) \\ & +[(e^{h(\lambda^\alpha)^s} - 1)(e^{mh(1-\lambda^\alpha)} - 1)](f(\phi_1 g(\phi_2))) \\ & +[(e^{mh(1-\lambda^\alpha)^s} - 1)(e^{h(\lambda^\alpha)^s} - 1)](f(\phi_2)g(\phi_1)) \\ & +[(e^{mh(1-\lambda^\alpha)^s} - 1)(e^{mh(1-\lambda^\alpha)^s} - 1)](f(\phi_2)g(\phi_2)). \end{aligned}$$

and this gives,

$$f(\frac{\phi_1\phi_2}{\lambda\phi_2+m(1-\lambda)\phi_1})g(\frac{\phi_1\phi_2}{\lambda\phi_2+m(1-\lambda)\phi_1})$$
$$\leqslant (e^{h(\lambda^\alpha)^s}-1)^2[f(\phi_1)g(\phi_1)]$$
$$+(e^{m^2(h(1-\lambda^\alpha)^s)^2}-1)[f(\phi_2)g(\phi_2)]$$
$$+(e^{h(\lambda^\alpha)^s mh(1-\lambda^\alpha)^s}-1)[f(\phi_1)g(\phi_2)]$$
$$+(e^{mh(1-\lambda^\alpha)^s h(\lambda^\alpha)^s}-1)[f(\phi_2)g(\phi_1)],$$

this implies

$$f(\frac{\phi_1\phi_2}{\lambda\phi_2+m(1-\lambda)\phi_1})g(\frac{\phi_1\phi_2}{\lambda\phi_2+m(1-\lambda)\phi_1})$$
$$\leqslant (e^{[h(\lambda^\alpha)^s]^2}-1)[f(\phi_1 g(\phi_1))] \tag{3.8}$$
$$+(e^{m^2(h(1-\lambda^\alpha)^s)^2}-1)[f(\phi_2)g(\phi_2)]$$
$$+(e^{mh(\lambda^\alpha)^s h(1-\lambda^\alpha)^s}-1)[f(\phi_1)g(\phi_2)+f(\phi_2)g(\phi_1)].$$

Integrating inequalities (3.8) on interval $[0,1]$ gives:

$$\int_0^1[f(\frac{\phi_1\phi_2}{\lambda\phi_2+m(1-\lambda)\phi_1})g(\frac{\phi_1\phi_2}{\lambda\phi_2+m(1-\lambda)\phi_1})]$$
$$\leqslant \int_0^1(e^{[h(\lambda^\alpha)^s]^2}-1)[f(\phi_1)g(\phi_1)]d\lambda$$
$$+\int_0^1(e^{m^2(h(1-\lambda^\alpha)^s)^2}-1)[f(\phi_2)g(\phi_2)]d\lambda$$
$$+\int_0^1(e^{mh(\lambda^\alpha)^s h(1-\lambda^\alpha)^s}-1)[f(\phi_1)g(\phi_2)+f(\phi_2)g(\phi_1)].$$

By proper substitution of (3.6), we have

$$\int_0^1[f(\frac{\phi_1\phi_2}{\lambda\phi_2+m(1-\lambda)})g(\frac{\phi_1\phi_2}{\lambda\phi_2+m(1-\lambda)\phi_1})]$$
$$\leqslant \int_0^1(e^{[h(\lambda^\alpha)^s]^2}-1)[q(\phi_1\phi_1)]d\lambda$$
$$+[q(\phi_2\phi_2)]\int_0^1(e^{m^2(h(1-\lambda^\alpha)^s)^2}-1)d\lambda$$
$$+[q(\phi_1\phi_2)]\int_0^1(e^{mh(\lambda^\alpha)^s h(1-\lambda^\alpha)^s}-1)d\lambda.$$

By appropriate substitution of (3.7), we have

$$\int_0^1 [f(\frac{\phi_1\phi_2}{\lambda\phi_2+m(1-\lambda)\phi_1})g(\frac{\phi_1\phi_2}{\lambda\phi_2+m(1-\lambda)\phi_1})]d\lambda \leqslant q(\phi_1\phi_1)p+q(\phi_2\phi_2)N+q(\phi_1\phi_2)R. \tag{3.9}$$

Taking $x=\lambda\phi_2+m(1-\lambda)\phi_1$ and with the use of lemma 3.2 in inequalities (3.9), we have:

$$\frac{1}{m\phi_1-\phi_2}\int_{\phi_2}^{m\phi_1} f(x)g(x) \quad \leqslant \quad q(\phi_1\phi_1 p)+q(\phi_2\phi_2)N+q(\phi_1\phi_2)R. \tag{3.10}$$

□

REMARK 3.2. (1) If $m=\alpha=1$, $e^{h(\lambda^\alpha)^s+h(1-\lambda^\alpha)^s}\leqslant 1$ following the given conditions in theorem 3.1, inequality (3.5) gives the result obtained in [20].

(2)If $h(\lambda)=I$ and $\dfrac{xy}{\lambda y+(1-\lambda)x}\leqslant e^{h(\lambda)x+h(1-\lambda)y}-1$ in theorem 3.1, inequality (3.5) gives the result obtained in theorem 2 of [19].

THEOREM 3.2. . *Let f and g be two exponential type harmonically $m(\alpha,s)$-convex functions, suppose f and g are similarly ordered functions and* $e^{h(\lambda^\alpha)^s+mh(1-\lambda^\alpha)^s}-1\leqslant 1$, $e^{h(\lambda^\alpha)^s-1}\geqslant 0$ *and* $e^{h(a)}+e^{h(b)}\geqslant 2$, *where a and b are positive functions. Then, the product fg is also exponential type harmonically $m(\alpha,s)$-convex function.*

PROOF. Since f and g are exponential type harmonically $m(\alpha,s)$-convex functions, it follows

$$\begin{aligned}&f(\frac{\phi_1\phi_2}{\lambda\phi_2+m(1-\lambda)\phi_1})g(\frac{\phi_1\phi_2}{\lambda\phi_2+m(1-\lambda)\phi_1})\\&\leqslant [(e^{h(\lambda^\alpha)^s}-1)f(\phi_1)+(e^{mh(1-\lambda^\alpha)^s}-1)f(\phi_2)]\\&\times[(e^{h(\lambda^\alpha)^s}-1)g(\phi_1)+(e^{mh(1-\lambda^\alpha)^s}-1)g(\phi_2)]\end{aligned} \tag{3.11}$$

and this gives,

$$\begin{aligned}&f(\frac{\phi_1\phi_2}{\lambda\phi_2+m(1-\lambda)\phi_1})g(\frac{\phi_1\phi_2}{\lambda\phi_2+m(1-\lambda)\phi_1})\\&\leqslant (e^{h(\lambda^\alpha)^s}-1)(e^{h(\lambda^\alpha)^s}-1)[f(\phi_1)g(\phi_1)]\\&+(e^{h(\lambda^\alpha)^s}-1)(e^{mh(1-\lambda^\alpha)^s}-1)[f(\phi_1)g(\phi_2)]\\&+(e^{mh(1-\lambda^\alpha)^s}-1)(e^{h(\lambda^\alpha)^s}-1)[f(\phi_2)g(\phi_1)]\\&+(e^{mh(1-\lambda^\alpha)^s}-1)(e^{mh(1-\lambda^\alpha)^s}-1)[f(\phi_2)g(\phi_2)],\end{aligned}$$

which implies

$$f(\frac{\phi_1\phi_2}{\lambda\phi_2+m(1-\lambda)\phi_1})g(\frac{\phi_1\phi_2}{\lambda\phi_2+m(1-\lambda)\phi_1})$$

$$\begin{aligned}&\leqslant (e^{h(\lambda^{2\alpha})^s}-1)[f(\phi_1)g(\phi_1)]\\&+(e^{h(\lambda^\alpha)^s mh(1-\lambda^\alpha)^s}-1)[f(\phi_1)g(\phi_2)]\\&+(e^{mh(1-\lambda^\alpha)^s h(\lambda^\alpha)^s}-1)[f(\phi_2)g(\phi_1)]\\&+(e^{mh(1-\lambda^\alpha)^{2s}}-1)[f(\phi_2)g(\phi_2)].\end{aligned}$$

By solving, we have

$$f(\frac{\phi_1\phi_2}{\lambda\phi_2+m(1-\lambda)\phi_1})g(\frac{\phi_1\phi_2}{\lambda\phi_2+m(1-\lambda)\phi_1})$$

$$\leqslant (e^{h((\lambda^{2\alpha})^s}-1)f(\phi_1)g(\phi_1)$$

$$+(e^{h(\lambda^\alpha)^s mh(1-\lambda^\alpha)^s}-1)[f(\phi_1)g(\phi_1)+f(\phi_2)g(\phi_2)]$$

$$+(e^{mh(1-\lambda^\alpha)^{2s}}-1)f(\phi_2)g(\phi_2),$$

which further leads to,

$$f(\frac{\phi_1\phi_2}{\lambda\phi_2+m(1-\lambda)\phi_1})g(\frac{\phi_1\phi_2}{\lambda\phi_2+m(1-\lambda)\phi_1})$$

$$=(e^{h(\lambda^\alpha)^s}-1)[f(\phi_1)g(\phi_1)]+(e^{mh(1-\lambda^\alpha)^s}-1)$$

$$\times[f(\phi_2)g(\phi_2)][(e^{h(\lambda^\alpha)}-1)+(e^{mh(1-\lambda^\alpha)^s}-1)].$$

Following the given conditions, we obtain

$$\begin{aligned}f(\frac{\phi_1\phi_2}{\lambda\phi_2+m(1-\lambda)\phi_1})g(\frac{\phi_1\phi_2}{\lambda\phi_2+m(1-\lambda)\phi_1}) &\leqslant (e^{h(\lambda^\alpha)^s}-1)[f(\phi_1)g(\phi_1)]\\ &+(e^{mh(1-\lambda^\alpha)^s}-1)[f(\phi_2)g(\phi_2)].\end{aligned} \tag{3.12}$$

Hence the proof. □

REMARK 3.3. If $m=1$ in theorem 3.2, it gives the result obtained in [1].

THEOREM 3.3. *Let f and g be two exponential type harmonically $m(\alpha,s)$-convex functions, if $fg\in L[\phi_1,\phi_2]$ and*

$f(\frac{\phi_1\phi_2}{\lambda\phi_2+(1-\lambda)\phi_1})\leqslant f(\lambda\phi_1+(1-\lambda)\phi_2)$. *Given* $(e^a-1)(e^b-1)\leqslant e^{ab}-1$ *with* $(e^a-1)(e^a-1)\leqslant e^{a^2}-1$, *where* $a,b>0$. *Then the following hold*

$$2p(\frac{\phi_1+m\phi_2}{2})g(\frac{\phi_1+m\phi_2}{2})$$
$$\leqslant \int_0^1 f(\lambda\phi_1+m(1-\lambda)\phi_2)g(\lambda\phi_1+m(1-\lambda)\phi_2)d\lambda \tag{3.13}$$
$$+[Q(\phi_1\phi_1)+m^2Q(\phi_2\phi_2)]R+\frac{1}{2}[q(\phi_1\phi_2)]V.$$

where

$$\begin{aligned} R &= \int_0^1 (e^{h(\lambda^\alpha)^s h(1-\lambda^\alpha)^s}-1)d\lambda \\ V &= \int_0^1 (e^{m[h(\lambda^\alpha)^s]^2}-1)+(e^{[h(1-\lambda^\alpha)^s]^2}-1), \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} Q(\phi_1,\phi_1) &= f(\phi_1)f(\phi_2) \\ Q(\phi_2,\phi_2) &= f(\phi_2)g(\phi_2) \\ Q(\phi_1,\phi_2) &= f(\phi_1)g(\phi_2)+f(\phi_2)g(\phi_1). \end{aligned} \tag{3.15}$$

Proof. Since f and g are exponential type harmonically $m(\alpha,s)$-convex functions, we have

$$\begin{aligned} f(\frac{\phi_1+m\phi_2}{2})g(\frac{\phi_1+m\phi_2}{2}) &= f(\frac{\lambda\phi_1+m(1-\lambda)\phi_2}{2})+\frac{(1-\lambda)\phi_1+m\lambda\phi_2}{2}\times \\ &\quad g(\frac{\lambda\phi_1+m(1-\lambda)\phi_2}{2})+\frac{(1-\lambda)\phi_1+m\lambda\phi_2}{2} \\ &\leqslant \frac{1}{4}[f(\lambda\phi_1+m(1-\lambda)\phi_2)+f((1-\lambda)\phi_1+m\lambda\phi_2)] \\ &\quad \times\frac{1}{4}[g(\lambda\phi_1+m(1-\lambda)\phi_2)+g((1-\lambda)\phi_1+m\lambda\phi_2)] \end{aligned}$$

$$f(\frac{\phi_1+m\phi_2}{2})g(\frac{\phi_1+m\phi_2}{2})$$
$$=\frac{1}{4}[f(\lambda\phi_1+m(1-\lambda)\phi_2)g(\lambda\phi_1+m(1-\lambda)\phi_2)$$
$$+f((1-\lambda)\phi_1+m\lambda\phi_2)g((1-\lambda)\phi_1+m\lambda\phi_2)] \tag{3.16}$$
$$+\frac{1}{4}[f(\lambda\phi_1+m(1-\lambda)\phi_2)g(1-\lambda)\phi_1+m\lambda\phi_2$$
$$+f((1-\lambda)\phi_1+m\lambda\phi_2)g(\lambda\phi_1+m(1-\lambda)\phi_2)],$$

and this gives

$$
\begin{aligned}
&f(\frac{\phi_1+m\phi_2}{2})g(\frac{\phi_1+m\phi_2}{2})\\
&\leqslant \frac{1}{4}[f(\lambda\phi_1+m(1-\lambda)\phi_2)g(\lambda\phi_1+m(1-\lambda)\phi_2)\\
&+f(1-\lambda)\phi_1+m\lambda\phi_2)g(1-\lambda)\phi_1+m\lambda\phi_2)]\\
&+\frac{1}{4}[(e^{h(\lambda^\alpha)^s}-1)f(\phi_1)+(e^{mh(1-\lambda^\alpha)^s}-1)f(\phi_2)]\\
&\times[(e^{h(1-\lambda^\alpha)^s}-1)g(\phi_1)+(e^{mh(\lambda^\alpha)^s}-1)g(\phi_2)]\\
&\leqslant \frac{1}{4}[f(\lambda\phi_1+m(1-\lambda)\phi_2)g(\lambda\phi_1+m(1-\lambda)\phi_2)\\
&+f(1-\lambda)\phi_1+m\lambda\phi_2)g((1-\lambda)\phi_1+m\lambda\phi_2)]\\
&+\frac{1}{4}(e^{h(\lambda^\alpha)^s h(1-\lambda^\alpha)^s}-1)[f(\phi_1)g(\phi_1)]\\
&+(e^{m[h(\lambda^\alpha)^s]^2}-1)[f(\phi_1 g(\phi_2))]\\
&+(e^{m[h(1-\lambda^\alpha)^s]^2}-1)[f(\phi_2)g(\phi_1)]\\
&+(e^{m^2h(1-\lambda^\alpha)^s h(\lambda^\alpha)^s}-1)[f(\phi_1)g(\phi_1)].
\end{aligned}
$$

Thus

$$
\begin{aligned}
&f(\frac{\phi_1+m\phi_2}{2})g(\frac{\phi_1+m\phi_2}{2})\\
&\leqslant \frac{1}{4}[f(\lambda\phi_1+m(1-\lambda)\phi_2)g(\lambda\phi_1+m(1-\lambda)\phi_2)\\
&+f((1-\lambda)\phi_1+m\lambda\phi_2)g((1-\lambda)\phi_1+m\lambda\phi_2)] \qquad (3.17)\\
&+\frac{1}{2}[(e^{h(\lambda^\alpha)^s h(1-\lambda^\alpha)^s}-1)](f(\phi_1)f(\phi_2)+m^2f(\phi_2)g(\phi_2)\\
&+\frac{1}{4}[(e^{m[h(\lambda^\alpha)^s]^2}-1)+(e^{[h(1-\lambda^\alpha)^s]^2}-1)](F(\phi_1)g(\phi_2)+f(\phi_2)g(\phi_1)).
\end{aligned}
$$

Integrating (3.17) on $[0,1]$, gives

$$
\begin{aligned}
& f(\frac{\phi_1+m\phi_2}{2})g(\frac{\phi_1+m\phi_2}{2}) \\
& \leqslant \frac{1}{4}\int_0^1[f(\lambda\phi_1+m(1-\lambda)\phi_2)g(\lambda\phi_1+m(1-\lambda)\phi_2) \\
& +f((1-\lambda)\phi_1+m\lambda\phi_2)g((1-\lambda)\phi_1+m\lambda\phi_2)] \\
& +\frac{1}{2}\int_0^1(e^{h(\lambda^\alpha)^s h(1-\lambda^\alpha)^s}-1)[f(\phi_1)f(\phi_2)+m^2f(\phi_2)g(\phi_2)] \\
& +\frac{1}{4}\int_0^1[(e^{m[h(\lambda^\alpha)^s]^2}-1)+(e^{h[(1-\lambda^\alpha)^s]^2}-1)]d\lambda \\
& +[f(\phi_1)g(\phi_2)f(\phi_2)g(\phi_1)].
\end{aligned} \tag{3.18}
$$

Substituting (3.14)and (3.15) gives

$$
\begin{aligned}
f(\frac{\phi_1+m\phi_2}{2})g(\frac{\phi_1+m\phi_2}{2}) \quad \leqslant \quad & \frac{1}{2}\int_0^1 f(\lambda\phi_1+m(1-\lambda)\phi_2)g(\lambda\phi_1+m(1-\lambda)\phi_2)d\lambda \\
& +\frac{1}{2}[Q(\phi_1,\phi_1)+m^2Q(\phi_2,\phi_2)]R+\frac{1}{4}[Q(\phi_1,\phi_2)]V.
\end{aligned}
$$

Taking $x=\lambda\phi_1+m(1-\lambda)\phi_2$, result to

$$
\begin{aligned}
& 2f(\frac{\phi_1+m\phi_2}{2})g(\frac{\phi_1+m\phi_2}{2}) \leqslant [\frac{1}{\phi_2-\phi_1}\int_{\phi_1}^{\phi_2} f(x)g(x)d\lambda \\
& +[Q(\phi_1,\phi_1)+m^2Q(\phi_2,\phi_2)]R+\frac{1}{2}[Q(\phi_1,\phi_2)]V.
\end{aligned} \tag{3.19}
$$

□

REMARK 3.4. (1) If $e^{h(a)}=h(a)+1$ and $s=1$ in theorem 3.3, then it reduces to inequality (14) of [19].

(2) If we take $s=0$ and $e^{h(a)}=h(a)+1$ in theorem 3.3, the following results for exponential type harmonically p-convex functions will be obtain

COROLLARY 3.1. *Let f and g be two exponential type harmonically p-convex functions. If $fg\in L[\phi_1,\phi_2]$ and* $f(\frac{\phi_1\phi_2}{\lambda\phi_2+(1-\lambda)\phi_1})\leqslant f(\lambda\phi_1+(1-\lambda)\phi_2)$, *where* $(e^a-1)(e^b-1) \leqslant e^{ab}-1$ *with* $(e^a-1)(e^a-1) \leqslant e^{a^2}-1$. *Then,*

$$\begin{aligned} 2p(\frac{\phi_1+m\phi_2}{2}) &\leqslant \int_0^1 f(\lambda\phi_1+m(1-\lambda)\phi_2)g(\lambda\phi_1+m(1-\lambda)\phi_2)d\lambda \\ &+Q(\phi_1,\phi_1)+m^2Q(\phi_2,\phi_2)+\frac{1}{2}Q(\phi_1,\phi_2), \end{aligned} \tag{3.20}$$

where, $Q(\phi_1,\phi_1)$, $Q(\phi_2,\phi_2)$ *and* $Q(\phi_1,\phi_2)$ *are as stated in theorem 3.2 .*

THEOREM 3.4. *Let* $f,g : I \longrightarrow \mathbb{R}$ *be two exponential type harmonically* $m(\alpha,s)$*-convex functions. Where* $a,b \in I$ *with* $a<b$, $\alpha \in (0,1], s \in [0,1]$ *and* $(e^a-1)(e^b-1) \leqslant e^{ab}-1$, $e^{h(\lambda^\alpha)^s}=0$. *If* $fg \in L[a,b]$. *Then*

$$\frac{1}{m\phi_2-\phi_1}\int_{\phi_1}^{m\phi_2} f(x)g(x)dx \leqslant \xi_1\mu_1+\xi_2\mu_2+\xi_3\mu_3, \tag{3.21}$$

where

$$\begin{aligned} \xi_1 &= f(\phi_1)g(\phi_1) \\ \xi_2 &= f(\phi_2)g(\phi_2) \\ \xi_3 &= f(\phi_1)g(\phi_2)+f(\phi_2)g(\phi_1), \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} \mu_1 &= \int_0^1 (e^{h(\lambda^{2\alpha})^s}-1)d\lambda \\ \mu_2 &= \int_0^1 (e^{mh(1-\lambda^\alpha)^{2s}}-1)d\lambda \\ \mu_3 &= \int_0^1 (e^{mh(\lambda^\alpha)^s h(1-\lambda^\alpha)^s}-1)d\lambda. \end{aligned} \tag{3.23}$$

PROOF. Assume $f,g : I \longrightarrow \mathbb{R}$ to be an exponential type harmonically $m(\alpha,s)$-convex functions, then

$$\begin{aligned} &f(\lambda\phi_1+m(1-\lambda)\phi_2)g(\lambda\phi_1+m(1-\lambda)\phi_2) \\ &\leqslant [(e^{h(\lambda^\alpha)^s}-1)f(\phi_1)+(e^{mh(1-\lambda^\alpha)^s}-1)f(\phi_2)] \\ &\times[(e^{h(\lambda^\alpha)^s}-1)g(\phi_1)+(e^{mh(1-\lambda^\alpha)^s}-1)g(\phi_2)] \\ &= (e^{h(\lambda^{2\alpha})^s}-1)[f(\phi_1)g(\phi_1)]+(e^{mh(1-\lambda^\alpha)^{2s}}-1)[f(\phi_2)g(\phi_2)] \\ &+(e^{mh(\lambda^\alpha)^s h(1-\lambda^\alpha)^s}-1)[f(\phi_1)g(\phi_2)+f(\phi_2)g(\phi_1)]. \end{aligned} \tag{3.24}$$

Integrating (3.24) on interval $[0,1]$ gives,

$$\int_0^1 f(\lambda\phi_1+m(1-\lambda)\phi_2)g(\lambda\phi_1+m(1-\lambda)\phi_2)$$
$$\leqslant \int_0^1 (e^{h(\lambda^\alpha)^s}-1)[f(\phi_1)g(\phi_1)+(e^{mh(1-\lambda^\alpha)^{2s}}-1)]f(\phi_2)g(\phi_2)$$
$$+\int_0^1 (e^{mh(\lambda^\alpha)^s h(1-\lambda^\alpha)^s}-1)[f(\phi_1)g(\phi_2)+f(\phi_2)g(\phi_1)],$$

then

$$\int_0^1 f(\lambda\phi_1+m(1-\lambda)\phi_2)g(\lambda\phi_1+m(1-\lambda)\phi_2)$$
$$\leqslant f(\phi_1)g(\phi_1)\int_0^1 (e^{h(\lambda^\alpha)^s}-1)d\lambda+f(\phi_2)g(\phi_2)\int_0^1 (e^{mh(1-\lambda^\alpha)^s}-1)d\lambda \tag{3.25}$$
$$+[f(\phi_1)g(\phi_2)+f(\phi_2)g(\phi_1)]\int_0^1 (e^{mh(\lambda^\alpha)^s h(1-\lambda^\alpha)^s}-1)d\lambda.$$

thus

$$\int_0^1 f(\lambda\phi_1+m(1-\lambda)\phi_2)g(\lambda\phi_1+m(1-\lambda)\phi_2)$$
$$\leqslant \xi_1\int_0^1 (e^{h(\lambda^{2\alpha})^s}-1)d\lambda+\xi_2\int_0^1 (e^{mh(1-\lambda^\alpha)^{2s}}-1)d\lambda \tag{3.26}$$
$$+\xi_3\int_0^1 (e^{mh(\lambda^\alpha)^s h(1-\lambda^\alpha)^s}-1)d\lambda.$$

Taking $x=\lambda\phi_1+m(1-\lambda)\phi_2$, result to

$$\frac{1}{m\phi_2-\phi_1}\int_{\phi_1}^{m\phi_2} f(x)g(x)dx \quad \leqslant \quad \xi_1\mu_1+\xi_2\mu_2+\xi_3\mu_3. \tag{3.27}$$

□

THEOREM 3.5. *Let f and g be two exponential type harmonically $m(\alpha,s)$-convex functions, if $fg\in L[\phi_1,\phi_2]$ and $(e^a-1)(e^b-1)\leqslant e^{ab}-1$. Then, the following inequalities hold*

$$\frac{1}{(\phi_2-\phi_1)^2}\int_{\phi_1}^{\phi_2}\int_{\phi_1}^{\phi_2}\int_0^1 f(\lambda x+m(1-\lambda)y)g(\lambda x+m(1-\lambda)y)d\lambda dydx$$
$$\leqslant (P+T)\frac{1}{\phi_2-\phi_1}\int_{\phi_1}^{\phi_2} f(x)g(x)ds \tag{3.28}$$
$$+\frac{1}{2}\frac{1}{(\phi_2-\phi_1)^2}J[Q(\phi_1,\phi_1)+Q(\phi_2,\phi_2)+Q(\phi_1,\phi_2)],$$

where

$$
(3.29)\qquad
\begin{aligned}
P &= \int_0^1 (e^{[h(\lambda^\alpha)^s]^2} - 1)d\lambda \\
T &= \int_0^1 (e^{m^2[h(1-\lambda^\alpha)^s]^2} - 1)d\lambda \\
J &= \int_0^1 (e^{mh(\lambda^\alpha)^s h(1-\lambda^\alpha)^s} - 1)d\lambda \\
Q(\phi_1, \phi_1) &= f(\phi_1)g(\phi_1).
\end{aligned}
$$

and

$Q(\phi_2, \phi_2)$, $Q(\phi_1, \phi_2)$ *are as stated in theorem 3.4.*

Proof. Since f, g are exponential type harmonically $m(\alpha, s)$-convex functions, using the exponential type $m(\alpha, s)$-convexity, we have

$$
\begin{aligned}
&f(\lambda x + m(1-\lambda)y)g(\lambda x + m(1-\lambda)y) \\
&\leqslant [(e^{h(\lambda^\alpha)^s} - 1)f(x) + (e^{mh(1-\lambda^\alpha)^s} - 1)f(y)][(e^{h(\lambda^\alpha)^s} - 1)g(x) + (e^{mh(1-\lambda^\alpha)^s} - 1)g(y)].
\end{aligned}
$$

this gives

$$
(3.30)\qquad
\begin{aligned}
&f(\lambda x + m(1-\lambda)y)g(\lambda x + m(1-\lambda)y) \\
&= (e^{[h(\lambda^\alpha)^s]^2} - 1)[f(x)g(x)] + (e^{m^2[h(1-\lambda^\alpha)^s]^2} - 1)[f(y)g(y)] \\
&+ (e^{mh(\lambda^\alpha)^s h(1-\lambda^\alpha)^s} - 1)[f(x)g(y) + f(y)g(x)].
\end{aligned}
$$

Integrating (3.30) on $[0, 1]$, gives

$$
\begin{aligned}
&\int_0^1 f(\lambda x + m(1-\lambda)y)g(\lambda x + m(1-\lambda)y)d\lambda \\
&\leqslant [f(x)g(x)] \int_0^1 (e^{[h(\lambda^\alpha)^s]^2} - 1)d\lambda \\
&+ [f(x)g(y)] \int_0^1 (e^{m^2[h(1-\lambda^\alpha)^s]^2} - 1)d\lambda \\
&+ [f(x)g(y) + f(y)g(x)] \int_0^1 (e^{mh(\lambda^\alpha)^s h(1-\lambda^\alpha)^s} - 1)d\lambda.
\end{aligned}
$$

thus

$$
(3.31)\qquad
\begin{aligned}
&\int_0^1 f(\lambda x + m(1-\lambda)y)g(\lambda x + m(1-\lambda)y)d\lambda \\
&\leqslant P[f(x)g(x)] + T[f(y)g(y)] + J[f(x)g(y) + f(y)g(x)].
\end{aligned}
$$

Integrating (3.31) on the rectangle $[0,1]\times[0,1]$, we have

$$\int_{\phi_1}^{\phi_2}\int_{\phi_1}^{\phi_2}\int_0^1 f(\lambda x+m(1-\lambda)y)g(\lambda x+m(1-\lambda)y)d\lambda dydx$$

$$\leqslant P(\phi_2-\phi_1)\int_{\phi_1}^{\phi_2} f(x)g(x)dx+T(\phi_2-\phi)\int_{\phi_1}^{\phi_2} f(y)g(y)dy \tag{3.32}$$

$$+J[\int_{\phi_1}^{xi_2} f(x)dx\int_{\phi_1}^{\phi_2} g(y)dy+\int_{\phi_1}^{\phi_2} f(y)dy\int_{\phi_1}^{\phi_2} g(x)dx].$$

By the application of Hermite-Hadamard's inequality from right half, the result becomes

$$\int_{\phi_1}^{\phi_2}\int_{\phi_1}^{\phi_2}\int_0^1 f(\lambda x+m(1-\lambda)y)g(\lambda x+m(1-\lambda)y)d\lambda dydx$$

$$\leqslant [P+T](\phi_2-\phi_1)\int_{\phi_1}^{\phi_2} f(x)g(x)dx+\frac{1}{2}J[f(\phi_1)g(\phi_1)+f(\phi_2)g(\phi_2)$$

$$+f(\phi_1)g(\phi_2)+f(\phi_2)g(\phi_1)]. \tag{3.33}$$

$$=[P+T](\phi_2-\phi_1)\int_{\phi_1}^{\phi_2} f(x)g(x)dx$$

$$+\frac{1}{2}T[Q(\phi_1,\phi_1)+Q(\phi_2,\phi_2)+Q(\phi_1,\phi_2)].$$

thus, we obtain

$$\frac{1}{(\phi_2-\phi_1)^2}\int_{\phi_1}^{\phi_2}\int_{\phi_1}^{\phi_2}\int_0^1 f(\lambda x+m(1-\lambda)y)g(\lambda x+m(1-\lambda)y)d\lambda dydx$$

$$\leqslant \frac{1}{\phi_2-\phi_1}[P+T]\int_{\phi_1}^{\phi_2} f(x)g(x)dx+\frac{1}{(\phi_2-\phi_1)^2} \tag{3.34}$$

$$J[Q(\phi_1,\phi_1)+Q(\phi_2,\phi_2)+Q(\phi_1,\phi_2)].$$

□

4. Applications

In this section, we made an application to some special means.

(1) The geometric mean:

$$GM=\sqrt[n]{x_1,x_2,...,x_n} \tag{4.1}$$

which equally can be written as,

$$GM = (x_1, x_2, ..., x_n)^{1/n}$$

$$logGM = \frac{1}{n} log(x_1, x_2, ..., x_n). \tag{4.2}$$

In short, we have

$$GM = GM(xy) = \sqrt{xy}, \quad x \neq y. \tag{4.3}$$

(2) The logarithmic mean:

$$LM = L(x, y) = \frac{x - y}{lnx - lny}, \quad \forall \ x, y > 0. \tag{4.4}$$

(3) The power mean:

$$M_p = M_p(x, y) = (\frac{x^p + y^p}{2})^{1/p}, \quad \forall \ x, y > 0 \ and \ for \ all \ real \ number \ p \neq 0.$$

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