JOURNAL OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4866, ISSN (o) 2303-4947
www.imvibl.org /JOURNALS/JOURNAL
J. Int. Math. Virtual Inst., **14**(1)(2024), 49-70
DOI: 10.7251/JIMVI2401049H

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

ANALYSIS OF A QUASI-STATIC ELECTRO-ELASTIC-VISCOPLASTIC CONTACT PROBLEM WITH DAMAGE AND ADHESION

Ahmed Hamidat, Adel Aissaoui, and Hakim Bagua

ABSTRACT. In this paper, we focus on investigating a mathematical problem that characterizes the adhesive contact between an electro-elastic-viscoplastic material with damage and a foundation. The contact phenomenon is represented using the Signorini condition, while the progression of damage is delineated through a parabolic-type inclusion. Additionally, the adhesion process is rendered via a bonding field applied to the contact surface. The entire process occurs quasistatically. We establish a weak formulation of the system and subsequently demonstrate the existence of a unique weak solution to the problem. The proof relies on a comprehensive result concerning evolution equations involving maximal monotone operators, parabolic inequalities, differential equations, and fixed points.

1. Introduction

Contact between deformable bodies or between deformable and rigid bodies is prevalent in both industrial applications and daily life. Consequently, significant efforts have been dedicated to modeling and analyzing these contacts. Due to their inherent intricacy, contact phenomena give rise to novel and compelling mathematical models. The mathematical scrutiny of contact problems relies on fundamental physical principles and necessitates expertise in partial differential equations, nonlinear analysis, and numerical methods.

Despite the extensive research in this field, there is a dearth of mathematical results addressing contact problems involving piezoelectric materials. Thus, there

2020 *Mathematics Subject Classification.* Primary 74M10, Secondary 74M15, 49J40.
Key words and phrases. Piezoelectric materials, elastic-viscoplastic, quasistatic, Signorini's condition, adhesion, fixed point.
Communicated by Dusko Bogdanic.

is a pressing need to expand existing models for contact with deformable bodies to encompass the coupling between mechanical and electrical properties. General models for elastic materials with piezoelectric effects can be derived from sources such as [**2**, **3**, **13**–**15**, **22**, **23**], and related references. Additionally, our own contributions to this area are documented in the sources [**7**–**11**].

A piezoelectric material is characterized by its ability to generate an electrical charge in response to mechanical stress, such as compression or stretching. Conversely, applying an electric field to the material induces mechanical deformation, causing it to expand or contract. This class of materials is commonly encountered in various industrial applications, including radiotronic switches, electroacoustics, and measurement equipment. The initial confirmation of the direct piezoelectric effect dates back to 1880, attributed to Pierre and Jacques Curie, who not only predicted but also experimentally verified the existence of piezoelectricity.

Adhesion processes play a crucial role in numerous industrial scenarios, particularly in the bonding of nonmetallic components. Consequently, there has been a notable surge in attention to this subject within the realm of mathematical literature. Models for adhesive contact are analyzed in works such as [**16**, **17**], as well as in the comprehensive monographs presented in [**19**]. The application of adhesive contact theory extends to the medical field, with a focus on prosthetic limbs explored in [**18**]. In this context, the significance of the bond between the bone-implant and the tissue is emphasized, as any debonding may result in a reduction in the individual's ability to effectively use the artificial limb or joint.

The consideration of damage holds significant importance in design engineering as it directly impacts the lifespan of the engineered structure or component. Extensive literature within engineering delves into this subject. Mathematical investigations have explored models that account for the influence of internal material damage on the contact process. In particular, innovative general models for damage were formulated in [**5**] based on the virtual power principle. The mathematical analysis of one-dimensional problems pertaining to damage can be found in [**6**]. Contact problems involving damage have undergone examination in [**12**] and related references.

Our focus lies in describing and analyzing a physical process that encompasses contact, adhesion, damage, and the piezoelectric effect. The process is quasi-static, and Signorini-type boundary conditions were employed, as demonstrated in [**4**].

The remainder of the paper is structured as follows. In Section 2, we provide the necessary preliminaries. Section 3 introduces the mechanical problem, outlines assumptions on the data, and derives the variational formulation of the problem. The existence and uniqueness of the solution are demonstrated in Section 4.

2. Preliminaries

In this short section, we present the notation we shall use and some preliminary material. We denote by $\mathbb{S}^d$ the space of symmetric tensors on $\mathbb{R}^d$. We define the inner product and the Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$, respectively, by

$$\boldsymbol{\vartheta}\cdot\boldsymbol{w}=\vartheta_i w_i,\quad \forall\boldsymbol{\vartheta},\boldsymbol{w}\in\mathbb{R}^d \quad\text{and}\quad \boldsymbol{\varsigma}\cdot\boldsymbol{\delta}=\varsigma_{ij}\delta_{ij}\quad\forall\boldsymbol{\varsigma},\boldsymbol{\delta}\in\mathbb{S}^d,$$
$$\|\boldsymbol{\vartheta}\|=(\boldsymbol{\vartheta}\cdot\boldsymbol{\vartheta})^{\frac{1}{2}},\quad\forall\boldsymbol{\vartheta}\in\mathbb{R}^d\quad\text{and}\quad\|\boldsymbol{\varsigma}\|=(\boldsymbol{\varsigma}\cdot\boldsymbol{\varsigma})^{\frac{1}{2}},\quad\forall\boldsymbol{\varsigma}\in\mathbb{S}^d.$$

In the expressions provided, the indices i and j range from 1 to d, and the summation convention over repeated indices is employed. Additionally, an index following a comma indicates a partial derivative concerning the corresponding component of the independent variable.

Let $\Omega\subset\mathbb{R}^d$ represent a bounded domain with a smooth boundary Γ, and let ν symbolize the unit outward normal vector on Γ. The function spaces are defined as follows

$$H=L^2(\Omega)^d=\left\{\boldsymbol{\vartheta}=(\vartheta_i)\mid\vartheta_i\in L^2(\Omega)\right\},\quad H_1=\{\boldsymbol{\vartheta}=(\vartheta_i)\mid\varepsilon(\boldsymbol{\vartheta})\in\mathcal{H}\},$$
$$\mathcal{H}=\left\{\boldsymbol{\varsigma}=(\varsigma_{ij})\mid\varsigma_{ij}=\varsigma_{ji}\in L^2(\Omega)\right\},\quad\mathcal{H}_1=\{\boldsymbol{\varsigma}\in\mathcal{H}\mid\operatorname{Div}\boldsymbol{\varsigma}\in H\}.$$

Here, ε represents the deformation operator, and Div stands for the divergence operator. These operators are defined as follows

$$\varepsilon(\boldsymbol{\vartheta})=(\varepsilon_{ij}(\boldsymbol{\vartheta})),\quad\varepsilon(\boldsymbol{\vartheta})=\frac{1}{2}\left(\boldsymbol{\vartheta}_{i,j}+\boldsymbol{\vartheta}_{j,i}\right),\quad\operatorname{Div}(\boldsymbol{\varsigma})=\varsigma_{ij,j}.$$

The sets H, H_1, $\mathcal{H}$, and $\mathcal{H}_1$ are real Hilbert spaces equipped with the canonical inner products

$$(\boldsymbol{\vartheta},\boldsymbol{w})_H=\int_\Omega\vartheta_i w_i dx\quad\forall\boldsymbol{\vartheta},\mathbf{w}\in H,\quad(\boldsymbol{\varsigma},\boldsymbol{\delta})_{\mathcal{H}}=\int_\Omega\varsigma_{ij}\delta_{ij}dx\quad\forall\boldsymbol{\varsigma},\boldsymbol{\delta}\in\mathcal{H},$$

$$(\boldsymbol{\vartheta},\boldsymbol{w})_{H_1}=(\boldsymbol{\vartheta},\boldsymbol{w})_H+(\varepsilon(\boldsymbol{\vartheta}),\varepsilon(\boldsymbol{w}))_{\mathcal{H}},\quad\forall\boldsymbol{\vartheta},\boldsymbol{w}\in H_1,$$
$$(\boldsymbol{\varsigma},\boldsymbol{\delta})_{\mathcal{H}_1}=(\boldsymbol{\varsigma},\boldsymbol{\delta})_{\mathcal{H}}+(\operatorname{Div}\boldsymbol{\varsigma},\operatorname{Div}\boldsymbol{\delta})_H,\quad\boldsymbol{\varsigma},\boldsymbol{\delta}\in\mathcal{H}_1.$$

The corresponding norms are represented as $\|.\|_H$, $\|.\|_{H_1}$, $\|.\|_{\mathcal{H}}$ and $\|.\|_{\mathcal{H}_1}$. Let $H_\Gamma=H^{\frac{1}{2}}(\Gamma)^d$ and $\gamma:H_1\to H_\Gamma$ be the trace map. For every element $\boldsymbol{\vartheta}\in H_1$, We use the notation $\boldsymbol{\vartheta}$ to represent the trace $\gamma\boldsymbol{\vartheta}$ of $\boldsymbol{\vartheta}$ on the surface Γ. Additionally, we refer to ϑ_ν and $\boldsymbol{\vartheta}_\tau$ as the normal and tangential components of $\boldsymbol{\vartheta}$ on Γ respectively. These components are defined as follows

$$\vartheta_\nu=\boldsymbol{\vartheta}.\boldsymbol{\nu},\quad\boldsymbol{\vartheta}_\tau=\boldsymbol{\vartheta}-\vartheta_\nu\boldsymbol{\nu}. \tag{2.1}$$

We recall that when $\boldsymbol{\varsigma}$ is a regular function then the normal component and the tangential part of the stress field $\boldsymbol{\varsigma}$ on the boundary are defined by

$$\varsigma_\nu=\varsigma\nu\cdot\boldsymbol{\nu},\quad\boldsymbol{\varsigma}_\tau=\boldsymbol{\varsigma}\boldsymbol{\nu}-\varsigma_\nu\boldsymbol{\nu}, \tag{2.2}$$

and for all $\boldsymbol{\varsigma}\in\mathcal{H}_1$ the following Green's formula holds

$$(\boldsymbol{\varsigma},\varepsilon(\boldsymbol{v}))_{\mathcal{H}}+(\operatorname{Div}\boldsymbol{\varsigma},\boldsymbol{v})_H=\int_\Gamma\boldsymbol{\varsigma}\nu.\boldsymbol{v}da\quad\forall\boldsymbol{v}\in H_1. \tag{2.3}$$

In the context of any real Hilbert space V, we employ the conventional symbols for the spaces $L^p(0,T;V)$ and $W^{k,p}(0,T;V)$, where $1\leqslant p\leqslant\infty$ and $k\geqslant 1$.

For a function $\Psi : V \rightarrow]-\infty, \infty]$ we use the notation $D(\Psi)$ and $\partial\Psi$ for the effective domain and the subdifferential of Ψ, i.e.

$$D(\Psi) = \{\boldsymbol{\vartheta} \in V \mid \Psi(\boldsymbol{\vartheta}) \neq \infty\}.$$
$$\partial\Psi(\boldsymbol{\vartheta}) = \{\boldsymbol{f} \in V \mid \Psi(\boldsymbol{w}) - \Psi(\boldsymbol{\vartheta}) \geqslant (\boldsymbol{f}, \boldsymbol{w} - \boldsymbol{\vartheta})_V,\ \forall \boldsymbol{w} \in V\}, \quad \forall \boldsymbol{\vartheta} \in V.$$

We denote by $D(A)$ is the domain of A given by

$$D(A) = \{\boldsymbol{\vartheta} \in V \mid A\boldsymbol{\vartheta} \neq \emptyset\}.$$

3. Mechanical and variational formulations

We describe the model for the process, we present its variational formulation. Here's the context: Consider an electroelastic-viscoplastic body occupying a limited region $\Omega \subset \mathbb{R}^d (d = 2, 3)$ with an outer Lipschitz surface Γ. The body experiences body forces with a density of f_0 and volume electric charges with a density of q_0. It is subject to mechanical and electrical constraints on the boundary. Γ is divided into three disjoint parts Γ_1, Γ_2, and Γ_3, and further subdivided into open portions Γ_a and Γ_b, with meas $(\Gamma_1) > 0$ and meas $(\Gamma_a) > 0$. For a specified time interval $T > 0$ within $[0, T]$, the body is fixed on $\Gamma_1 \times (0, T)$, resulting in zero displacement there. A surface traction of density f_2 acts on $\Gamma_2 \times (0, T)$. Additionally, the electrical potential is zero on $\Gamma_a \times (0, T)$, and a surface electric charge of density q_2 is prescribed on $\Gamma_b \times (0, T)$. The body is in adhesive contact with a rigid insulator obstacle, termed the foundation, on Γ_3. This contact is frictionless and is modeled using the Signorini condition.

The classical formulation of the mechanical problem for an electro-elastic-viscoplastic material with damage and adhesive properties can be stated as follows.

Problem P**.** Find a displacement field $\boldsymbol{u} : \Omega \times (0, T) \rightarrow \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega \times (0, T) \rightarrow \mathbb{S}^d$, an electric potential field $\varphi : \Omega \times (0, T) \rightarrow \mathbb{R}$, an electric displacement field $\boldsymbol{D} : \Omega \times (0, T) \rightarrow \mathbb{R}^d$, a damage field $\beta : \Omega \times (0, T) \rightarrow \mathbb{R}$, and a bonding field $\alpha : \Gamma_3 \times (0, T) \rightarrow \mathbb{R}$ such that

$$\begin{aligned} \boldsymbol{\sigma}(t) &= \mathcal{L}\varepsilon(\dot{\boldsymbol{u}}(t)) + \mathcal{N}\varepsilon(\boldsymbol{u}(t)) + \mathcal{Z}^* E(\varphi(t)) \\ &+ \int_0^t \mathcal{Q}\left(\boldsymbol{\sigma}(s) - \mathcal{L}\varepsilon(\dot{\boldsymbol{u}}(s) - \mathcal{Z}^* E(\varphi(s))), \varepsilon\left(\boldsymbol{u}(s)\right), \beta(s)\right) ds \quad \text{in } \Omega \times (0, T), \end{aligned} \tag{3.1}$$

$$\boldsymbol{D} = \mathcal{Z}\varepsilon(\boldsymbol{u}) + \boldsymbol{M}E(\varphi) \qquad \text{in } \Omega \times (0, T), \tag{3.2}$$

$$\dot{\beta} - k\Delta\beta + \partial\psi_K(\beta) \ni \mathcal{C}\left(\boldsymbol{\sigma} - \mathcal{L}\varepsilon(\dot{\boldsymbol{u}}) - \mathcal{Z}^*\nabla(\varphi), \varepsilon(\boldsymbol{u}), \beta\right), \quad \text{in } \Omega \times (0, T), \tag{3.3}$$

$$\operatorname{Div}\boldsymbol{\sigma} + f_0 = 0 \qquad \text{in } \Omega \times (0,T), \tag{3.4}$$

$$\operatorname{div}\boldsymbol{D} - q_0 = 0 \qquad \text{in } \Omega \times (0,T), \tag{3.5}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \Gamma_1 \times (0,T), \tag{3.6}$$

$$\boldsymbol{\sigma\nu} = f_2 \qquad \text{on } \Gamma_2 \times (0,T), \tag{3.7}$$

$$\begin{cases} u_\nu \leqslant 0 \\ \sigma_\nu - \gamma_\nu \alpha^2 R_\nu(u_\nu) \leqslant 0, \\ \left(\sigma_\nu - \gamma_\nu \alpha^2 R_\nu(u_\nu)\right) u_\nu = 0 \end{cases} \qquad \text{on } \Gamma_3 \times (0,T), \tag{3.8}$$

$$-\boldsymbol{\sigma}_\tau = p_\tau(\alpha) R_\tau(\boldsymbol{u}_\tau) \qquad \text{on } \Gamma_3 \times (0,T), \tag{3.9}$$

$$\dot{\alpha}(t) = -\left(\gamma_\nu \alpha(t) R_\nu(u_\nu(t))^2 - \varepsilon_a\right)_+ \qquad \text{on } \Gamma_3 \times (0,T), \tag{3.10}$$

$$\frac{\partial \beta}{\partial \nu} = 0 \qquad \text{on } \Gamma \times (0,T), \tag{3.11}$$

$$\varphi = 0 \qquad \text{on } \Gamma_a \times (0,T), \tag{3.12}$$

$$\boldsymbol{D}.\nu = q_2 \qquad \text{on } \Gamma_b \times (0,T), \tag{3.13}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \beta(0) = \beta_0, \qquad \text{in } \Omega. \tag{3.14}$$

$$\alpha(0) = \alpha_0, \qquad \text{on } \Gamma_3. \tag{3.15}$$

Equations (3.1) and (3.2) delineate the electroelastic-viscoplastic constitutive law with damage, where $\mathcal{L}$ and $\mathcal{N}$ represent the viscosity and elasticity operators, respectively. The nonlinear constitutive function $\mathcal{Q}$ characterizes the viscoplastic behavior of the material, with β serving as an internal variable describing the material damage resulting from elastic deformations. Here, $E(\varphi) = -\nabla\varphi$ denotes the electric field, and $\mathcal{Z} = (z_{ijk})$ represents the third-order piezoelectric tensor, with $\mathcal{Z}^*$ denoting its transposition.

The evolution of the damage field is controlled by a parabolic-type inclusion as described in relation (3.3), where K represents the set of admissible damage functions defined by.

$$K = \left\{\zeta \in H^1(\Omega) \mid 0 \leqslant \zeta \leqslant 1 \text{ a.e. in } \Omega\right\},$$

Here, $\partial\psi_K$ denotes the subdifferential of the indicator function for the set K, and $\mathcal{C}$ is a prescribed constitutive function that characterizes the sources of damage within the system.

Equations (3.4) and (3.5) depict the equilibrium equations governing the stress and electric displacement fields. Equations (3.6)-(3.7) are the displacement-traction conditions.

We assume that the resistance to tangential motion is generated only by the glue, and is assumed to depend on the adhesion field and on the tangential displacement (see (3.9)), condition (3.8) represents the Signorini contact condition with adhesion, where u_ν is the normal displacement, σ_ν represents the normal stress, γ_ν indicate a given adhesion coefficient, and R_ν, R_τ are the truncation operators defined by

$$R_\nu(s)=\begin{cases}-p & \text{if } l<-p\\ -l & \text{if } -p\leqslant l\leqslant 0\\ 0 & \text{if } l>0\end{cases},\qquad R_\tau(w)=\begin{cases}w & \text{if } \|w\|\leqslant p,\\ p\frac{w}{\|w\|} & \text{if } \|w\|>p.\end{cases}$$

Here $p>0$ is the characteristic length of the bond, beyond which it does not offer any additional traction (see [**19**]). $p_\tau(\alpha)$ acts as the solidity or spring constant, mounting with (α), and the traction is in the direction adverse to the displacement. The maximal modulus of the tangential traction is $p_\tau(1)p$.

Equation (3.10) constitutes the ordinary differential equation describing the evolution of the bonding field. The equation includes positive parameters γ_ν and ε_a, where $r_+=\max 0,r$. Relation (3.11) specifies a homogeneous Neumann boundary condition, while (3.12) and (3.13) delineate the electric boundary conditions.

Finally, The functions $\boldsymbol{u}_0$, β_0 and α_0 in (3.14) and (3.15) are the initial data.

To formulate the variational approach for the problem (3.1)-(3.15), the set for the bonding field is defined as follows

$$\mathcal{X}=\left\{\theta\in L^\infty\left(0,T;L^2\left(\Gamma_3\right)\right):0\leqslant\theta(t)\leqslant 1,\forall t\in[0,T],\ \text{a.e. on } \Gamma_3\right\},$$

and for the displacement field we need the closed subspace of $H^1(\Omega)^d$ defined by

$$V=\left\{\boldsymbol{\vartheta}\in H^1(\Omega)^d\mid \boldsymbol{\vartheta}=0\text{ on }\Gamma_1\right\}.$$

Given that meas $(\Gamma_1)>0$, Korn's inequality is applicable, ensuring the existence of a constant $C_0>0$ depending solely on Ω and Γ_1. This constant satisfies the inequality

$$\|\varepsilon(\boldsymbol{\vartheta})\|_{\mathcal{H}}\geqslant C_0\|\boldsymbol{\vartheta}\|_{H^1(\Omega)^d},\quad \forall\boldsymbol{\vartheta}\in V.$$

On V, we consider the inner product defined by

$$(\boldsymbol{\vartheta},\boldsymbol{w})_V=(\mathcal{L}\varepsilon(\boldsymbol{\vartheta}),\varepsilon(\boldsymbol{w}))_{\mathcal{H}}, \tag{3.16}$$

and let $\|.\|_V$ be the associated norm.

We employ the Hilbert space for the electric displacement field, defined as

$$\mathcal{W}=\left\{\mathbf{D}=(D_i)\mid D_i\in L^2(\Omega),\operatorname{div}\mathbf{D}\in L^2(\Omega)\right\},$$

equipped with the inner product

$$(\mathbf{D},\mathbf{E})_{\mathcal{W}}=\int_\Omega\mathbf{D}\cdot\mathbf{E}dx+\int_\Omega\operatorname{div}\mathbf{D}\cdot\operatorname{div}\mathbf{E}dx,$$

and the corresponding norm $\|.\|_{\mathcal{W}}$. We are seeking the electric potential field within

$$W=\left\{\Theta\in H^1(\Omega),\Theta=0\text{ on }\Gamma_a\right\}.$$

Since meas $(\Gamma_a)>0$, the Friedrichs-Poincaré inequality holds

$$\|\nabla\Theta\|_H\geqslant c_{Fd}\|\Theta\|_{H^1(\Omega)},\quad \forall\Theta\in W, \tag{3.17}$$

Here, $c_{Fd}>0$ represents a constant that relies solely on Ω and Γ_a. On W we use the inner product

$$(\phi,\Theta)_W=(\nabla\phi,\nabla\Theta)_H, \tag{3.18}$$

Additionally, $\|.\|_W$ denotes the corresponding norm. From (3.17), it is evident that $\|.\|_{H^1(\Omega)}$ and $\|.\|_W$ are equivalent norms on W, establishing $(W, \|.\|_W)$ as a Hilbert space. Furthermore, according to the Sobolev trace theorem, there exist two positive constants c_1 and c_2 such that

$$\|\boldsymbol{\vartheta}\|_{L^2(\Gamma_3)^d} \leqslant c_1\|\boldsymbol{\vartheta}\|_V, \quad \forall \boldsymbol{\vartheta} \in V, \quad \|\Theta\|_{L^2(\Gamma_3)} \leqslant c_2\|\Theta\|_W, \quad \forall \Theta \in W. \tag{3.19}$$

Furthermore, if $\boldsymbol{D} \in \mathcal{W}$ is a regular function, the subsequent Green's type formula is valid

$$(\boldsymbol{D}, \nabla\kappa)_H + (\operatorname{div}\boldsymbol{D}, \kappa)_{L^2(\Omega)} = \int_\Gamma \boldsymbol{D}\cdot\boldsymbol{\nu}\kappa da, \quad \forall \kappa \in H^1(\Omega). \tag{3.20}$$

When analysing problem P, we take into account the following assumptions $\mathcal{L} : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d$ is a viscosity operator such that

(3.21)
$$\begin{cases} \text{(a) } \mathcal{L}(\boldsymbol{x}, \boldsymbol{\upsilon}) = (a_{ijkl}(\boldsymbol{x})\upsilon_{kl}) \text{ for all } \boldsymbol{\upsilon} \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(b) } a_{ijkl} = a_{klij} = a_{jikl} \in L^\infty(\Omega), \\ \text{(c) } a_{ijkl}\upsilon_{ij}\upsilon_{kl} \geqslant m_{\mathcal{L}}\|\boldsymbol{\upsilon}\|^2, \text{ for all } \boldsymbol{\upsilon} = (\upsilon_{ij}) \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega, \text{ with } m_{\mathcal{L}} > 0. \end{cases}$$

$\mathcal{N} : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d$ is an elasticity operator such that

(3.22)
$$\begin{cases} \text{(a) } \|\mathcal{N}(\boldsymbol{x}, \boldsymbol{\upsilon}_1) - \mathcal{N}(\boldsymbol{x}, \boldsymbol{\upsilon}_2)\| \leqslant L_{\mathcal{N}}\|\boldsymbol{\upsilon}_1 - \boldsymbol{\upsilon}_2\|, \forall \boldsymbol{\upsilon}_1, \boldsymbol{\upsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega, \text{ with } L_{\mathcal{N}} > 0. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto \mathcal{N}(\boldsymbol{x}, \boldsymbol{\upsilon}) \text{ is Lebesgue measurable on } \Omega, \\ \text{for any } \boldsymbol{\upsilon} \in \mathbb{S}^d. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{N}(\boldsymbol{x}, \boldsymbol{0}) \in \mathcal{H}. \end{cases}$$

$\mathcal{Q} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \longrightarrow \mathbb{R}$ is a visco-plasticity operator such that

$$\begin{cases} \text{(a) } \|\mathcal{Q}(\boldsymbol{x}, \boldsymbol{\delta}_1, \boldsymbol{\varsigma}_1, \rho_1) - \mathcal{Q}(\boldsymbol{x}, \boldsymbol{\delta}_2, \boldsymbol{\varsigma}_2, \rho_2)\| \\ \qquad \leqslant L_{\mathcal{Q}}\left(\|\boldsymbol{\delta}_1 - \boldsymbol{\delta}_2\| + \|\boldsymbol{\varsigma}_1 - \boldsymbol{\varsigma}_2\| + \|\rho_1 - \rho_2\|\right), \\ \text{for all } \boldsymbol{\varsigma}_1, \boldsymbol{\varsigma}_2, \boldsymbol{\delta}_1, \boldsymbol{\delta}_2 \in \mathbb{S}^d, \text{ for all } \rho_1, \rho_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Omega, \text{ with } L_{\mathcal{Q}} > 0 \\ \text{(b) The mapping } \boldsymbol{x} \mapsto \mathcal{Q}(\boldsymbol{x}, \boldsymbol{\delta}, \boldsymbol{\varsigma}, \rho) \text{ is Lebesgue measurable on } \Omega, \\ \qquad \text{for all } \boldsymbol{\delta}, \boldsymbol{\varsigma} \in \mathbb{S}^d, \text{ for all } \rho \in \mathbb{R}. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{Q}(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{0}, 0) \in \mathcal{H}. \end{cases} \tag{3.23}$$

$\mathcal{C} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \longrightarrow \mathbb{R}$ is a function such that

$$(3.24)\quad \begin{cases} \|\mathcal{C}(\boldsymbol{x},\boldsymbol{\delta}_1,\boldsymbol{\varsigma}_1,\rho_1)-\mathcal{C}(\boldsymbol{x},\boldsymbol{\delta}_2,\boldsymbol{\varsigma}_2,\rho_2)\| \\ \qquad \leqslant L_{\mathcal{C}}\left(\|\boldsymbol{\delta}_1-\boldsymbol{\delta}_2\|+\|\boldsymbol{\varsigma}_1-\boldsymbol{\varsigma}_2\|+\|\rho_1-\rho_2\|\right), \\ \text{for all } \boldsymbol{\varsigma}_1,\boldsymbol{\varsigma}_2,\boldsymbol{\delta}_1,\boldsymbol{\delta}_2\in\mathbb{S}^d, \text{ for all } \rho_1,\rho_2\in\mathbb{R}, \text{ a.e. } \boldsymbol{x}\in\Omega, \text{ with } L_{\mathcal{C}}>0. \\ \text{(b) The mapping } \boldsymbol{x}\mapsto\mathcal{C}(\boldsymbol{x},\boldsymbol{\delta},\boldsymbol{\varsigma},\rho) \text{ is Lebesgue measurable on } \Omega, \\ \qquad \text{for all } \boldsymbol{\delta},\boldsymbol{\varsigma}\in\mathbb{S}^d, \text{ for all } \rho\in\mathbb{R}. \\ \text{(c) The mapping } \boldsymbol{x}\mapsto\mathcal{C}(\boldsymbol{x},\mathbf{0},\mathbf{0},0)\in L^2(\Omega). \end{cases}$$

$\boldsymbol{M}=(m_{ij}):\Omega\times\mathbb{R}^d\to\mathbb{R}^d$ is an electric permittivity operator such that

$$(3.25)\quad \begin{cases} \text{(a) } \boldsymbol{M}(\boldsymbol{x},E)=(m_{ij}(\boldsymbol{x})E_j) \text{ for all } E=(E_i)\in\mathbb{R}^d, \text{ a.e. } \boldsymbol{x}\in\Omega. \\ \text{(b) } m_{ij}=m_{ji}\in L^\infty(\Omega), 1\leqslant i,j\leqslant d. \\ \text{(c) There exists a constant } C_{\boldsymbol{M}}>0 \text{ such that} \\ \qquad \boldsymbol{M}E.E\geqslant m_{\boldsymbol{M}}\|E\|^2, \text{ for all } E=(E_i)\in\mathbb{R}^d, \text{ a.e. in } \Omega, \text{with } C_{\boldsymbol{M}}>0. \end{cases}$$

$\mathcal{Z}:\Omega\times\mathbb{S}^d\to\mathbb{R}^d$ is a piezoelectric operator such that

$$(3.26)\quad \begin{cases} \text{(a) } \mathcal{Z}(\boldsymbol{x},\boldsymbol{\varsigma})=(f_{ijk}(\boldsymbol{x})\varsigma_{jk}), \quad \text{for all } \boldsymbol{\varsigma}=(\varsigma_{ij})\in\mathbb{S}^d, \text{ a.e. } \boldsymbol{x}\in\Omega. \\ \text{(b) } f_{ijk}=f_{ikj}\in L^\infty(\Omega), \quad 1\leqslant i,j,k\leqslant d. \end{cases}$$

$p_\tau:\Gamma_3\times\mathbb{R}\longrightarrow\mathbb{R}_+$ is a tangential contact function such that

$$(3.27)\quad \begin{cases} \text{(a) } \|p_\tau(\boldsymbol{x},\rho_1)-p_\tau(\boldsymbol{x},\rho_2)\|\leqslant L_\tau\|\rho_1-\rho_2\|, \\ \text{for all } \rho_1,\rho_2\in\mathbb{R}, \text{ a.e. } \boldsymbol{x}\in\Gamma_3, \text{ with } L_\tau>0 \\ \text{(b) } \|p_\tau(\boldsymbol{x},\rho)\|\leqslant M_\tau, \text{ for all } \rho\in\mathbb{R}, \text{ a.e. } \boldsymbol{x}\in\Gamma_3, \text{ with } M_\tau>0. \\ \text{(c) For any } \rho\in\mathbb{R}, \boldsymbol{x}\mapsto p_\tau(\boldsymbol{x},\rho) \text{ is measurable on } \Gamma_3. \\ \text{(d) The mapping } \boldsymbol{x}\mapsto p_\tau(\boldsymbol{x},0) \text{ belongs to } L^2(\Gamma_3). \end{cases}$$

The adhesion coefficients and the limit bound satisfies

$$(3.28)\quad \gamma_\nu\in L^\infty(\Gamma_3), \quad \varepsilon_a\in L^2(\Gamma_3), \quad \gamma_\nu,\gamma_\tau,\varepsilon_a\geqslant 0 \text{ a.e. on } \Gamma_3.$$

The initial bonding field satisfies

$$(3.29)\quad \alpha_0\in L^2(\Gamma_3), \quad 0\leqslant\alpha_0\leqslant 1 \text{ a.e. on } \Gamma_3.$$

and the initial damage field satisfies

$$(3.30)\quad \beta_0\in K.$$

For the Signorini problem, we employ the convex subset of permissible displacement fields, as indicated by the following expression

$$(3.31)\quad U_{ad}=\{\boldsymbol{u}\in V\mid u_\nu\leqslant 0 \text{ on } \Gamma_3\},$$

Additionally, we assume the regularity condition in our analysis

$$(3.32)\quad \boldsymbol{u}_0\in U_{ad}.$$

The provided potential achieve the specified condition.

$$\varphi_0 \in L^2(\Gamma_3). \tag{3.33}$$

The forces, traction, volume, and surface free charge densities meet the following conditions

$$f_0 \in W^{1,1}\left(0,T;L^2(\Omega)^d\right), \quad f_2 \in W^{1,1}\left(0,T;L^2\left(\Gamma_2\right)^d\right), \tag{3.34}$$

$$q_0 \in C\left(0,T;L^2(\Omega)\right), \quad q_2 \in C\left(0,T;L^2\left(\Gamma_b\right)\right). \tag{3.35}$$

$$q_2(t) = 0 \text{ on } \Gamma_3, \quad \forall t \in (0,T). \tag{3.36}$$

It is crucial to note that assumption (3.36) is imposed for physical reasons. Specifically, the foundation is assumed to be an insulator

We establish the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ through the following definition

$$a(\xi,\zeta) = k\int_\Omega \nabla\xi.\nabla\zeta dx, \tag{3.37}$$

The microcrack diffusion coefficient satisfies the following condition

$$k > 0. \tag{3.38}$$

Next. We define three mappings $j : L^2(\Gamma_3) \times V \times V \to \mathbb{R}$, , $f : [0,T] \to V$ and $q : [0,T] \to W$, respectively, by

$$j(\alpha,\boldsymbol{u},\boldsymbol{v}) = \int_{\Gamma_3} p_\tau(\alpha)R_\tau\left(\boldsymbol{u}_\tau\right)\cdot\boldsymbol{v}_\tau da - \int_{\Gamma_3}\gamma_\nu\alpha^2 R_\nu\left(u_\nu\right)v_\nu da, \tag{3.39}$$

$$(\boldsymbol{f}(t),\boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t)\cdot\boldsymbol{v}dx + \int_{\Gamma_2}\boldsymbol{f}_2(t)\cdot\boldsymbol{v}da, \tag{3.40}$$

$$(q(t),\zeta)_W = \int_\Omega q_0(t)\zeta dx - \int_{\Gamma_b} q_2(t)\zeta da, \tag{3.41}$$

for all $\boldsymbol{u},\boldsymbol{v} \in V$, $\varphi,\zeta \in W$ and $t \in (0,T)$. Note that

$$\boldsymbol{f} \in W^{1,1}\left(0,T;V\right), \quad q \in C(0,T;W). \tag{3.42}$$

Using a standard procedure that relies on Green's formula, we can deduce the following variational formulation for the contact problem (3.1)-(3.15).

Problem PV**.** Find a displacement field $\boldsymbol{u} : (0,T) \to V$, a stress field $\boldsymbol{\sigma} : (0,T) \to \mathcal{H}$, an electric potential $\varphi : (0,T) \to W$, a damage field $\beta : (0,T) \to H^1(\Omega)$, and a bonding field $\alpha : (0,T) \to L^2(\Gamma_3)$, such that

(3.43)
$$\begin{aligned}\boldsymbol{\sigma}(t) =&\mathcal{L}\varepsilon(\dot{\boldsymbol{u}}(t)) + \mathcal{N}(\varepsilon \boldsymbol{u}(t)) + \mathcal{Z}^*\nabla\varphi(t)\\ &+\int_0^t \mathcal{Q}\left(\boldsymbol{\sigma}(s) - \mathcal{L}\varepsilon(\dot{\boldsymbol{u}}(s) - \mathcal{Z}^*\nabla\varphi(t)), \varepsilon\left(\boldsymbol{u}(s), \beta(s)\right)\right) ds,\end{aligned}$$

(3.44)
$$\begin{aligned}&\boldsymbol{u}(t) \in U_{ad}, (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{\vartheta}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_{\mathcal{H}} + j(\alpha(t), \boldsymbol{u}(t), \boldsymbol{\vartheta} - \boldsymbol{u}(t))\\ &\qquad \geqslant (\boldsymbol{f}(t), \boldsymbol{\vartheta} - \boldsymbol{u}(t))_V, \quad \forall \boldsymbol{\vartheta} \in U_{ad}, \quad \text{a.e. } t \in (0,T),\end{aligned}$$

(3.45)
$$(\boldsymbol{M}\nabla\varphi(t), \nabla\Theta)_H - (\mathcal{Z}\varepsilon(\boldsymbol{u}(t)), \nabla\Theta)_H = (q(t), \Theta)_W \quad \forall \Theta \in W, t \in (0,T),$$

(3.46)
$$\begin{aligned}&\beta(t) \in K, (\dot{\beta}(t), \kappa - \beta(t))_{L^2(\Omega)} + a(\beta(t), \kappa - \beta(t))\\ &\qquad \geqslant \left(\mathcal{C}\left(\boldsymbol{\sigma}(t) - \mathcal{L}\varepsilon(\dot{\boldsymbol{u}}(t)) - \mathcal{Z}^*\nabla\varphi(t), \varepsilon(\boldsymbol{u}(t)), \beta(t)\right), \kappa - \beta(t)\right)_{L^2(\Omega)},\\ &\qquad \forall \kappa \in K, t \in (0,T),\end{aligned}$$

(3.47)
$$\dot{\alpha}(t) = -\left(\gamma_\nu \alpha(t) R_\nu \left(u_\nu(t)\right)^2 - \varepsilon_a\right)_+ . \text{ a.e. } t \in (0,T),$$

(3.48)
$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \beta(0) = \beta_0, \quad \alpha(0) = \alpha_0.$$

We consider this note, essential for later use.

Remark 3.1. We observe that in Problem P and Problem PV, there is no explicit need to impose the constraint $0 \leqslant \alpha \leqslant 1$. Indeed, (3.48) ensures that $\alpha(\boldsymbol{x}, t) \leqslant \alpha_0(\boldsymbol{x})$, and thus, assumption (3.29) implies that $\alpha(\boldsymbol{x}, t) \leqslant 1$ for $t \geqslant 0$, almost everywhere for $\boldsymbol{x} \in \Gamma_3$. Conversely, if $\alpha(\boldsymbol{x}, t_0) = 0$ at time t_0, then from (3.48), it follows that $\dot{\alpha}(\boldsymbol{x}, t) = 0$ for all $t \geqslant t_0$, and consequently, $\alpha(\boldsymbol{x}, t) = 0$ for all $t \geqslant t_0$, almost everywhere for $\boldsymbol{x} \in \Gamma_3$. We conclude that $0 \leqslant \alpha(\boldsymbol{x}, t) \leqslant 1$ for all $t \in [0,T]$, a.e. $\boldsymbol{x} \in \Gamma_3$.

4. Existence and uniqueness

Theorem 4.1. *Suppose that conditions (3.21)-(3.36) are satisfied. In such a case, there exists a singular solution* $(\boldsymbol{u}, \boldsymbol{\sigma}, \varphi, \beta, \alpha, \boldsymbol{D})$ *to problem* PV*. Furthermore, this solution exhibits a distinctive level of regularity.*

$$\boldsymbol{u} \in W^{1,\infty}(0,T;V), \tag{4.1}$$

$$\varphi \in C(0,T;W), \tag{4.2}$$

$$\boldsymbol{\sigma} \in W^{1,\infty}(0,T;\mathcal{H}), \quad Div\boldsymbol{\sigma} \in W^{1,\infty}(0,T;H), \tag{4.3}$$

$$\beta \in W^{1,2}\left(0,T;L^2(\Omega)\right) \cap L^2\left(0,T;H^1(\Omega)\right), \tag{4.4}$$

$$\alpha \in W^{1,\infty}\left(0,T;L^2\left(\Gamma_3\right)\right) \cap \mathcal{Q}. \tag{4.5}$$

$$\boldsymbol{D} \in C(0,T;\mathcal{W}). \tag{4.6}$$

The functions $\boldsymbol{u}$, $\boldsymbol{\sigma}$, φ, β, α, and $\boldsymbol{D}$ which satisfy (3.43)-(3.48) are called a weak solution of the contact problem P. We conclude that, under the assumptions (3.21)-(3.36), the mechanical problem (3.1)-(3.15) has a unique weak solution satisfying (4.1)-(4.6).

The demonstration of Theorem 4.1 unfolds in several steps and relies on a broader outcome concerning evolution equations featuring maximal monotone operators, parabolic inequalities, differential equations, and fixed points.

We designate by C a constant, and its value may vary from line to line, provided that there is no potential for confusion.

Suppose $\boldsymbol{\eta} \in L^\infty(0,T;\mathcal{H})$. In the initial step, we examine the following variational problem.

Problem $\mathcal{P}_\eta^1$. Find a displacement field $\mathbf{u}_\eta : (0,T) \to V$, such that

(4.7)
$$(\mathcal{L}\varepsilon(\dot{\boldsymbol{u}}_\eta(t)), \varepsilon(\boldsymbol{\vartheta} - \boldsymbol{u}_\eta(t)))_{\mathcal{H}} + (\mathcal{N}\varepsilon(\boldsymbol{u}_\eta(t)), \varepsilon(\boldsymbol{\vartheta} - \boldsymbol{u}_\eta(t)))_{\mathcal{H}} + (\boldsymbol{\eta}(t), \boldsymbol{\vartheta} - \boldsymbol{u}_\eta(t))_{\mathcal{H}}$$
$$\geqslant (\boldsymbol{f}(t), \boldsymbol{\vartheta} - \boldsymbol{u}_\eta(t))_V, \forall \boldsymbol{\vartheta} \in V, \text{ a.e. } t \in (0,T),$$

(4.8)
$$\boldsymbol{u}_\eta(0) = \boldsymbol{u}_0.$$

We have the following result for $\mathcal{P}_\eta^1$

Lemma 4.1. *A solution to Problem $\mathcal{P}_\eta^1$ with the regularity (4.1) exists and is unique.*

Proof. We employ the Riesz representation theorem to establish a definition.

$$(N\boldsymbol{u}, \boldsymbol{\vartheta})_V = (\mathcal{N}\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{\vartheta}))_{\mathcal{H}}, \quad \forall \boldsymbol{u}, \boldsymbol{\vartheta} \in V. \tag{4.9}$$

Now, from (3.16), (3.21), (3.22) and (4.9), we obtain that

$$\|N\boldsymbol{u}_1 - N\boldsymbol{u}_2\|_V \leqslant \frac{L_{\mathcal{N}}}{m_{\mathcal{L}}} \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V, \quad \forall \boldsymbol{u}_1, \boldsymbol{u}_2 \in V, \tag{4.10}$$

In other words, N is an operator that exhibits Lipschitz continuity. Furthermore, the operator

$$N + \frac{L_{\mathcal{N}}}{m_{\mathcal{L}}} I_V : V \to V,$$

constitutes a monotonic and Lipschitz continuous operator over the space V.

The function $\Psi_{U_{ad}} : V \to]-\infty; +\infty]$ represents the indicator function of the set U_{ad}. Let $\partial\Psi_{U_{ad}}$ denote the subdifferential of $\Psi_{U_{ad}}$. Since U_{ad} is a nonempty, convex, closed subset of V, it follows that $\partial\Psi_{U_{ad}}$ is a maximal monotone operator on V, with $D(\partial\Psi_{U_{ad}}) = U_{ad}$. Moreover, consider the sum

$$\partial\Psi_{U_{ad}} + N + \frac{L_{\mathcal{N}}}{m_{\mathcal{L}}} I_V : U_{ad} \subset V \to 2^V,$$

which also constitutes a maximal monotone operator.

Consider the function $f_{\boldsymbol{\eta}}(t)$ defined in the vector space V.

$$(f_{\boldsymbol{\eta}}(t), \boldsymbol{\vartheta})_V = (f(t), \boldsymbol{\vartheta})_V - (\boldsymbol{\eta}(t), \boldsymbol{\vartheta})_{\mathcal{H}}, \quad \forall \boldsymbol{\vartheta} \in U_{ad},$$

keeping in mind $\boldsymbol{\eta} \in L^{\infty}(0,T;\mathcal{H})$, it follow that $f_{\boldsymbol{\eta}} \in W^{1,\infty}(0,T;V)$.

Let $L = \partial\Psi_{U_{ad}} + N$ with $D(L) = U_{ad} \subset V$, satisfying the conditions (3.32). This enables us to invoke a classical result on evolution equations involving maximal monotone operators, as outlined in ([**1**], p. 32). Consequently, there exists a unique element $\boldsymbol{u}_\eta \in W^{1,\infty}(0,T;V)$ such that

$$\begin{aligned} &\dot{\boldsymbol{u}}_\eta(t) + L\boldsymbol{u}_\eta(t) \ni f_{\boldsymbol{\eta}}(t), \\ &\boldsymbol{u}_\eta(0) = \boldsymbol{u}_0. \end{aligned}$$

Therefore

$$\dot{\boldsymbol{u}}_\eta(t) + \partial\Psi_{U_{ad}}(\boldsymbol{u}_\eta(t)) + N\boldsymbol{u}_\eta(t) \ni f_\eta(t) \text{ a.e.t } \in (0,T), \tag{4.11}$$

$$\boldsymbol{u}_\eta(0) = \boldsymbol{u}_0. \tag{4.12}$$

For any $\boldsymbol{u}, \boldsymbol{h} \in V$, we obtain

$$\boldsymbol{h} \in \partial\Psi_{Uad}(\boldsymbol{u}) \Leftrightarrow \boldsymbol{u} \in U_{ad}, (\boldsymbol{h}, \boldsymbol{\vartheta} - \boldsymbol{u})_V \leqslant 0, \quad \forall \boldsymbol{\vartheta} \in U_{ad},$$

Following the inclusion (4.11), we can assert that $\boldsymbol{u}\eta(t) \in Uad$, leading to the subsequent variational inequality

$$\begin{aligned} &(\dot{\boldsymbol{u}}_\eta(t), \boldsymbol{\vartheta} - \boldsymbol{u}_\eta(t))_V + (N\boldsymbol{u}_\eta(t), \boldsymbol{\vartheta} - \boldsymbol{u}_\eta(t))_V \\ &\qquad \geqslant (f_\eta(t), \boldsymbol{\vartheta} - \boldsymbol{u}_\eta(t))_V, \forall \boldsymbol{\vartheta} \in U_{ad} \text{ a.e. } t \in (0,T). \end{aligned} \tag{4.13}$$

By employing (3.16) and (4.9), we conclude that there exists a unique solution to problem $\mathcal{P}^1_\eta$, and it meets the condition (4.1). □

In the subsequent step, we utilise the displacement field acquired in Lemma 4.1 to formulate the following variational problem for the electrical potential.

Problem $\mathcal{P}^2_\eta$. Find an electrical potential $\varphi_\eta : (0,T) \to W$ such that

$$(\boldsymbol{M}\nabla\varphi_\eta(t), \nabla\Theta)_H - (\mathcal{P}\varepsilon(\boldsymbol{u}_\eta(t)), \nabla\Theta)_H = (q(t), \Theta)_W, \quad \forall\Theta \in W, t \in (0,T). \tag{4.14}$$

The following outcome holds for problem $\mathcal{P}^2_\eta$.

LEMMA 4.2. *Problem (4.14) possesses a unique solution φ_η that adheres to the regularity condition (4.2).*

PROOF. We examine the bilinear form $b(\cdot,\cdot) : W \times W \to \mathbb{R}$ defined as follows

$$b(\varphi, \Theta) = (\boldsymbol{M}\nabla\varphi, \nabla\Theta)_H \quad \forall\varphi, \Theta \in W. \tag{4.15}$$

By employing (4.15), (3.17), and (3.25), we establish that the bilinear form b is continuous, symmetric, and coercive on W. Additionally, leveraging (3.41) and the Riesz representation Theorem, we can define an element $q_\eta : [0,T] \to W$ such that

$$(q_\eta(t), \phi)_W = (q(t), \phi)_W + (\mathcal{Z}\varepsilon(\boldsymbol{u}_\eta(t)), \nabla\phi)_H \quad \forall\phi \in W, t \in (0,T).$$

Applying the Lax-Milgram Theorem, we conclude that there exists a unique element $\varphi_\eta(t) \in W$ such that

$$b(\varphi_\eta(t), \Theta) = (q_\eta(t), \Theta)_W \quad \forall\Theta \in W. \tag{4.16}$$

The implication of (4.16) is that $\varphi_\eta(t)$ constitutes a solution to $\mathcal{P}^2_\eta$. Let $t_1, t_2 \in [0,T]$, it follows from (4.14) that

$$\|\varphi_\eta(t_1) - \varphi_\eta(t_2)\|_W \leqslant C\left(\|\boldsymbol{u}_\eta(t_1) - \boldsymbol{u}_\eta(t_2)\|_V + \|q(t_1) - q(t_2)\|_W\right),$$

and the previous inequality, the regularity of $\boldsymbol{u}_\eta$ and q imply that $\varphi_\eta \in C(0,T;W)$. □

In the third step, we consider $\theta \in L^2(0,T;L^2(\Omega))$

Problem $\mathcal{P}_\theta$. Find the damage field $\beta_\theta : (0,T) \to H^1(\Omega)$ such that $\beta_\theta(t) \in K$ and

$$\begin{aligned}(4.17)\qquad &\left(\dot\beta_\theta(t), \kappa - \beta_\theta\right)_{L^2(\Omega)} + a\left(\beta_\theta(t), \kappa - \beta_\theta(t)\right)\\ &\qquad \geqslant \left(\theta(t), \kappa - \beta_\theta(t)\right)_{L^2(\Omega)}, \quad \forall \kappa \in K, \text{ a.e. } t \in (0,T),\end{aligned}$$

$$(4.18)\qquad \beta_\theta(0) = \beta_0.$$

The following result is established.

Lemma 4.3. *A unique solution β_θ to the auxiliary problem $\mathcal{P}_\theta$ is guaranteed, meeting the condition (4.4).*

Proof. The continuous inclusion mapping from $\left(H^1(\Omega), |\cdot|H^1(\Omega)\right)$ into $\left(L^2(\Omega), |\cdot|L^2(\Omega)\right)$ ensures that its range is dense. The dual space of $H^1(\Omega)$ is denoted as $(H^1(\Omega))'$, and by identifying the dual of $L^2(\Omega)$ with itself, we establish the Gelfand triple as follows

$$H^1(\Omega) \subset L^2(\Omega) \subset \left(H^1(\Omega)\right)'.$$

The duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$ is denoted by $(\cdot,\cdot)_{(H^1(\Omega))'\times H^1(\Omega)}$.

$$(\beta,\kappa)_{(H^1(\Omega))'\times H^1(\Omega)} = (\beta,\kappa)_{L^2(\Omega)}, \quad \forall \beta \in L^2(\Omega), \kappa \in H^1(\Omega),$$

We observe that K is a closed convex set in $H^1(\Omega)$ while making use of the definition (3.37) of the bilinear form a, for all $\varrho, \kappa \in H^1(\Omega)$, we have $a(\varrho,\kappa) = a(\kappa,\varrho)$ and

$$|a(\varrho,\kappa)| \leqslant k\|\nabla\varrho\|_H \|\nabla\kappa\|_H \leqslant c\|\varrho\|_{H^1(\Omega)}\|\kappa\|_{H^1(\Omega)},$$

Hence, a is both continuous and symmetric. Consequently, for any $\varphi \in H^1(\Omega)$, we obtain

$$a(\varrho,\varrho) = k\|\nabla\varrho\|_H^2,$$

so

$$a(\varrho,\varrho) + (k+1)\|\varrho\|^2_{L^2(\Omega)} \geqslant k\left(\|\nabla\varrho\|_H^2 + \|\varrho\|^2_{L^2(\Omega)}\right),$$

which implies

$$a(\varrho,\varrho) + c_0 l\|\varrho\|^2_{L^2(\Omega)} \geqslant c_1\|\varrho\|^2_{H^1(\Omega)} \text{ with } c_0 = k+1 \text{ and } c_1 = k.$$

Lastly, we employ (3.24), (3.30) to see that $\theta \in L^2(0,T;L^2(\Omega))$ and $\beta_0 \in K$, and using classical arguments of functional analysis concerning parabolic inequalities ([**1**], p. 124), implies that $\mathcal{P}_\theta$ has a unique solution β_θ having the regularity (4.4). This completes the proof. □

Problem $\mathcal{P}_\alpha$. Find a bonding field $\alpha : (0,T) \to L^2(\Gamma_3)$ such that

$$\dot{\alpha}_\eta(t) = -\left(\gamma_\nu \alpha_\eta(t) R_\nu \left(u_{\eta\nu}(t)\right)^2 - \varepsilon_a\right)_+ \text{a.e. } t \in (0,T), \tag{4.19}$$

$$\alpha_\eta(0) = \alpha_0. \tag{4.20}$$

The following result is obtained.

LEMMA 4.4. *A unique solution α_η to Problem $\mathcal{P}\alpha$ exists, and it meets the conditions $\alpha\eta \in W^{1,\infty}(0,T; L^2(\Gamma_3)) \cap \mathcal{X}$.*

PROOF. For simplicity, we omit the dependence of various functions on Γ_3. Let's consider the mapping $F_\eta : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$ defined as follows:

$$F_\eta(t,\alpha) = -\left(\alpha\left(\gamma_\nu \left(R_\nu \left(u_{\eta\nu}(t)\right)\right)^2 - \varepsilon_a\right) +\right),$$

for all $t \in [0,T]$ and $\alpha \in L^2(\Gamma 3)$. We demonstrate that F_η is Lipschitz continuous concerning the second variable, uniformly in time. Additionally, for all $\alpha \in L^2(\Gamma_3)$, the mapping $t \mapsto F_\eta(t,\alpha)$ belongs to $L^\infty(0,T; L^2(\Gamma_3))$. Consequently, by utilizing a version of the classical Cauchy-Lipschitz theorem given in ([**21**], p. 60), we infer the existence of a unique function $\alpha_\eta \in W^{1,\infty}(0,T; L^2(\Gamma_3))$ as a solution to Problem $\mathcal{P}_\alpha$. Moreover, the arguments outlined in Remark 3.1 demonstrate that $0 \leqslant \alpha_\eta(t) \leqslant 1$ for all $t \in [0,T]$, almost everywhere on Γ_3. Thus, from the definition of the set $\mathcal{X}$, we conclude that $\alpha_\eta \in \mathcal{X}$, thereby completing the proof of Lemma 4.4.

□

Now, let's examine the auxiliary problem

Problem $\mathcal{P}_{\eta,\theta}$. Find the stress field $\boldsymbol{\sigma}_{\eta,\theta} : (0,T) \to \mathcal{H}$ which is a solution of the problem

(4.21)

$$\boldsymbol{\sigma}_{\eta,\theta}(t) = \mathcal{N}\left(\varepsilon\left(\mathbf{u}_\eta(t)\right)\right) + \int_0^t \mathcal{Q}\left(\boldsymbol{\sigma}_{\eta,\theta}(s), \varepsilon\left(\mathbf{u}_\eta(s)\right), \beta_\theta(s)\right) ds, \qquad \text{a.e. } t \in (0,T).$$

LEMMA 4.5. *The problem $\mathcal{P}_{\eta,\theta}$ possesses a unique solution $\boldsymbol{\sigma}_{\eta,\theta} \in W^{1,\infty}(0,T;\mathcal{H})$. Furthermore, if $\boldsymbol{\sigma}_{\eta_i,\theta_i}$, $\boldsymbol{u}_{\eta_i}$, β_{η_i} represent the solutions to Problems $\mathcal{P}_{\eta,\theta}$, $\mathcal{P}^1_\eta$, and $\mathcal{P}_\theta$ respectively, for $(\boldsymbol{\eta}_i, \theta_i) \in W^{1,\infty}(0,T;\mathcal{H} \times L^2(\Omega))$, $i = 1,2$, then there exists $C > 0$ such that*

(4.22)

$$\begin{aligned}\|\boldsymbol{\sigma}_{\eta_1,\theta_1}(t) - \boldsymbol{\sigma}_{\eta_2,\theta_2}(t)\|^2_{\mathcal{H}} \leqslant & C\left(\|\boldsymbol{u}_{\eta_1}(t) - \boldsymbol{u}_{\eta_2}(t)\|^2_V + \int_0^t \|\boldsymbol{u}_{\eta_1}(s) - \boldsymbol{u}_{\eta_2}(s)\|^2_V\right.\\ & \left. + \int_0^t \|\beta_{\theta_1}(s) - \beta_{\theta_2}(s)\|^2_{L^2(\Omega)}\, ds\right).\end{aligned}$$

PROOF. Consider the operator $\boldsymbol{\Pi}_{\eta,\theta} : W^{1,\infty}(0,T;\mathcal{H}) \to W^{1,\infty}(0,T;\mathcal{H})$ defined as

$$\boldsymbol{\Pi}_{\eta,\theta}\boldsymbol{\sigma}(t) = \mathcal{N}\varepsilon\left(\boldsymbol{u}_\eta(t)\right) + \int_0^t \mathcal{Q}\left(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}\left(\boldsymbol{u}_\eta(s)\right), \beta_\theta(s)\right) ds. \tag{4.23}$$

Given $\boldsymbol{\sigma}_i \in W^{1,\infty}(0,T;\mathcal{H})$ for $i=1,2$ and $t_1 \in (0,T)$, employing hypothesis (3.23) and Holder's inequality, we obtain

$$\|\boldsymbol{\Pi}_{\eta,\theta}\boldsymbol{\sigma}_1(t_1) - \boldsymbol{\Pi}_{\eta,\theta}\boldsymbol{\sigma}_2(t_1)\|_{\mathcal{H}}^2 \leqslant L_{\mathcal{Q}}^2 T \int_0^{t_1} \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}^2\, ds.$$

Integration on the time interval $(0,t_2) \subset (0,T)$, it follows that

$$\int_0^{t_2} \|\boldsymbol{\Pi}_{\eta,\theta}\boldsymbol{\sigma}_1(t_1) - \boldsymbol{\Pi}_{\eta,\theta}\boldsymbol{\sigma}_2(t_1)\|_{\mathcal{H}}^2\, dt_1 \leqslant L_{\mathcal{Q}}^2 T \int_0^{t_2}\int_0^{t_1} \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}^2\, ds dt_1.$$

Therefore

$$\|\boldsymbol{\Pi}_{\eta,\theta}\boldsymbol{\sigma}_1(t_2) - \boldsymbol{\Pi}_{\eta,\theta}\boldsymbol{\sigma}_2(t_2)\|_{\mathcal{H}}^2 \leqslant L_{\mathcal{Q}}^4 T^2 \int_0^{t_2}\int_0^{t_1} \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}^2\, ds dt_1.$$

For any $t_1, t_2, ..., t_n$ within the interval $(0,T)$, we extend the described procedure through recurrence on n. This leads to the derivation of the following inequality

$$\begin{aligned}&\|\boldsymbol{\Pi}_{\eta,\theta}\boldsymbol{\sigma}_1(t_n) - \boldsymbol{\Pi}_{\eta,\theta}\boldsymbol{\sigma}_2(t_n)\|_{\mathcal{H}}^2\\ &\qquad\leqslant L_{\mathcal{Q}}^{2n} T^n \int_0^{tn}\cdots\int_0^{t_2}\int_0^{t_1} \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}^2\, ds dt_1 \ldots dt_{n-1}.\end{aligned}$$

Which implies

$$\|\boldsymbol{\Pi}_{\eta,\theta}\boldsymbol{\sigma}_1(t_n) - \boldsymbol{\Pi}_{\eta,\theta}\boldsymbol{\sigma}_2(t_n)\|_{\mathcal{H}}^2 \leqslant \frac{L_{\mathcal{Q}}^{2n} T^{n+1}}{n!} \int_0^{T} \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}^2\, ds.$$

Therefore, by integrating over the time interval (0, T), we can deduce that

$$\|\boldsymbol{\Pi}_{\eta,\theta}\boldsymbol{\sigma}_1 - \boldsymbol{\Pi}_{\eta,\theta}\boldsymbol{\sigma}_2\|_{W^{1,\infty}(0,T;\mathcal{H})}^2 \leqslant \frac{L_{\mathcal{Q}}^{2n} T^{n+2}}{n!} \|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|_{W^{1,\infty}(0,T;\mathcal{H})}^2.$$

For sufficiently large n, the operator $\boldsymbol{\Pi}_{\eta,\theta}^n$ becomes a contraction in the Banach space $W^{1,\infty}(0,T;\mathcal{H})$, as implied by the given inequality. Consequently, there exists a unique element $\boldsymbol{\sigma} \in W^{1,\infty}(0,T;\mathcal{H})$ satisfying $\boldsymbol{\Pi}_{\eta,\theta}\boldsymbol{\sigma} = \boldsymbol{\sigma}$. Furthermore, $\boldsymbol{\sigma}$ represents the exclusive solution to Problem $\mathcal{P}_{\eta,\theta}$. Now, consider $(\boldsymbol{\eta}_1,\theta_1), (\boldsymbol{\eta}_2,\theta_2) \in W^{1,\infty}(0,T;\mathcal{H}\times L^2(\Omega))$ and for $i=1,2$ let $\boldsymbol{u}_{\eta_i} = \boldsymbol{u}_i$, $\beta_{\theta_i} = \beta_i$, and $\boldsymbol{\sigma}_{\eta_i,\theta_i} = \boldsymbol{\sigma}_i$. We have

$$\boldsymbol{\sigma}_i(t) = \mathcal{N}\varepsilon(\boldsymbol{u}_i(t)) + \int_0^t \mathcal{Q}(\boldsymbol{\sigma}_i(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_i(s)), \beta_i(s))\, ds, \quad \text{a.e. } t \in (0,T). \tag{4.24}$$

□

Employing the properties (3.22) and (3.23) of $\mathcal{N}$ and $\mathcal{Q}$, we derive the following

$$\begin{aligned}&\|\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)\|_{\mathcal{H}}^2\\&\leqslant C\left(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V^2+\int_0^t\|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}^2\,ds\right.\\&\quad\left.+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V^2\,ds+\int_0^t\|\beta_1(s)-\beta_2(s)\|_{L^2(\Omega)}^2\,ds\right),\quad\forall t\in(0,T).\end{aligned}\tag{4.25}$$

Applying the Gronwall argument to the preceding inequality, we deduce (4.22), thereby completing the proof of Lemma 4.5.

Ultimately, as an outcome of these findings and leveraging the properties of the operator $\mathcal{Q}$ the operator $\mathcal{P}$, and the function S for $t\in(0,T)$, we contemplate the element.

$$\mathcal{T}(\boldsymbol{\eta},\theta)(t)=\big(\mathcal{T}^1(\boldsymbol{\eta},\theta)(t),\mathcal{T}^2(\boldsymbol{\eta},\theta)(t)\big)\in\mathcal{H}\times L^2(\Omega),\tag{4.26}$$

defined by

$$\begin{aligned}(\mathcal{T}(\boldsymbol{\eta},\theta)(t),\boldsymbol{\vartheta})_{\mathcal{H}\times V}&=(\mathcal{Z}^*\nabla\varphi_\eta(t),\boldsymbol{\varepsilon}(\boldsymbol{\vartheta}))_{\mathcal{H}}+j(\alpha_{\boldsymbol{\eta}}(t),\boldsymbol{u}_\eta(t),\boldsymbol{\vartheta})\\&\quad+\left(\int_0^t\mathcal{Q}\left(\boldsymbol{\sigma}_{\eta,\theta}(s),\boldsymbol{\varepsilon}\left(\boldsymbol{u}_\eta(s)\right),\beta_\theta(s)\right)ds,\boldsymbol{\varepsilon}(\boldsymbol{\vartheta})\right)_{\mathcal{H}},\forall\boldsymbol{\vartheta}\in V,\end{aligned}\tag{4.27}$$

$$\mathcal{T}^2\left(\eta,\theta\right)(t)=S\left(\boldsymbol{\sigma}_{\eta,\theta},\varepsilon\left(\boldsymbol{u}_\eta(t)\right),\beta_\theta(t)\right).\tag{4.28}$$

In the given context, for every $(\boldsymbol{\eta},\theta)\in W^{1,\infty}(0,T;\mathcal{H}\times L^2(\Omega))$. $\boldsymbol{u}_\eta$, φ_η, β_θ, $\alpha_{\boldsymbol{\eta}}$ and $\boldsymbol{\sigma}_{\eta,\theta}$ represent the displacement field, the electric potential field, the damage, a bonding field, and the stress field, respectively. These fields are obtained in Lemmas 4.1, 4.2, 4.3, 4.4, and 4.5. The following result ensues.

Lemma 4.6. *The mapping $\mathcal{T}$ possesses a fixed point $(\boldsymbol{\eta}^*,\theta^*)\in W^{1,\infty}(0,T;\mathcal{H}\times L^2(\Omega))$, satisfying $\mathcal{T}(\boldsymbol{\eta}^*,\theta^*)=(\boldsymbol{\eta}^*,\theta^*)$.*

Proof. For $t\in(0,T)$ and $(\boldsymbol{\eta}_1,\theta_1),(\boldsymbol{\eta}_2,\theta_2)\in W^{1,\infty}\left(0,T;\mathcal{H}\times L^2(\Omega)\right)$. we adopt the notation $\boldsymbol{u}_{\eta_i}=\boldsymbol{u}_i$, $\beta_{\theta_i}=\beta_i$, $\varphi_{\eta_i}=\varphi_i$, $\alpha_{\boldsymbol{\eta}}=\alpha_i$ and $\boldsymbol{\sigma}_{\eta_i,\theta_i}=\boldsymbol{\sigma}_i$ for $i=1,2$.

Commencing with the utilization of (3.19), along with the hypotheses (3.23), (3.26), (3.27), (3.28), and the definition of R_ν, R_τ and Remark 3.1, we obtain

$$\begin{aligned}&\left\|\mathcal{T}^1\left(\boldsymbol{\eta}_1,\theta_1\right)(t)-\mathcal{T}^1\left(\boldsymbol{\eta}_2,\theta_2\right)(t)\right\|_{\mathcal{H}}^2\\&\leqslant\|\mathcal{Z}^*\nabla\varphi_1(t)-\mathcal{Z}^*\nabla\varphi_2(t)\|_{\mathcal{H}}^2\\&\quad+\int_0^t\|\mathcal{Q}\left(\boldsymbol{\sigma}_1(s),\varepsilon\left(\boldsymbol{u}_1(s)\right),\beta_1(s)\right)-\mathcal{Q}\left(\boldsymbol{\sigma}_2(s),\boldsymbol{\varepsilon}\left(\boldsymbol{u}_2(s)\right),\beta_2(s)\right)\|_{\mathcal{H}}^2\,ds\\&\quad+C\left(\left\|\alpha_1^2(t)R_\nu\left(u_{1\eta v}(t)\right)-\alpha_2^2(t)R_\nu\left(u_{2\eta v}(t)\right)\right\|_{L^2(\Gamma_3)}^2\right)\\&\quad+C\left(\|p_\tau\left(\alpha_1(t)\right)R_\tau\left(u_{1\eta\tau}(t)\right)-p_\tau\left(\alpha_2(t)\right)R_\tau\left(u_{2\eta}\tau(t)\right)\|_{L^2(\Gamma_3)}^2\right),\end{aligned}\tag{4.29}$$

so we obtain

$$\begin{aligned}
&\left\|\mathcal{T}^1(\boldsymbol{\eta}_1,\theta_1)(t)-\mathcal{T}^1(\boldsymbol{\eta}_2,\theta_2)(t)\right\|_{\mathcal{H}}^2\\
&\leqslant C\left(\|\varphi_1(t)-\varphi_2(t)\|_W^2+\int_0^t\|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}^2\,ds\right.\\
&+\int_0^t\|\beta_1(s)-\beta_2(s)\|_{L^2(\Omega)}^2+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V^2\,ds\\
&\left.+\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V^2+\|\alpha_1(t)-\alpha_2(t)\|_{L^2(\Omega)}^2\right).
\end{aligned}\tag{4.30}$$

We use estimate (4.22) to obtain

$$\begin{aligned}
&\left\|\mathcal{T}^1(\boldsymbol{\eta}_1,\theta_1)(t)-\mathcal{T}^1(\boldsymbol{\eta}_2,\theta_2)(t)\right\|_{\mathcal{H}}^2\\
&\leqslant C\left(\|\varphi_1(t)-\varphi_2(t)\|_W^2+\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V^2+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V^2\,ds\right.\\
&\quad\left.+\|\alpha_1(s)-\alpha_2(s)\|_{L^2(\Gamma_3)}^2+\int_0^t\|\beta_1(s)-\beta_2(s)\|_{L^2(\Omega)}^2\,ds\right).
\end{aligned}\tag{4.31}$$

By similar arguments, from (4.28), (4.22) and (3.24) we obtain

$$\begin{aligned}
&\left\|\mathcal{T}^2(\boldsymbol{\eta}_1,\theta_1)(t)-\mathcal{T}^2(\boldsymbol{\eta}_2,\theta_2)(t)\right\|_{\mathcal{H}}^2\\
&\leqslant C\left(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V^2+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V^2\right.\\
&\quad\left.+\|\beta_1(t)-\beta_2(t)\|_{L^2(\Omega)}^2+\int_0^t\|\beta_1(s)-\beta_2(s)\|_{L^2(\Omega)}^2\,ds\right),\qquad\text{a.e. } t\in(0,T).
\end{aligned}\tag{4.32}$$

Hence,

$$\begin{aligned}
&\|\mathcal{T}(\boldsymbol{\eta}_1,\theta_1)(t)-\mathcal{T}(\boldsymbol{\eta}_2,\theta_2)(t)\|_{\mathcal{H}\times L^2(\Omega)}^2\leqslant C\left(\|\varphi_1(t)-\varphi_2(t)\|_W^2\right.\\
&+\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V^2+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V^2\,ds+\|\beta_1(t)-\beta_2(t)\|_{L^2(\Omega)}^2\\
&\left.+\int_0^t\|\beta_1(s)-\beta_2(s)\|_{L^2(\Omega)}^2\,ds+\|\alpha_1(t)-\alpha_2(t)\|_{L^2(\Gamma_3)}^2\right)\qquad\text{a.e. } t\in(0,T).
\end{aligned}\tag{4.33}$$

To express the electric potential, we apply (4.14) with $\boldsymbol{\eta}=\boldsymbol{\eta}_1$ and obtain

$$(\boldsymbol{M}\nabla\varphi_1,\nabla\Theta)_H=(\mathcal{Z}\varepsilon(\boldsymbol{u}_1),\nabla\Theta)_H+(q(t),\Theta)_W\quad\forall\Theta\in W,\tag{4.34}$$

and for $\boldsymbol{\eta}=\boldsymbol{\eta}_2$, we have

$$(\boldsymbol{M}\nabla\varphi_2,\nabla\Theta)_H=(\mathcal{Z}\varepsilon(\boldsymbol{u}_2),\nabla\Theta)_H+(q(t),\Theta)_W\quad\forall\Theta\in W,\tag{4.35}$$

we set $\Theta=\varphi_1-\varphi_2$ and subtract (4.34) and (4.35) let's find

$$(\boldsymbol{M}\nabla(\varphi_1-\varphi_2),\nabla(\varphi_1-\varphi_2))_H=(\mathcal{Z}\varepsilon(\boldsymbol{u}_1-\boldsymbol{u}_2),\nabla(\varphi_1-\varphi_2))_H,$$

from (3.25) (3.26) we have

$$\|\varphi_1(t)-\varphi_2(t)\|_W\leqslant C\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V.\tag{4.36}$$

Next, we use (4.13) to find that

$$(\dot{\boldsymbol{u}}_1(t)-\dot{\boldsymbol{u}}_2(t),\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t))_V \leqslant (\boldsymbol{\eta}_2(t)-\boldsymbol{\eta}_1(t),\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t))_{\mathcal{H}} + (N\boldsymbol{u}_2(t)-N\boldsymbol{u}_1(t),\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t))_V\,,$$

By employing the Cauchy-Schwarz inequality and (4.10), we achieve

$$(\dot{\boldsymbol{u}}_1(t)-\dot{\boldsymbol{u}}_2(t),\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t))_V \leqslant \|\boldsymbol{\eta}_1(t)-\boldsymbol{\eta}_2(t)\|_{\mathcal{H}}\,\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V + \frac{L_{\mathcal{N}}}{m_{\mathcal{L}}}\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V^2\,.$$

By integrating this inequality with respect to time and applying the initial conditions $\boldsymbol{u}_1(0)=\boldsymbol{u}_2(0)=\boldsymbol{u}_0$, we determine that

$$\frac{1}{2}\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V^2 \leqslant \int_0^t \|\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)\|_{\mathcal{H}}\,\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V\,ds + \frac{L_{\mathcal{N}}}{m_{\mathcal{L}}}\int_0^t \|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V^2\,ds.$$

Applying the given inequality

$$ab \leqslant \frac{1}{2}a^2+\frac{1}{2}b^2, \quad a,b\in\mathbb{R},$$

we establish that

$$\frac{1}{2}\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V^2 \leqslant \frac{1}{2}\int_0^t \|\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)\|_{\mathcal{H}}^2\,ds + \frac{1}{2}\int_0^t \|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V^2\,ds + \frac{L_{\mathcal{N}}}{m_{\mathcal{L}}}\int_0^t \|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V^2\,ds$$

where we obtain

$$\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V^2 \leqslant C\int_0^t \|\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)\|_{\mathcal{H}}^2\,ds + C\int_0^t \|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V^2\,ds,$$

Following a Gronwall argument, we derive

$$\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V^2 \leqslant C\int_0^t \|\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)\|_{\mathcal{H}}^2\,ds. \tag{4.37}$$

On the other hand, considering the Cauchy problem (4.19)-(4.20) we can write

$$\alpha_i(t)=\alpha_0-\int_0^t \left(\gamma_\nu\alpha_i(s)R_\nu\left(u_{i\nu}(s)\right)^2-\varepsilon_a\right)_+ ds.$$

Now employing the definition of R_ν, the inequality $|R_\nu(u_\nu)|\leqslant L$, and expressing α_1 as $\alpha_1-\alpha_2+\alpha_2$, we derive

$$\|\alpha_1(t)-\alpha_2(t)\|_{L^2(\Gamma_3)} \leqslant C\int_0^t \|\alpha_1(s)-\alpha_2(s)\|_{L^2(\Gamma_3)}\,ds + C\int_0^t \|u_{1\nu}(s)-u_{2\nu}(s)\|_{L^2(\Gamma_3)}\,ds.$$

We apply Gronwall's inequality to deduce that

$$\|\alpha_1(t)-\alpha_2(t)\|_{L^2(\Gamma_3)} \leqslant C\int_0^t \|u_{1\nu}(s)-u_{2\nu}(s)\|_{L^2(\Gamma_3)}\,ds,$$

and, using (3.19) we obtain

$$\|\alpha_1(t)-\alpha_2(t)\|_{L^2(\Gamma_3)} \leqslant C\int_0^t \|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V\,ds. \tag{4.38}$$

Form (4.17), we deduce that

$$(\dot\beta_1-\dot\beta_2,\beta_1-\beta_2)_{L^2(\Omega)}+a(\beta_1-\beta_2,\beta_1-\beta_2)\leqslant(\theta_1-\theta_2,\beta_1-\beta_2)_{L^2(\Omega)},\quad \forall t\in(0,T). \tag{4.39}$$

By integrating the preceding inequality with respect to time and considering the initial conditions $\beta_1(0)=\beta_2(0)=\beta_0$, along with the given inequality $a(\beta_1-\beta_2,\beta_1-\beta_2)\geqslant 0$, we aim to determine

$$\frac{1}{2}\|\beta_1(t)-\beta_2(t)\|^2_{L^2(\Omega)}\leqslant\int_0^t(\theta_1(s)-\theta_2(s),\beta_1(s)-\beta_2(s))_{L^2(\Omega)}\,ds. \tag{4.40}$$

This inequality, when coupled with Gronwall's inequality, results in

$$\|\beta_1(t)-\beta_2(t)\|^2_{L^2(\Omega)}\leqslant C\int_0^t\|\theta_1(s)-\theta_2(s)\|^2_{L^2(\Omega)}\,ds,\quad\forall t\in(0,T), \tag{4.41}$$

Based on the preceding inequality and the assessments provided in (4.33), (4.36), (4.37), (4.38), and (4.41), it can be deduced that

$$\begin{aligned}&\|\mathcal{T}(\boldsymbol{\eta}_1,\theta_1)(t)-\mathcal{T}(\boldsymbol{\eta}_2,\theta_2)(t)\|^2_{\mathcal{H}\times L^2(\Omega)}\\&\qquad\leqslant C\int_0^T\|(\boldsymbol{\eta}_1,\theta_1)(s)-(\boldsymbol{\eta}_2,\theta_2)(s)\|^2_{\mathcal{H}\times L^2(\Omega)}\,ds.\end{aligned} \tag{4.42}$$

Reiterating this inequality m times we obtain

$$\begin{aligned}&\|\mathcal{T}^m(\boldsymbol{\eta}_1,\theta_1)-\mathcal{T}^m(\boldsymbol{\eta}_2,\theta_2)\|^2_{W^{1,\infty}(0,T;\mathcal{H}\times L^2(\Omega))}\\&\qquad\leqslant\frac{C^mT^m}{m!}\|(\boldsymbol{\eta}_1,\theta_1)-(\boldsymbol{\eta}_2,\theta_2)\|^2_{W^{1,\infty}(0,T;\mathcal{H}\times L^2(\Omega))}.\end{aligned}$$

Hence, for a sufficiently large value of m, the operator $\mathcal{T}^m$ exhibits contraction properties within the Banach space $W^{1,\infty}(0,T;\mathcal{H}\times L^2(\Omega))$. Consequently, the operator $\mathcal{T}$ possesses a distinctive fixed point. □

Now, all the necessary elements are in place to establish the validity of Theorem 4.1.

Existence. Let $(\boldsymbol{\eta}^*,\theta^*)\in W^{1,\infty}(0,T;\mathcal{H}\times L^2(\Omega))$ be the fixed point of $\mathcal{T}$ and

$$\boldsymbol{u}=\boldsymbol{u}_{\eta^*},\quad \varphi_{\eta^*}=\varphi,\quad \boldsymbol{\sigma}=\mathcal{L}\varepsilon(\dot{\boldsymbol{u}})+\mathcal{Z}^*\nabla\varphi(t)+\boldsymbol{\sigma}_{\eta^*\lambda^*}, \tag{4.43}$$

$$\beta=\beta_{\theta^*},\quad \alpha=\alpha_{\boldsymbol{\eta}^*}. \tag{4.44}$$

$$\boldsymbol{D}=\mathcal{Z}\varepsilon(\boldsymbol{u})+\boldsymbol{M}\nabla(\varphi). \tag{4.45}$$

We establish the satisfaction of (3.43)-(3.48) and (4.1)-(4.6) by the tuple $(\boldsymbol{u}, \boldsymbol{\sigma}, \varphi, \beta, \alpha, \boldsymbol{D})$ Specifically, we invoke (4.21) with $\boldsymbol{\eta}^* = \boldsymbol{\eta}, \quad \theta^* = \theta$ utilising (4.43)-(4.44). This leads to the fulfillment of (3.43). Subsequently, we examine (4.7) with $\boldsymbol{\eta}^* = \boldsymbol{\eta}$ and employ the first equality in (4.43), yielding the satisfaction of the respective condition.

(4.46)

$$(\mathcal{L}\varepsilon(\dot{\boldsymbol{u}}_\eta(t)), \varepsilon(\boldsymbol{\vartheta} - \boldsymbol{u}_\eta(t)))_{\mathcal{H}} + (\mathcal{N}\varepsilon(\boldsymbol{u}_\eta(t)), \varepsilon(\boldsymbol{\vartheta} - \boldsymbol{u}_\eta(t)))_{\mathcal{H}} + (\boldsymbol{\eta}^*(t), \varepsilon(\boldsymbol{\vartheta} - \boldsymbol{u}_{\boldsymbol{\eta}}(t)))_{\mathcal{H}} \geqslant (\boldsymbol{f}(t), \boldsymbol{\vartheta} - \boldsymbol{u}_\eta(t))_V, \forall \boldsymbol{\vartheta} \in V, \text{ a.e. } t \in (0,T),$$

The equalities $\mathcal{T}^1(\boldsymbol{\eta}^*, \theta^*) = \boldsymbol{\eta}^*$, and $\mathcal{T}^2(\boldsymbol{\eta}^*, \theta^*) = \theta^*$,

, in conjunction with (4.27)-(4.28), (4.43), and (4.44), demonstrate that for all $\boldsymbol{\vartheta} \in V$

$$\begin{aligned}(\boldsymbol{\eta}^*(t), \boldsymbol{\vartheta})_{\mathcal{H}\times V} &= (\mathcal{Z}^*\nabla\varphi(t), \varepsilon(\boldsymbol{\vartheta}))_{\mathcal{H}} + j(\alpha(t), \boldsymbol{u}(t), \boldsymbol{\vartheta}) \\ &+ \left(\int_0^t \mathcal{Q}(\boldsymbol{\sigma}(s) - \mathcal{L}\varepsilon(\dot{\boldsymbol{u}}(s)) - \mathcal{Z}^*\nabla\varphi(t), \varepsilon(\boldsymbol{u}(s)), \beta(s))ds, \varepsilon(\boldsymbol{\vartheta})\right)_{\mathcal{H}},\end{aligned} \tag{4.47}$$

$$\theta^*(t) = S(\boldsymbol{\sigma}(s) - \mathcal{L}\varepsilon(\dot{\boldsymbol{u}}(s) - \mathcal{Z}^*\nabla\varphi(t), \varepsilon(\boldsymbol{u}(t)), \beta(t)). \tag{4.48}$$

We substitute (4.47) in (4.46)) and use (3.43) to see that (3.44) is satisfied. We express (4.17) for $\theta = \theta^*$ and employ (4.44) and (4.48) to establish (3.46). We apply (4.14), (4.19) for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$ and use (4.43)-(4.44) to confirm that (3.45), (3.47) are satisfied. Additionally, (3.48) and the regularities (4.1), (4.2), (4.4), and (4.5) follow from Lemmas 4.1, 4.2, 4.3, 4.4, and the regularity (4.3) follows from Lemma 4.5. Now, let $t_1, t_2 \in [0,T]$; from (3.25), (3.26), (3.17), and (4.45), we conclude that there exists a positive constant $C > 0$ such that

$$\|\boldsymbol{D}(t_1) - \boldsymbol{D}(t_2)\|_H \leqslant C\left(\|\varphi(t_1) - \varphi(t_2)\|_W + \|\boldsymbol{u}(t_1) - \boldsymbol{u}(t_2)\|_V\right).$$

The regularity of $\boldsymbol{u}$ and φ given by (4.1) and (4.2) implies

$$\boldsymbol{D} \in C(0,T;H). \tag{4.49}$$

Choosing $\phi \in D(\Omega)^d$ in (3.45) and using (3.41), we find

$$\operatorname{div} \boldsymbol{D}_*(t) = q_0(t), \quad \forall t \in [0,T], \tag{4.50}$$

By (3.35) and (4.49), we obtain

$$\boldsymbol{D} \in C(0,T;\mathcal{W}).$$

This concludes the existence part of the theorem.

References

[1] V. Barbu, *Optimal control of variational inequalities.* Res. Notes Math. **100**, (1984), 38–57 .

[2] T. Buchukuri and T. Gegelia, *Some dynamic problems of the theory of electroelasticity*, Mem. Differential Equations Math. Phys. **10** (1997), 1-53.

[3] O. Chau, J.R. Fernandez, M. Shillor, and M. Sofonea, *Variational and numerical analysis of a quasistatic viscoelastic contact problem with adhesion*, J. Comput. Appl. Math, **159** (2003), 431-465.

[4] N. Chougui and S. Drabla, *A quasistatic electro-viscoelastic contact problem with adhesion. Bulletin of the Malaysian Mathematical Sciences Society*, (2016), 1439-1456.

[5] M. Fremond and B. Nedjar, *Damage, gradient of damage and principle of virtual work Internat.* J. Solids Struct. **33** (1996), 1083-1103.

[6] M. Fremond, KL. Kuttler, B. Nedjar, and M. Shillor, *One-dimensional models of damage*, Adv. Math. Sci. Appl. **8** (1998), ´ 541-570.

[7] A. Hamidat and A. Aissaoui, *A Quasistatic frictional contact problem with normal damped response for thermo-electro-elastic-viscoelastic bodies*, Advances in Mathematics: Scientific Journal.10 (2021), no.12.

[8] A. Hamidat and A. Aissaoui. *A quasi-static contact problem with friction in electro viscoelasticity with long-term memory body with damage and thermal effects.* International Journal of Nonlinear Analysis and Applications (2022).

[9] A. Hamidat and A. Aissaoui. *quasistatic frictional contact problem with damage for thermo–electro–elastic-viscoplastic bodies.* Differential Equations and Applications 4 (2023), 361-380.

[10] A. Hamidat, . *Analysis of a class of frictional contact problem for elastic-viscoplastic piezoelectric thermal materials.* Mathematical Analysis and its Contemporary Applications, 5(4),(2023) 11-31.

[11] A. Hamidat, *quasistatic evolution of damage in thermo-electro-elastic-viscoplastic materials with normal compliance.* Annals of Communications in Mathematics, 7(1),(2024) 21.

[12] W. Han and M. Sofonea, *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity*, Studies in Advanced Mathematics, Americal Mathematical Society and International Press, (2002).

[13] R.D. Mindlin, *Polarisation gradient in elastic dielectrics*, Int. J. Solids Struct. **4** (1968), 637-642.

[14] R.D. Mindlin, *Continuum and lattice theories of influence of electromechanical coupling on capacitance of thin dielectric films*, Int. J. Solids Struct. **4** (1969), 1197-1213.

[15] R.D. Mindlin, *Elasticity, piezoelectricity and crystal lattice dynamics*, J. Elasticity, **2** (1972), 217-282.

[16] M. Raous, L. Cangemi, and M. Cocu, *A consistent model coupling adhesion, friction, and unilateral contact*, Comput. Methods Appl. Mech. Eng., **177**,(1999), 383–399.

[17] J. Rojek, J. J. Telega, *Contact problems with friction, adhesion and wear in orthopaedic biomechanics.* I: General developments, J. Theor. Appl. Mech., **39**,(2001), 655–677.

[18] J. Rojek, J.J. Telega, and S. Stupkiewicz, *Contact problemswithfriction, adhesion andwear in orthopaedic biomechanics.* II : numerical implementation and application to implanted knee joints. J. Theor. Appl. Mech. **39**, (2001), 679–706.

[19] M. Sofonea, W. Han, and M. Shillor, *Analysis and Approximation of Contact Problems with Adhesion or Damage.* Pure and Applied Mathematics, **276**. Chapman-Hall/CRC Press, New York (2006).

[20] M. Shillor, M. Sofonea, and J. J. Telega, *Models and Analysis of Quasistatic Contact*, Lecture Notes in Physics 655, Springer, Berlin, 2004.

[21] P. Suquet, *Plasticité et homogénéisation*, Ph.D. thesis, Université Pierre et Marie Curie, Paris 6. 1982.

[22] R.A. Toupin, *The elastic dielectric*, J. Ration. Mech. Anal. **5** (1966), 849-915.

[23] R.A. Toupin, *A dynamical theory of elastic dielectrics*, Int. J. Engrg. Sci. **1** (1963).

Received by editors 26.12.2023; Revised version 14.6.2024; Available online 30.6.2024.

Ahmed Hamidat, Laboratory of Operator Theory and PDE: Foundations and Applications, Faculty of Exact Sciences, University of El Oued, 39000, El Oued, Algeria.
Email address: hamidat-ahmed@univ-eloued.dz

Adel Aissaoui, Laboratory of Operator Theory and PDE: Foundations and Applications, Faculty of Exact Sciences, University of El Oued, 39000, El Oued, Algeria.
Email address: aissaoui-adel@univ-eloued.dz

Hakim Bagua, Faculty of Science and Technology University of Djelfa, 17000, Djelfa, Algeria.
Email address: h.bagua@univ-djelfa.dz