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PARA-SASAKIAN MANIFOLDS ADMITTING ALMOST η- RICCI SOLITONS ON SOME SPECIAL CURVATURE TENSORS

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Abstract. In this paper, we consider Ricci-pseudosymmetric para-Sasakian manifolds admitting almost $\eta-$Ricci solitons by means of some curvature tensors. Ricci pseudosymmetry concept of para-Sasakian manifolds admitting $\eta-$Ricci soliton is introduced according to the choice of the curvature tensor. Then, again according to these curvature tensor, necessary and sufficient conditions are given for a para-sasakian manifold admitting $\eta-$Ricci soliton to be Ricci semi-symmetric. Then some characterizations are obtained and some classifications are made.

1. Introduction

Hamilton revealed a new concept the Ricci flow in 1982. He obtained the canonical metric of a smooth manifold with the concept of Ricci flow. In the following years, Ricci flow played a very active role in the study of Riemann manifolds. It has become very useful especially for Riemannian manifolds with positive curvature. Poincare conjecture proven by Perelman using Ricci flow in [**16**], [**17**]. Ricci flow, which is an evolution equation for metrics in Riemannian manifolds, defined as

$$\frac{\partial}{\partial t}g\left(t\right)=-2S\left(g\left(t\right)\right).$$

The limit of solutions of Ricci flow is called Ricci soliton. If the Ricci flow moves with the only one set of parameters, it is called a solution of the Ricci flow.

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Especially in the last 20 years, Ricci solitons occupied a very important place for geometry and managed to attract the attention of many authors. In addition, the Ricci soliton was solved by Perelman in 1904 with the help of Ricci soliton, and the long-standing Poincare conjecture turned Ricci solitons into a very important one. In [**18**], Sharma studied the Ricci solitons in contact geometry. Thereafter Ricci solitons in contact metric manifolds have been studied by various authors such as Bagewadi et al. in [**1**], [**2**], [**4**], [**12**], Bejan and Crasmareanu in [**5**], Blaga in [**6**], Chandra et al. in [**7**], Chen and Deshmukh in [**8**], Deshmukh et al. in [**10**], He and Zhu [**11**], Atçeken et al. in [**3, 13, 14**], Nagaraja and Premalatta in [**15**], Tripathi in [**20**] and many others.

Motivated by the above studies, in this paper, we consider Ricci-pseudosymmetric para-Sasakian manifolds admitting almost $\eta-$Ricci solitons by means of some curvature tensors. Ricci pseudosymmetry concept of para-Sasakian manifolds admitting $\eta-$Ricci soliton is introduced according to the choice of the curvature tensor. Then, again according to these curvature tensor, necessary and sufficient conditions are given for a para-sasakian manifold admitting $\eta-$Ricci soliton to be Ricci semi-symmetric. Then some characterizations are obtained and some classifications are made.

2. Preliminaries

A $(2n+1)$-dimensional smooth manifold M^{2n+1} has an almost paracontact structure (ϕ,ξ,η) if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying the following conditions;

$$\phi^2\varpi=\varpi-\eta(\varpi)\xi,\quad \eta(\xi)=1,\quad \phi\xi=0,\quad \eta\circ\phi=0. \tag{1}$$

If an almost paracontact manifold is endowed with a semi-Riemannian metric tensor g such that

$$g(\phi\varpi,\phi\rho)=-g(\varpi,\rho)+\eta(\varpi)\eta(\rho), \tag{2}$$

for all vector fields ϖ,ρ on M^{2n+1}, then $M^{2n+1}(\phi,\xi,\eta,g)$ is said to be almost paracontact metric manifold. It is clear that

$$g(\xi,\varpi)=\eta(\varpi). \tag{3}$$

The fundamental 2-form Φ of an almost paracontact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ is defined by

$$\Phi(\varpi,\rho)=g(\varpi,\phi\rho). \tag{4}$$

If $d\eta=\Phi$, then almost paracontact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ is called paracontact metric manifold. If a paracontact metric structure is normal, this structure is called para-Sasakian. That means

$$d\eta(\varpi,\rho)=\frac{1}{2}\left[\eta(\rho)\varpi-\eta(\varpi)\rho-\eta([\varpi,\rho])\right]=0. \tag{5}$$

It is well known that an almost paracontact metric manifold is a para-Sasakian if and only if

$$\left(\bar{\nabla}_{\varpi}\phi\right)\rho=-g(\varpi,\rho)\xi-\eta(\rho)\varpi+2\eta(\varpi)\eta(\rho)\xi, \tag{6}$$

for all ϖ. From (6), one can easily to see that

$$\bar{\nabla}_{\varpi}\xi = -\phi\varpi, \tag{7}$$

where $\bar{\nabla}$ denote the Levi-Civita connection on M^{2n+1}. If the relation

$$\left(\bar{\nabla}_{\varpi}\eta\right)\rho = -g\left(\varpi,\rho\right) + \eta\left(\varpi\right)\eta\left(\rho\right)$$

is satisfied specifically, then para-Sasakian manifold is called the special para-Sasakian manifold or the Sp-Sasakian manifold.

Lemma 2.1. *In a $(2n+1)-$dimensional para-Sasakian manifold, the following relations holds:*

$$R(\varpi,\rho)\xi = \eta\left(\varpi\right)\rho - \eta\left(\rho\right)\varpi, \tag{8}$$

$$R(\xi,\varpi)\rho = \eta\left(\rho\right)\varpi - g\left(\varpi,\rho\right)\xi, \tag{9}$$

$$\eta\left(R\left(\varpi,\rho\right)\tau\right) = g\left(\eta\left(\rho\right)\varpi - \eta\left(\varpi\right)\rho,\tau\right), \tag{10}$$

$$S(\varpi,\xi) = -2n\eta(\varpi), \tag{11}$$

$$Q\xi = -2n\xi, \tag{12}$$

$$\left(\nabla_{\varpi}\eta\right)\rho = g\left(\varpi,\phi\rho\right) \tag{13}$$

for any vector fields ϖ,ρ on M^{2n+1}, where ∇ is the Levi-Civita connection, R and S denote the Riemannian curvature tensor and Ricci tensor of M^{2n+1}, respectively.

Precisely, a Ricci soliton on a Riemannian manifold (M,g) is defined as a triple (g,ξ,λ) on M satisfying

$$L_{\xi}g + 2S + 2\lambda g = 0, \tag{14}$$

where L_{ξ} is the Lie derivative operator along the vector field ξ and λ is a real constant. We note that if ξ is a Killing vector field, then Ricci soliton reduces to an Einstein metric (g,λ). Futhermore, in [**9**], as generalization the notion of $\eta-$Ricci soliton is defined by J.T. Cho and M. Kimura as a quadruple (g,ξ,λ,μ) satisfying

$$L_{\xi}g + 2S + 2\lambda g + 2\mu\eta\oplus\eta = 0, \tag{15}$$

where λ and μ denote real constants and η is the dual of ξ and S denotes the Ricci tensor of M. Furthermore if λ and μ are smooth functions on M, then it called an almost $\eta-$Ricci soliton on M [**9**].

Suppose the (g, ξ, λ, μ) is almost $\eta-$Ricci soliton on M.
· If $\lambda < 0$, then M is shriking.
· If $\lambda = 0$, then M is steady.
· If $\lambda > 0$, then M is expanding.

3. Almost $\eta-$Ricci solitons on Ricci pseudosymmetric para-Sasakian manifolds

Now let (g, ξ, λ, μ) be almost $\eta-$Ricci soliton on para-Sasakian manifold. Then we have

$$(L_\xi g)(\varpi, \rho) = L_\xi g(\varpi, \rho) - g(L_\xi \varpi, \rho) - g(\varpi, L_\xi \rho)$$

$$= \xi g(\varpi, \rho) - g([\xi, \varpi], \rho) - g(\varpi, [\xi, \rho])$$

$$= g(\nabla_\xi \varpi, \rho) - g(\varpi, \nabla_\xi \rho) - g(\nabla_\xi \varpi, \rho)$$

$$+g(\nabla_\varpi \xi, \rho) - g(\nabla_\xi \rho, \varpi) + g(\varpi, \nabla_\rho \xi),$$

for all $\varpi, \rho \in \Gamma(TM)$. By using ϕ is anti-symmetric and (7), we have

$$(L_\xi g)(\varpi, \rho) = 0. \tag{16}$$

Thus, in a para-Sasakian manifolds, from (15) and (16), we have

$$S(\varpi, \rho) = -\lambda g(\varpi, \rho) - \mu\eta(\varpi)\eta(\rho). \tag{17}$$

It is clear from (17) that the $(2n+1)-$dimensional para-Sasakian admitting $\eta-$Ricci soliton $\left(M^{2n+1}, g, \xi, \lambda, \mu\right)$ is an $\eta-$Einstein manifold.

For $\rho = \xi$ in (17), it implies that

$$S(\xi, \varpi) = -(\lambda + \mu)\eta(\varpi). \tag{18}$$

Taking into account of (11), we conclude that

$$\lambda + \mu = 2n. \tag{19}$$

Again it is clear from (17)

$$Q\rho = -\lambda\rho - \mu\eta(\rho)\xi. \tag{20}$$

For an $(2n+1)-$dimensional semi-Riemann manifold M, the W_1-curvature tensor is defined as

$$W_1(\varpi, \rho)\tau = R(\varpi, \rho)\tau + \frac{1}{2n}[S(\rho, \tau)\varpi - S(\varpi, \tau)\rho]. \tag{21}$$

For an $(2n+1)-$dimensional para-Sasakian manifold, if we choose $\tau = \xi$ in (21), we can easily see

$$W_1(\varpi, \rho)\xi = 2[\eta(\varpi)\rho - \eta(\rho)\varpi], \tag{22}$$

in the same way, if we take the inner product of both sides of (21) by ξ, we get

$$\eta(W_1(\varpi, \rho)\tau) = 2g(\eta(\rho)\varpi - \eta(\varpi)\rho, \tau) \tag{23}$$

DEFINITION 3.1. Let M^{2n+1} be an $(2n+1)-$dimensional para-Sasakian manifold. If $W_1 \cdot S$ and $Q(g,S)$ are linearly dependent, then the manifold is said to be W_1-Ricci pseudosymmetric.

In this case, there exists a function F_{W_1} on M^{2n+1} such that

$$W_1 \cdot S = F_{W_1} Q(g,S).$$

In particular, if $F_{W_1} = 0$, the manifold M^{2n+1} is said to be W_1-Ricci semi-symmetric.

Let us now consider W_1-Ricci pseudosymmetry case of the $(2n+1)-$dimensional para-Sasakian manifold.

THEOREM 3.1. *Let M^{2n+1} be para-Sasakian manifold and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a W_1-Ricci pseudosymmetric, then $F_{W_1} = -2$ or $\lambda = 2n$.*

PROOF. Let's assume that para-Sasakian manifold M^{2n+1} be W_1-Ricci pseudosymmetric and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on para-Sasakian manifold M^{2n+1}. That's mean

$$(W_1(\varpi,\rho)\cdot S)(\upsilon,\varrho) = F_{W_1} Q(g,S)(\upsilon,\varrho;\varpi,\rho),$$

for all $\varpi,\rho,\upsilon,\varrho \in \Gamma(TM)$. From the last equation, we can easily write

$$\begin{aligned} & S(W_1(\varpi,\rho)\upsilon,\varrho) + S(\upsilon,W_1(\varpi,\rho)\varrho) \\ & = F_{W_1}\{S((\varpi\wedge_g\rho)\upsilon,\varrho) + S(\upsilon,(\varpi\wedge_g\rho)\varrho)\}. \end{aligned} \tag{24}$$

If we choose $\varrho = \xi$ in (24), we get

$$\begin{aligned} & S(W_1(\varpi,\rho)\upsilon,\xi) + S(\upsilon,W_1(\varpi,\rho)\xi) \\ & = F_{W_1}\{S(g(\rho,\upsilon)\varpi - g(\varpi,\upsilon)\rho,\xi) \\ & +S(\upsilon,\eta(\rho)\varpi - \eta(\varpi)\rho)\}. \end{aligned} \tag{25}$$

If we make use of (11) and (22) in (25), we have

$$\begin{aligned} & -2n\eta(W_1(\varpi,\rho)\upsilon) + 2S(\upsilon,\eta(\varpi)\rho - \eta(\rho)\varpi) \\ & = F_{W_1}\{-2ng(\eta(\varpi)\rho - \eta(\rho)\varpi,\upsilon) + S(\upsilon,\eta(\rho)\varpi - \eta(\varpi)\rho)\}. \end{aligned} \tag{26}$$

If we use (23) in (26), we get

$$\begin{aligned} & -4ng(\eta(\rho)\varpi - \eta(\varpi)\rho,\upsilon) + S(\eta(\varpi)\rho - \eta(\rho)\varpi,\upsilon) \\ & = F_{W_1}\{-2ng(\eta(\varpi)\rho - \eta(\rho)\varpi,\upsilon) + S(\upsilon,\eta(\rho)\varpi - \eta(\varpi)\rho)\}. \end{aligned} \tag{27}$$

If we use (17) in (27), we reach at

$$[(2n-\lambda)(2+F_{W_1})]\,g(\eta(\rho)\varpi - \eta(\varpi)\rho,\upsilon) = 0, \tag{28}$$

which proves our assertions. □

We can give the results obtained from this theorem as follows.

Corollary 3.1. *Let M^{2n+1} be para-Sasakian manifold and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a W_1-Ricci semi-symmetric then $\lambda = 2n$, M^{2n+1} is an Einstein manifold and almost $\eta-$Ricci soliton (g,ξ,λ,μ) induces Ricci soliton (g,ξ,λ).*

Corollary 3.2. *Let M^{2n+1} be para-Sasakian manifold and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a W_1-Ricci semisymmetric, then M^{2n+1} is an expanding.*

For an $(2n+1)-$dimensional semi-Riemann manifold M, the W_1^*-curvature tensor is defined as

$$W_1^*(\varpi,\rho)\tau = R(\varpi,\rho)\tau - \frac{1}{2n}\left[S(\rho,\tau)\varpi - S(\varpi,\tau)\rho\right]. \tag{29}$$

For an $(2n+1)-$dimensional para-Sasakian manifold, if we choose $\tau=\xi$ in (29), we can write

$$W_1^*(\varpi,\rho)\xi = 0, \tag{30}$$

so, we have

$$\eta\left(W_1^*(\varpi,\rho)\tau\right) = 0. \tag{31}$$

Definition 3.2. Let M^{2n+1} be an $(2n+1)-$dimensional para-Sasakian manifold. If $W_1^*\cdot S$ and $Q(g,S)$ are linearly dependent, then the manifold is said to be W_1^*-Ricci pseudosymmetric.

In this case, there exists a function $F_{W_1^*}$ on M^{2n+1} such that

$$W_1^*\cdot S = F_{W_1^*}Q(g,S).$$

In particular, if $F_{W_1^*}=0$, the manifold M^{2n+1} is said to be W_1^*-Ricci semisymmetric.

Let us now investigate the W_1^*-Ricci pseudosymmetry case of the $(2n+1)-$dimensional para-Sasakian manifold under condition the almost $\eta-$Ricci soliton.

Theorem 3.2. *Let M^{2n+1} be para-Sasakian manifold and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a W_1^*-Ricci pseudosymmetric, then M^{2n+1} is either Ricci semisymmetric or $\lambda=2n$ and in this case almost $\eta-$Ricci soliton (g,ξ,λ,μ) reduces Ricci soliton (g,ξ,λ).*

Proof. Let's assume that para-Sasakian manifold M^{2n+1} be W_1^*-Ricci pseudosymmetric and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on para-Sasakian manifold M^{2n+1}. That's mean

$$\left(W_1^*(\varpi,\rho)\cdot S\right)(\upsilon,\varrho) = F_{W_1^*}Q(g,S)(\upsilon,\varrho;\varpi,\rho),$$

for all $\varpi,\rho,\upsilon,\varrho\in\Gamma(TM)$. From the last equation, we can easily write

$$\begin{aligned} &S\left(W_1^*(\varpi,\rho)\upsilon,\varrho\right) + S\left(\upsilon,W_1^*(\varpi,\rho)\varrho\right) \\ &= F_{W_1^*}\left\{S\left((\varpi\wedge_g\rho)\upsilon,\varrho\right) + S\left(\upsilon,(\varpi\wedge_g\rho)\varrho\right)\right\}. \end{aligned} \tag{32}$$

If we putting $\varrho = \xi$ in (32), we get

$$
\begin{aligned}
& S\left(W_1^*\left(\varpi,\rho\right)\upsilon,\xi\right) + S\left(\upsilon, W_1^*\left(\varpi,\rho\right)\xi\right) \\
& = F_{W_1^*}\left\{S\left(g\left(\rho,\upsilon\right)\varpi - g\left(\varpi,\upsilon\right)\rho,\xi\right)\right. \\
& \left. + S\left(\upsilon,\eta\left(\rho\right)\varpi - \eta\left(\varpi\right)\rho\right)\right\}.
\end{aligned} \tag{33}
$$

If making use of (11) and (30) in (33), we have

$$
\begin{aligned}
& F_{W_1^*}\left\{-2ng\left(\eta\left(\varpi\right)\rho - \eta\left(\rho\right)\varpi,\upsilon\right) + S\left(\upsilon,\eta\left(\rho\right)\varpi - \eta\left(\varpi\right)\rho\right)\right\} \\
& = -2n\eta\left(W_1^*\left(\varpi,\rho\right)\upsilon\right).
\end{aligned} \tag{34}
$$

Setting (31) and (17) in (34), we get

$$
F_{W_1^*}\left(\lambda - 2n\right) g\left(\eta\left(\varpi\right)\rho - \eta\left(\rho\right)\varpi,\upsilon\right) = 0. \tag{35}
$$

This completes the proof. □

We can give the results obtained from this theorem as follows.

COROLLARY 3.3. *Let M^{2n+1} be para-Sasakian manifold and (g,ξ,λ,μ) be almost η−Ricci soliton on M^{2n+1}. If M^{2n+1} is a W_1^*−Ricci semi-symmetric, then $\lambda = 2n$ and M^{2n+1} is an Einstein manifold.*

COROLLARY 3.4. *Let M^{2n+1} be para-Sasakian manifold and (g,ξ,λ,μ) be almost η−Ricci soliton on M^{2n+1}. If M^{2n+1} is a W_1^*−Ricci semiymmetric, then M^{2n+1} is an expanding.*

For an $(2n+1)$ −dimensional semi-Riemann manifold, the W_2−curvature tensor is defined as

$$
W_2\left(\varpi,\rho\right)\tau = R\left(\varpi,\rho\right)\tau - \frac{1}{2n}\left[g\left(\rho,\tau\right)Q\varpi - g\left(\varpi,\tau\right)Q\rho\right]. \tag{36}
$$

For an $(2n+1)$ −dimensional para-Sasakian manifold, if we choose $\tau = \xi$ in (36), we can write

$$
\begin{aligned}
& W_2\left(\varpi,\rho\right)\xi = \left[\eta\left(\varpi\right)\rho - \eta\left(\rho\right)\varpi\right] \\
& -\tfrac{1}{2n}\left[\eta\left(\rho\right)Q\varpi - \eta\left(\varpi\right)Q\rho\right],
\end{aligned} \tag{37}
$$

by means of (37), we get

$$
\begin{aligned}
& \eta\left(W_2\left(\varpi,\rho\right)\tau\right) = g\left(\eta\left(\rho\right)\varpi - \eta\left(\varpi\right)\rho,\tau\right) \\
& +\tfrac{1}{2n}S\left(\eta\left(\rho\right)\varpi - \eta\left(\varpi\right)\rho,\tau\right).
\end{aligned} \tag{38}
$$

DEFINITION 3.3. Let M^{2n+1} be an $(2n+1)$ −dimensional para-Sasakian manifold. If $W_2 \cdot S$ and $Q(g,S)$ are linearly dependent, then the manifold is said to be W_2−Ricci pseudosymmetric.

In this case, there exists a function F_{W_2} on M^{2n+1} such that

$$W_2 \cdot S = F_{W_2} Q(g,S).$$

In particular, if $F_{W_2}=0$, M^{2n+1} is called W_2-Ricci semi-symmetric.

Next, let us now investigate the W_2-Ricci pseudosymmetry case of the $(2n+1)-$dimensional para-Sasakian manifold.

THEOREM 3.3. *Let M^{2n+1} be para-Sasakian manifold and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a W_2-Ricci pseudosymmetric, then $F_{W_2}=\dfrac{\lambda-2n}{2n}$ provided $\lambda-2n\neq 0$.*

PROOF. Let's assume that para-Sasakian manifold M^{2n+1} is a W_2-Ricci pseudosymmetric and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on para-Sasakian manifold. This means

$$(W_2(\varpi,\rho)\cdot S)(\upsilon,\varrho)=F_{W_2}Q(g,S)(\upsilon,\varrho;\varpi,\rho), \tag{39}$$

for all $\varpi,\rho,\upsilon,\varrho\in\Gamma(TM)$. From (39), we can easily write

$$\begin{aligned}&S(W_2(\varpi,\rho)\upsilon,\varrho)+S(\upsilon,W_2(\varpi,\rho)\varrho)\\&=F_{W_2}\{S((\varpi\wedge_g\rho)\upsilon,\varrho)+S(\upsilon,(\varpi\wedge_g\rho)\varrho)\}.\end{aligned} \tag{40}$$

If we choose $\varrho=\xi$ in (40), we get

$$\begin{aligned}&S(W_2(\varpi,\rho)\upsilon,\xi)+S(\upsilon,W_2(\varpi,\rho)\xi)\\&=F_{W_2}\{S(g(\rho,\upsilon)\varpi-g(\varpi,\upsilon)\rho,\xi)\\&+S(\upsilon,\eta(\rho)\varpi-\eta(\varpi)\rho)\}.\end{aligned} \tag{41}$$

Making use of (11) and (37) in (41), we have

$$\begin{aligned}&F_{W_2}\{-2ng(\eta(\varpi)\rho-\eta(\rho)\varpi,\upsilon)+S(\upsilon,\eta(\rho)\varpi-\eta(\varpi)\rho)\}\\&=-2n\eta(W_2(\varpi,\rho)\upsilon)+S(\upsilon,[\eta(\varpi)\rho-\eta(\rho)\varpi]\\&-\tfrac{1}{2n}[\eta(\rho)Q\varpi-\eta(\varpi)Q\rho]).\end{aligned} \tag{42}$$

If we use (38) in (42), we get

$$\begin{aligned}&F_{W_2}\{-2ng(\eta(\varpi)\rho-\eta(\rho)\varpi,\upsilon)+S(\upsilon,\eta(\rho)\varpi-\eta(\varpi)\rho)\}\\&=2ng(\eta(\rho)\varpi-\eta(\varpi)\rho,\upsilon)-2S(\eta(\rho)\varpi-\eta(\varpi)\rho,\upsilon)\\&-\tfrac{1}{2n}S(\upsilon,\eta(\rho)Q\varpi-\eta(\varpi)Q\rho).\end{aligned} \tag{43}$$

If we use (17) in (43), we have

$$\begin{aligned}&(2-\lambda)g(\eta(\rho)\varpi-\eta(\varpi)\rho,\upsilon)+\tfrac{\lambda}{2n}S(\eta(\rho)\varpi-\eta(\varpi)\rho,\upsilon)\\&=F_{W_2}(2n-\lambda)g(\eta(\rho)\varpi-\eta(\varpi)\rho,\upsilon).\end{aligned} \tag{44}$$

If we use (17) in (44), we have

$$\left[\frac{\lambda^2}{2n} - 2(\lambda - n) + (2n - \lambda) F_{W_2}\right] g(\eta(\rho)\varpi - \eta(\varpi)\rho, \upsilon) = 0. \tag{45}$$

This completes the proof. □

We can give the sub-results obtained from this theorem as follows.

COROLLARY 3.5. *Let M^{2n+1} be para-Sasakian manifold and (g, ξ, λ, μ) be almost $\eta-$Ricci soliton on M^{2n+1}. M^{2n+1} is not admits a W_2-Ricci semisymmetric.*

For an $(2n+1)-$dimensional semi-Riemann manifold M, the W_3-curvature tensor is defined as

$$W_3(\varpi, \rho)\tau = R(\varpi, \rho)\tau - \frac{1}{2n}\left[S(\varpi, \tau)\rho - g(\rho, \tau) Q\varpi\right]. \tag{46}$$

For an $(2n+1)-$dimensional para-Sasakian manifold, if we choose $\tau = \xi$ in (46), we have

$$W_3(\varpi, \rho)\xi = 2\eta(\varpi)\rho - \eta(\rho)\varpi + \frac{1}{2n}\eta(\rho) Q\varpi. \tag{47}$$

Furthermore, we take the inner product of both sides of (46) by ξ, we get

$$\begin{aligned} \eta(W_3(\varpi, \rho)\tau) &= g(\eta(\rho)\varpi - 2\eta(\varpi)\rho, \tau) \\ &- \tfrac{1}{2n} S(\varpi, \tau)\eta(\rho). \end{aligned} \tag{48}$$

DEFINITION 3.4. Let M^{2n+1} be an $(2n+1)-$dimensional para-Sasakian manifold. If $W_3 \cdot S$ and $Q(g, S)$ are linearly dependent, then the manifold is said to be W_3-Ricci pseudosymmetric.

In this case, there exists a function F_{W_3} on M^{2n+1} such that

$$W_3 \cdot S = F_{W_3} Q(g, S).$$

In particular, if $F_{W_3} = 0$, M^{2n+1} is said to be W_3-Ricci semi-symmetric.

Now, taking into account that M is a W_3-Ricci pseudosymmetry case of the $(2n+1)-$dimensional para-Sasakian manifold.

THEOREM 3.4. *Let M^{2n+1} be para-Sasakian manifold and (g, ξ, λ, μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a W_3-Ricci pseudosymmetric, then M^{2n+1} is either expanding or $F_{W_3} = -2$.*

PROOF. We suppose that para-Sasakian manifold M^{2n+1} is a W_3-Ricci pseudosymmetric and (g, ξ, λ, μ) is almost $\eta-$Ricci soliton on para-Sasakian manifold. Then we have

$$(W_3(\varpi, \rho) \cdot S)(\upsilon, \varrho) = F_{W_3} Q(g, S)(\upsilon, \varrho; \varpi, \rho),$$

for all $\varpi, \rho, \upsilon, \varrho \in \Gamma(TM)$. From the last equation, we can easily write

$$\begin{aligned} &S(W_3(\varpi, \rho)\upsilon, \varrho) + S(\upsilon, W_3(\varpi, \rho)\varrho) \\ &= F_{W_3}\left\{S((\varpi \wedge_g \rho)\upsilon, \varrho) + S(\upsilon, (\varpi \wedge_g \rho)\varrho)\right\}. \end{aligned} \tag{49}$$

If we choose $\varrho = \xi$ in (49) , we get

$$\begin{aligned} &S\left(W_3\left(\varpi,\rho\right)\upsilon,\xi\right)+S\left(\upsilon,W_3\left(\varpi,\rho\right)\xi\right) \\ &= F_{W_3}\left\{S\left(g\left(\rho,\upsilon\right)\varpi-g\left(\varpi,\upsilon\right)\rho,\xi\right)\right. \\ &\left.+S\left(\upsilon,\eta\left(\rho\right)\varpi-\eta\left(\varpi\right)\rho\right)\right\}. \end{aligned} \tag{50}$$

If we (11) and (47) are put in (50) , we have

$$\begin{aligned} &F_{W_3}\left\{-2ng\left(\eta\left(\varpi\right)\rho-\eta\left(\rho\right)\varpi,\upsilon\right)+S\left(\upsilon,\eta\left(\rho\right)\varpi-\eta\left(\varpi\right)\rho\right)\right\} \\ &= -2n\eta\left(W_3\left(\varpi,\rho\right)\upsilon\right)+S\left(\upsilon,2\eta\left(\varpi\right)\rho-\eta\left(\rho\right)\varpi\right. \\ &\left.+\tfrac{1}{2n}\eta\left(\rho\right)Q\varpi\right). \end{aligned} \tag{51}$$

If we use (48) in (51), we get

$$\begin{aligned} &4ng\left(\rho,\upsilon\right)\eta\left(\varpi\right)-2ng\left(\varpi,\upsilon\right)\eta\left(\rho\right)+S\left(\upsilon,2\eta\left(\varpi\right)\rho+\tfrac{1}{2n}\eta\left(\rho\right)Q\varpi\right) \\ &= F_{W_3}\left\{-2ng\left(\eta\left(\varpi\right)\rho-\eta\left(\rho\right)\varpi,\upsilon\right)+S\left(\upsilon,\eta\left(\rho\right)\varpi-\eta\left(\varpi\right)\rho\right)\right\}. \end{aligned} \tag{52}$$

Substituting (17) into (52), we have

$$\begin{aligned} &\left[\left(2n-\lambda\right)\left(F_{W_3}+2\right)\right]g\left(\rho,\upsilon\right)\eta\left(\varpi\right) \\ &+\left[\tfrac{\lambda^2}{2n}-\left(2n-\lambda\right)F_{W_3}-2n\right]g\left(\varpi,\upsilon\right)\eta\left(\rho\right) \\ &+\tfrac{(2n-\lambda)(\lambda-2n)}{2n}\eta\left(\varpi\right)\eta\left(\rho\right)\eta\left(\upsilon\right)=0. \end{aligned} \tag{54}$$

If we choose $\varpi = \xi$ in (54), we obtain

$$\begin{aligned} &\left[\left(2n-\lambda\right)\left(F_{W_3}+2\right)\right]g\left(\rho,\upsilon\right) \\ &-\left[\left(2n-\lambda\right)\left(F_{W_3}+2\right)\right]\eta\left(\rho\right)\eta\left(\upsilon\right)=0. \end{aligned} \tag{55}$$

From (55), one can easily to see

$$-\left(2n-\lambda\right)\left(F_{W_3}+2\right)g\left(\phi\rho,\phi\upsilon\right)=0.$$

This completes the proof. □

COROLLARY 3.6. *Let M^{2n+1} be para-Sasakian manifold and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a W_3-Ricci semisymmetric, then M^{2n+1} is an expanding.*

For an $(2n+1)-$dimensional M semi-Riemann manifold, the W_4-curvature tensor is defined as

$$W_4\left(\varpi,\rho\right)\tau=R\left(\varpi,\rho\right)\tau+\frac{1}{2n}\left[g\left(\varpi,\tau\right)Q\rho-g\left(\varpi,\rho\right)Q\tau\right]. \tag{56}$$

For an $(2n+1)-$dimensional para-Sasakian manifold, if we choose $\tau=\xi$ in (56), we can write

$$W_4(\varpi,\rho)\xi=[\eta(\varpi)\rho-\eta(\rho)\varpi]+\frac{1}{2n}\eta(\varpi)Q\rho+g(\varpi,\rho)\xi, \tag{57}$$

and similarly if we take the inner product of both sides of (56) by ξ, we get

$$\begin{aligned}\eta(W_4(\varpi,\rho)\tau)&=-g(\eta(\tau)\varpi+\eta(\varpi)\tau,\rho)\\&+g(\varpi,\tau)\eta(\rho)-\tfrac{1}{2n}S(\rho,\tau)\eta(\varpi).\end{aligned} \tag{58}$$

Definition 3.5. Let M^{2n+1} be an $(2n+1)-$dimensional para-Sasakian manifold. If $W_4\cdot S$ and $Q(g,S)$ are linearly dependent, then the manifold M^{2n+1} is said to be W_4-Ricci pseudosymmetric.

In this case, there exists a function F_{W_4} on M^{2n+1} such that

$$W_4\cdot S=F_{W_4}Q(g,S).$$

In particular, if $F_{W_4}=0$, the manifold M^{2n+1} is said to be W_4-Ricci semi-symmetric.

Let us now investigate the W_4-Ricci pseudosymmetry case of the $(2n+1)-$dimensional para-Sasakian manifold.

Theorem 3.5. *Let M^{2n+1} be para-Sasakian manifold and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a W_4-Ricci pseudosymmetric, then $F_{W_4}=\frac{\lambda-2n}{2n}$ provided $\lambda\neq 2n$.*

Proof. Let's assume that para-Sasakian manifold be W_4-Ricci pseudosymmetric and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on para-Sasakian manifold. That's mean

$$(W_4(\varpi,\rho)\cdot S)(\upsilon,\varrho)=F_{W_4}Q(g,S)(\upsilon,\varrho;\varpi,\rho),$$

for all $\varpi,\rho,\upsilon,\varrho\in\Gamma(TM)$. From the last equation, we can easily write

$$\begin{aligned}&S(W_4(\varpi,\rho)\upsilon,\varrho)+S(\upsilon,W_4(\varpi,\rho)\varrho)\\&=F_{W_4}\{S((\varpi\wedge_g\rho)\upsilon,\varrho)+S(\upsilon,(\varpi\wedge_g\rho)\varrho)\}.\end{aligned} \tag{59}$$

If we choose $\varrho=\xi$ in (59), we get

$$\begin{aligned}&S(W_4(\varpi,\rho)\upsilon,\xi)+S(\upsilon,W_4(\varpi,\rho)\xi)\\&=F_{W_4}\{S(g(\rho,\upsilon)\varpi-g(\varpi,\upsilon)\rho,\xi)\\&+S(\upsilon,\eta(\rho)\varpi-\eta(\varpi)\rho)\}.\end{aligned} \tag{60}$$

If making use of (11) and (57) in (60), we have

$$\begin{aligned}&F_{W_4}\{-2ng(\eta(\varpi)\rho-\eta(\rho)\varpi,\upsilon)+S(\upsilon,\eta(\rho)\varpi-\eta(\varpi)\rho)\}\\&=-2n\eta(W_4(\varpi,\rho)\upsilon)+S(\upsilon,g(\varpi,\rho)\xi+\eta(\varpi)\rho\\&-\eta(\rho)\varpi+\tfrac{1}{2n}\eta(\varpi)Q\rho).\end{aligned} \tag{61}$$

By using (58) in (61), we get

$$
\begin{aligned}
& F_{W_4}\left\{-2ng\left(\eta\left(\varpi\right)\rho-\eta\left(\rho\right)\varpi,\upsilon\right)+S\left(\upsilon,\eta\left(\rho\right)\varpi-\eta\left(\varpi\right)\rho\right)\right\} \\
& =2ng\left(\varpi,\rho\right)\eta\left(\upsilon\right)+2ng\left(\rho,\upsilon\right)\eta\left(\varpi\right)-2ng\left(\varpi,\upsilon\right)\eta\left(\rho\right) \\
& +2S\left(\rho,\upsilon\right)\eta\left(\varpi\right)+S\left(\upsilon,g\left(\varpi,\rho\right)\xi-\eta\left(\rho\right)\varpi\right. \\
& \left.+\tfrac{1}{2n}\eta\left(\varpi\right)Q\rho\right).
\end{aligned} \tag{62}
$$

If we use (17) in (62), we have

$$
\begin{aligned}
& -\left(2n-\lambda\right)\left(1+F_{W_4}\right)g\left(\varpi,\upsilon\right)\eta\left(\rho\right) \\
& +\left[\tfrac{\lambda^2}{2n}+2\left(n-\lambda\right)+\left(2n-\lambda\right)F_{W_4}\right]g\left(\rho,\upsilon\right)\eta\left(\varpi\right) \\
& +\left[\lambda-\tfrac{\lambda^2}{2n}\right]\eta\left(\varpi\right)\eta\left(\rho\right)\eta\left(\upsilon\right)=0.
\end{aligned} \tag{63}
$$

If we choose $\varpi=\xi$ in (63), we can easily to see

$$
\begin{aligned}
& \left[\tfrac{\lambda^2}{2n}+2\left(n-\lambda\right)+\left(2n-\lambda\right)F_{W_4}\right]g\left(\rho,\upsilon\right) \\
& -\left[\tfrac{\lambda^2}{2n}+2\left(n-\lambda\right)+\left(2n-\lambda\right)F_{W_4}\right]\eta\left(\rho\right)\eta\left(\upsilon\right)=0,
\end{aligned} \tag{64}
$$

which is equivalent to

$$
-\left[\frac{\lambda^2}{2n}+2\left(n-\lambda\right)+\left(2n-\lambda\right)F_{W_4}\right]g\left(\phi\rho,\phi\upsilon\right)=0. \tag{65}
$$

This completes the proof. □

Corollary 3.7. *Let M^{2n+1} be para-Sasakian manifold and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}.If M^{2n+1} is a W_4-Ricci semisymmetric, then M^{2n+1} is an expanding.*

For an $(2n+1)-$dimensional semi-Riemann manifold M, the W_5-curvature tensor is given by

$$
W_5\left(\varpi,\rho\right)\tau=R\left(\varpi,\rho\right)\tau-\frac{1}{2n}\left[S\left(\varpi,\tau\right)\rho-g\left(\varpi,\tau\right)Q\rho\right]. \tag{66}
$$

For an $(2n+1)-$dimensional para-Sasakian manifold, if we choose $\tau=\xi$ in (66), we can write

$$
W_5\left(\varpi,\rho\right)\xi=2\eta\left(\varpi\right)\rho-\eta\left(\rho\right)\varpi+\frac{1}{2n}\eta\left(\varpi\right)Q\rho, \tag{67}
$$

and similarly if we take the inner product of both sides of (67) by ξ, we get

$$
\begin{aligned}
& \eta\left(W_5\left(\varpi,\rho\right)\tau\right)=-2g\left(\rho,\tau\right)\eta\left(\varpi\right)+g\left(\varpi,\tau\right)\eta\left(\rho\right) \\
& -\tfrac{1}{2n}S\left(\rho,\tau\right)\eta\left(\varpi\right).
\end{aligned} \tag{68}
$$

DEFINITION 3.6. Let M^{2n+1} be an $(2n+1)-$dimensional para-Sasakian manifold. If $W_5 \cdot S$ and $Q(g,S)$ are linearly dependent, then the manifold is said to be W_5-Ricci pseudosymmetric.

In this case, there exists a function F_{W_5} on M^{2n+1} such that

$$W_5 \cdot S = F_{W_5} Q(g,S).$$

In particular, if $F_{W_5} = 0$, the manifold M^{2n+1} is said to be W_5-Ricci semisymmetric.

Let us now investigate the W_5-Ricci pseudosymmetry case of the $(2n+1)-$dimensional para-Sasakian manifold.

THEOREM 3.6. *Let M^{2n+1} be para-Sasakian manifold and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}.If M^{2n+1} is a W_5-Ricci pseudosymmetric, then $F_{W_5} = \frac{\lambda-4n}{2n}$ provided $\lambda \neq 2n$.*

PROOF. Let's assume that para-Sasakian manifold be W_5-Ricci pseudosymmetric and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on para-Sasakian manifold. That's mean

$$(W_5(\varpi,\rho)\cdot S)(\upsilon,\varrho) = F_{W_5} Q(g,S)(\upsilon,\varrho;\varpi,\rho),$$

for all $\varpi,\rho,\upsilon,\varrho \in \Gamma(TM)$. From the last equation, we can easily write

$$\begin{aligned} & S(W_5(\varpi,\rho)\upsilon,\varrho) + S(\upsilon, W_5(\varpi,\rho)\varrho) \\ & = F_{W_5}\{S((\varpi\wedge_g\rho)\upsilon,\varrho) + S(\upsilon,(\varpi\wedge_g\rho)\varrho)\}. \end{aligned} \tag{69}$$

If we choose $\varrho = \xi$ in (69), we get

$$\begin{aligned} & S(W_4(\varpi,\rho)\upsilon,\xi) + S(\upsilon, W_4(\varpi,\rho)\xi) \\ & = F_{W_4}\{S(g(\rho,\upsilon)\varpi - g(\varpi,\upsilon)\rho,\xi) \\ & +S(\upsilon,\eta(\rho)\varpi - \eta(\varpi)\rho)\}. \end{aligned} \tag{70}$$

If we make use of (11) and (67) in (70), we have

$$\begin{aligned} & F_{W_5}\{-2ng(\eta(\varpi)\rho - \eta(\rho)\varpi,\upsilon) + S(\upsilon,\eta(\rho)\varpi - \eta(\varpi)\rho)\} \\ & = -2n\eta(W_5(\varpi,\rho)\upsilon) + S(\upsilon, 2\eta(\varpi)\rho - \eta(\rho)\varpi \\ & +\tfrac{1}{2n}\eta(\varpi)Q\rho). \end{aligned} \tag{71}$$

If we use (68) in (71), we get

$$\begin{aligned} & 4ng(\rho,\upsilon)\eta(\varpi) - 2ng(\varpi,\upsilon)\eta(\rho) + S(\rho,\upsilon)\eta(\varpi) \\ & +S\left(\upsilon, 2\eta(\varpi)\rho - \eta(\rho)\varpi + \tfrac{1}{2n}\eta(\varpi)Q\rho\right) \\ & = F_{W_5}\{-2ng(\eta(\varpi)\rho - \eta(\rho)\varpi,\upsilon) + S(\upsilon,\eta(\rho)\varpi - \eta(\varpi)\rho)\}. \end{aligned} \tag{72}$$

If we use (17) in the (72), we have

$$
\begin{aligned}
&\left[\tfrac{\lambda^2}{2n}+4n-3\lambda+(2n-\lambda)F_{W_5}\right]g(\rho,\upsilon)\eta(\varpi)\\
&+(\lambda-2n)(1+F_{W_5})g(\varpi,\upsilon)\eta(\rho)\\
&+\left[\tfrac{\lambda}{2n}(2n-\lambda)-(2n-\lambda)\right]\eta(\varpi)\eta(\rho)\eta(\upsilon)=0
\end{aligned}
\tag{73}
$$

If we putting $\varpi=\xi$ in (73), we can write

$$
\begin{aligned}
&\left[\tfrac{\lambda^2}{2n}+4n-3\lambda+(2n-\lambda)F_{W_5}\right]g(\rho,\upsilon)\\
&-\left[\tfrac{\lambda^2}{2n}+4n-3\lambda+(2n-\lambda)F_{W_5}\right]\eta(\rho)\eta(\upsilon)=0.
\end{aligned}
\tag{74}
$$

It is clear from (74) that

$$
-\left[\frac{\lambda^2}{2n}+4n-3\lambda+(2n-\lambda)F_{W_5}\right]g(\phi\rho,\phi\upsilon)=0.
$$

This completes the proof. □

COROLLARY 3.8. *Let M^{2n+1} be para-Sasakian manifold and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a W_5-Ricci semisymmetric, then M^{2n+1} is an expanding.*

EXAMPLE 3.1. Let's take the $M=\mathbb{R}^3$ manifold and assume that this manifold is given in standard Cartesian coordinates ϖ,ρ,τ. We choose the vector fields

$$
e_1=e^{\varpi}\frac{\partial}{\partial\rho},e_2=e^{\varpi}\left(\frac{\partial}{\partial\rho}-\frac{\partial}{\partial\tau}\right),e_3=-\frac{\partial}{\partial\varpi},
$$

which are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$
g(e_i,e_j)=\delta_{ij}.
$$

Let η be the $1-$form defined by

$$
g(\tau,e_3)=\eta(\tau),
$$

for any vector field τ on M. We define the $(1,1)$-type tensor field ϕ as

$$
\phi(e_1)=e_1,\phi(e_2)=e_2 \text{ and } \phi(e_3)=0.
$$

Thus, $\xi=e_3,(\phi,\xi,\eta,g)$ defines an almost paracontact structure on the manifold M. Let ∇ be the Levi-Civita connection with respect to g. Then we have

$$
[e_1,e_2]=0,[e_1,e_3]=e_1,[e_2,e_3]=e_2.
$$

If we use Koszul's formula, we have

$$\nabla_{e_1}e_2 = 0, \qquad \nabla_{e_1}e_3 = e_1, \qquad \nabla_{e_1}e_1 = -e_3,$$
$$\nabla_{e_2}e_3 = e_2, \qquad \nabla_{e_2}e_2 = -e_3, \qquad \nabla_{e_2}e_1 = 0,$$
$$\nabla_{e_3}e_3 = 0, \qquad \nabla_{e_3}e_2 = 0, \qquad \nabla_{e_3}e_1 = 0.$$

This shows us that the (ϕ, ξ, η, g) structure is a para-Sasakian structure on the manifold M. and it is clear that $M(\phi, \xi, \eta, g)$ is a $3-$dimensional para-Sasakian manifold (see [**19**]). By using above result, we can easily obtain the following:

$$R(e_1, e_2)e_2 = -e_1, \qquad R(e_1, e_3)e_3 = -e_1, \qquad R(e_2, e_1)e_1 = -e_2,$$
$$R(e_2, e_3)e_3 = -e_2, \qquad R(e_3, e_1)e_1 = -e_3, \qquad R(e_3, e_2)e_2 = -e_3,$$
$$R(e_1, e_2)e_3 = 0, \qquad R(e_3, e_2)e_3 = e_2, \qquad R(e_3, e_1)e_2 = 0.$$

Tracing the Riemann curvature tensor, the components of the Ricci tensor is given by:

$$S(e_1, e_1) = -2, \qquad S(e_2, e_2) = -2, \qquad S(e_3, e_3) = -2,$$
$$S(e_1, e_2) = 0, \qquad S(e_1, e_3) = 0, \qquad S(e_2, e_3) = 0.$$

That is

$$S(\varpi, \rho) = -2g(\varpi, \rho).$$

Thus we have

$$\lambda = 2.$$

This manifold in the example is always expanding. Furthermore, for the above theorems, when it is a W_2-Ricci pseudosymmetric,

$$F_{W_2} = -\frac{2}{3},$$

when it is a W_4-Ricci pseudosymmetric,

$$F_{W_4} = -\frac{2}{3},$$

when it is a W_5-Ricci pseudosymmetric,

$$F_{W_5} = -\frac{5}{3}.$$

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