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STUDY ON (ANTI) FUZZY $k-$IDEALS OF ORDERED $\Gamma-$SEMIRINGS

M. Murali Krishna Rao, Noorbhasha Rafi, and Rajendra Kumar Kona

Abstract. In this paper, we introduce the notion of anti fuzzy ideals, anti fuzzy $k-$ideals of ordered $\Gamma-$semirings and study some of the properties. Characterize the ideals of an ordered $\Gamma-$semiring in terms of anti fuzzy $k-$ideals. We study homomorphic, anti homomorphic, and pre-image of anti fuzzy ideals of ordered $\Gamma-$semiring.

1. Introduction

In 1995, Murali Krishna Rao introduced the notion of a $\Gamma-$semiring which is a generalization of $\Gamma-$ring, ternary semiring, and semiring [**13**, **14**]. The notion of a semiring introduced by Vandiver [**18**] is a well known universal algebra. A universal algebra $(S, +, \cdot)$ is called a semiring if and only if $(S, +), (S, \cdot)$ are semigroups satisfying distributive laws, *i.e.*, $a(b+c) = ab+ac,\ (a+b)c = ac+bc,$ for all $a, b, c \in S$. In structure, semirings lie between semigroups and rings. The concept of a semiring is useful in studying optimization theory, graph theory, matrices, determinants, theory of automata, coding theory, analysis of computer programmes, etc.

Nobusawa introduced the notion of $\Gamma-$ring as a generalization of ring in 1964 [**16**]. As semiring is a generalization of a ring, ideals of semiring do not coincide with ring ideals. Ideals play an important role in advanced studies of algebraic structures. Generalization of ideals in algebraic structures is necessary for further study of algebraic structures. Many mathematicians proved significant results and charecterizations of algebraic structures by using the concept and the properties of

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a generalization of ideals in algebraic structures. Dedekind introduced the notion of ideals for the theory of algebraic numbers, which was generalized by Noether for associative rings. Ideals of semirings have wide applications in computer sciences, dynamical and logical systems, cryptography, graph theory, and artificial intelligence. Henriksen defined more restricted class of ideals called $k-$ideals in semirings to obtain analogous of ring results for semiring [**6**]. A semiring with additive idempotents, forms an ideal which is not a k-ideal. Sen and Adhikari studied k-ideals of semirings [**17**].

In many real world problems, we encounter the uncertainty in decision making, control engineering, management science, and artificial intelligence. To address such several theories were developed, like randomness and rough sets. One of the most appropriate theories in dealing with uncertainty is the fuzzy set theory developed by Zadeh in 1965 [**19**]. Applying the concept of fuzzy sets to the theory of ring, Y. B. Jun introduced the notion of fuzzy ideals in a ring and studied the properties of fuzzy ideals of a ring. D. Mandal [**12**] studied fuzzy ideals and fuzzy interior ideals in an ordered semiring. N. Kuroki [**9**] studied fuzzy interior ideals in semigroups. Biswas [**3**] introduced the concept of an anti-fuzzy subgroup of a group and studied the basic properties of a group in terms of anti-fuzzy subgroups. Akram and Dar [**1**] defined anti-fuzzy $h-$ideals in hemirings and discussed basic properties. Kehayopulu and Tsingelis studied fuzzy sets in ordered semigroups [**11**]. Hong and Jun [**7**] defined anti fuzzy ideal of a BCK-algebra.

This paper aims to introduce the notion of anti-fuzzy ideals($k-$ideals) of ordered $\Gamma-$Semirings and study some properties of anti-fuzzy ideals (anti fuzzy $k-$ ideals), homomorphic (anti homomorphic image) and pre-image of fuzzy ideals, of an ordered $\Gamma-$Semiring.

2. Preliminaries

In this section, we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

Definition 2.1. [**13**] Let $(M,+)$ and $(\Gamma,+)$ be commutative semigroups. Then we call M a $\Gamma-$semiring, if there exists a mapping $M\times\Gamma\times M\to M$ (images of (x,α,y) will be denoted by $x\alpha y, x,y\in M,\alpha\in\Gamma$) such that it satisfies the following axioms, for all $x,y,z\in M$ and $\alpha,\beta\in\Gamma$

(i) $x\alpha(y+z)=x\alpha y+x\alpha z$ (ii) $(x+y)\alpha z=x\alpha z+y\alpha z$
(iii) $x(\alpha+\beta)y=x\alpha y+x\beta y$ (iv) $x\alpha(y\beta z)=(x\alpha y)\beta z$.

Every semiring R is a $\Gamma-$semiring with $\Gamma=R$ and ternary operation $x\gamma y$ defined as the usual semiring multiplication.

Definition 2.2. [**4**] An ideal I of a $\Gamma-$semiring M is called a $k-$ideal, if $b\in M, a+b$ and $a\in I$ then $b\in I$.

Definition 2.3. [**13**] A function $f:R\to S$ where R and S are $\Gamma-$semirings is said to be a $\Gamma-$semiring homomorphism if $f(a+b)=f(a)+f(b)$,
$f(a\alpha b)=f(a)\alpha f(b),$ for all $a,b\in R,\alpha\in\Gamma$.

Definition 2.4. [**15**] Let μ be a fuzzy subset of S. For $t \in [0,1]$, the set $\mu_t = \{x \in S \mid \mu(x) \geqslant t\}$ is called a level subset of S with respect to μ.

Definition 2.5. [**15**] A fuzzy subset μ of a $\Gamma-$semiring M is called a fuzzy left(right)ideal of M if it satisfies $\mu(x+y) \geqslant min\{\mu(x),\mu(y)\}, \mu(x\alpha y) \geqslant \mu(y)$ $\big(\mu(x\alpha y) \geqslant \mu(x)\big)$, for all $x, y \in M, \alpha \in \Gamma$.

Definition 2.6. [**15**] If μ is a fuzzy left (right) ideal of a $\Gamma-$semiring M then $\mu(0) \geqslant \mu(x)$, for all $x \in M$.

Definition 2.7. [**15**] Let S and T be two sets and $\phi : S \to T$ be any function. A fuzzy subset μ of S is called a $\phi-$invariant if $\phi(x) = \phi(y)$, then $\mu(x) = \mu(y)$.

Definition 2.8. [**15**] Let μ and γ be two fuzzy subsets of an ordered $\Gamma-$semiring M and $x, y \in M$. Define

$$\mu \cup \gamma(x) = \max\big(\mu(x), \gamma(x)\big),\ \mu \cap \gamma(x) = \min\{\mu(x), \gamma(x)\},\ \text{for all } x \in M.$$

Definition 2.9. [**12**] A $\Gamma-$semiring M is called an ordered $\Gamma-$semiring if it admits a compatible relation $\leqslant$. i.e. $\leqslant$ is a partial ordering on M satisfies the following conditions. If $a \leqslant b$ and $c \leqslant d$ then

(i) $a + c \leqslant b + d$ (ii) $a\alpha c \leqslant b\alpha d$ (iii) $c\alpha a \leqslant d\alpha b$, for all $a, b, c, d \in M, \alpha \in \Gamma$.

Definition 2.10. [**12**] Let M be an ordered $\Gamma-$semiring. A fuzzy subset μ of M is called a fuzzy $\Gamma-$subsemiring of M if

(i) $\mu(x+y) \geqslant min\{\mu(x),\mu(y)\}$ (ii) $\mu(x\alpha y) \geqslant min\{\mu(x),\mu(y)\}$
(iii) $x \leqslant y \Rightarrow \mu(x) \geqslant \mu(y)$, for all $x, y \in M, \alpha \in \Gamma$.

Let μ be a non-empty fuzzy subset of M. Then μ is called a fuzzy left (right) ideal of M if

(i) $\mu(x+y) \geqslant min\{\mu(x),\mu(y)\}$ (ii) $\mu(x\alpha y) \geqslant \mu(y)\big(\mu(x)\big)$
(iii) $x \leqslant y \Rightarrow \mu(x) \geqslant \mu(y)$, for all $x, y \in M, \alpha \in \Gamma$.

By fuzzy ideal we mean, it is both a fuzzy left ideal as well as a fuzzy right ideal of an ordered $\Gamma-$semiring M.

Example 2.1. Let M be an abelian semigroup of all non positive integers and Γ be an abelian semigroup of all positive integers with usual addition, define the ternary operation $M \times \Gamma \times M \to M$ by $(a, \alpha, b) \to a\alpha b$ using usual multiplication. Then M is an ordered $\Gamma-$semiring with respect to an usual ordering $\leqslant$.

$$\text{Define } \mu : M \to [0,1] \text{ by } \mu(x) = \begin{cases} 1, & \text{if } x = 0; \\ 0.2, & \text{if } x \text{ is an even;} \\ 0.1, & \text{if } x \text{ is an odd .} \end{cases}$$

Then μ is a fuzzy ideal of M.

Example 2.2. Let M be the additive commutative semigroup of all non negative integers and Γ be the additive commutative semigroup of all natural numbers.

Then M is an ordered $\Gamma-$semiring with respect to an usual ordering $\leqslant$.
If $a\gamma b$ is defined as usual multiplication of integers a, γ, b, where $a, b \in M, \gamma \in \Gamma$.

$$\text{Define } \mu(x) = \begin{cases} \frac{3}{4}, & \text{if } x = 0; \\ \frac{1}{2}, & \text{if } x \in \{2, 4, 6, \cdots\}; \\ 0, & \text{otherwise.} \end{cases}$$

for all $x \in M$. Then μ is a fuzzy ideal of M.

3. Anti fuzzy k-ideal of ordered $\Gamma-$semirings

In this section, we introduce the notions of anti fuzzy ideals and anti fuzzy $k-$ideals of ordered $\Gamma-$semirings and study some of their properties.

Definition 3.1. A fuzzy subset μ of an ordered semiring M is called an anti fuzzy left(right) ideal of M if it satisfies $\mu(x + y) \leqslant max\{\mu(x), \mu(y)\}$, $\mu(x\alpha y) \leqslant \mu(y)\big(\mu(x\alpha y) \leqslant \mu(x)\big)$, for all $x, y \in M, \alpha \in \Gamma$, and $x \leqslant y \Rightarrow \mu(x) \leqslant \mu(y)$.

Definition 3.2. A fuzzy subset μ of an ordered semiring M is called an anti fuzzy ideal of M if μ is both an anti fuzzy left ideal and an anti fuzzy right ideal of M.

Definition 3.3. Let M be an ordered semiring. An anti fuzzy ideal μ of M is said to be an anti fuzzy-$k-$ideal of M if $\mu(x) \leqslant max\{\mu(x+y), \mu(y)\}$, for all $x, y \in M$.

Definition 3.4. Let M be an ordered $\Gamma-$semiring. If μ is an anti fuzzy-ideal of ordered $\Gamma-$semiring M, for any $t \in [0, 1], \mu_t$ is defined by $\mu_t = \{x \in M \mid \mu(x) \leqslant t\}$ then μ_t is called an anti level subset.

Example 3.1. Let M be the set of all non negative integers and Γ be the set of all natural numbers. Binary and ternary operations be defined by usual addition and multiplication respectively. Then M is an ordered $\Gamma-$semiring.
Define $\mu : M \to [0, 1]$, as

$$\mu(x) = \begin{cases} \frac{1}{4}, & \text{if } x = 0; \\ \frac{1}{2}, & \text{if } x \in \{2, 4, 6, \cdots\}; \\ 1, & \text{otherwise.} \end{cases}$$

Then μ is an anti fuzzy $k-$ideal of M.

Example 3.2. Let M be the set of all integers and Γ be the set of all natural numbers. Binary and ternary operations be defined by usual addition and multiplication respectively. Then M is an ordered $\Gamma-$semiring with respect to usual ordering S.
Define $\mu : M \to [0, 1]$, as

$$\mu(x) = \begin{cases} 0.2, & \text{if } x = 0; \\ \frac{1}{2}, & \text{if } x \in \{\cdots, -4, -2, 2, 4, \cdots\}; \\ 1, & \text{otherwise.} \end{cases}$$

Then μ is an anti fuzzy $k-$ideal of M.

THEOREM 3.1. *If μ is anti fuzzy $k-$ideal of an ordered $\Gamma-$semiring M then $M_\mu = \{x : x \in M \text{and } \mu(x) = 0\}$ is either empty or a $k-$ideal of an ordered $\Gamma-$semiring M.*

PROOF. Suppose μ is a anti fuzzy $k-$ ideal of M,
$M_\mu = \{x : x \in M\text{and } \mu(x) = 0\}$, $M_\mu \neq \phi, x, y \in M_\mu$ and $\alpha \in \Gamma$. Then
$\mu(x) = 0, \mu(y) = 0, \mu(x + y) \leqslant \max\{\mu(x), \mu(y)\} = 0,$ *then* $\mu(x + y) = 0.$
Therefore $x + y \in M_\mu$. Now $x \in M_\mu, y \in M, \alpha \in \Gamma,$ so $\mu(x) = 0$ and
$\mu(x\alpha y) \leqslant \min\{\mu(x), \mu(y)\} = 0,$ then $x\alpha y \in M_\mu$.
Let $x, x + y \in M_\mu$. Then $\mu(x) = 0, \mu(x + y) = 0$, hence $\mu(y) = 0$.
Therefore $y \in M_\mu$. Hence M_μ is a $k-$ideal of the ordered $\Gamma-$semiring M.
Let $x \in M, y \in M_\mu$ and $x \leqslant y$. Then $\mu(x) \leqslant \mu(y) = 0 \Rightarrow \mu(x) = 0$.
Therefore $x \in M_\mu$. □

THEOREM 3.2. *Let M be an ordered $\Gamma-$semiring. Then μ is an anti fuzzy $k-$ideal of an ordered $\Gamma-$semiring M, if and only if non-empty μ_t is a $k-$ideal of an ordered $\Gamma-$semiring M, for any $t \in (0, 1]$.*

PROOF. Suppose μ is an anti fuzzy ideal of M. Let $x, y \in \mu_t$. Then $\mu(x) \leqslant t, \mu(y) \leqslant t, \mu(x + y) \leqslant \max\{\mu(x), \mu(y)\} \leqslant t$,
$\mu(x\alpha y) \leqslant \max\{\mu(x), \mu(y)\} \leqslant t \Rightarrow x + y \in \mu_t, x\alpha y \in \mu_t$.
Therefore μ_t is a $k-$ideal of M. Let $y, x + y \in \mu_t$.
Then $\mu(y) \leqslant t,\ \mu(x + y) \leqslant t$. Since μ is an anti fuzzy $k-$ideal of M.
We have $\mu(x) \leqslant \max\{\mu(x + y), \mu(y)\} \leqslant t \Rightarrow \mu(x) \leqslant t \Rightarrow x \in \mu_t$.
Hence μ_t is a $k-$ideal of M. Suppose $a \in M, b \in \mu_t$ and $a \leqslant b$. Then $\mu(a) \leqslant \mu(b) \leqslant t \Rightarrow a \in \mu_t$.
Hence μ_t is an ideal of M.

Conversely suppose that μ_t is a $k-$ideal of M, for any $t \in (0, 1]$. Let $x, a \in M$ and $\mu(a) = t_1, \mu(x + a) = t_2$, put $t = max\{t_1, t_2\}$. Then $a \in \mu_t$ and $x + a \in \mu_t$. Since μ_t is a $k-$ideal, we have $x \in \mu_t$, therefore $\mu(x) \leqslant \max\{\mu(x + a), \mu(a)\}$ for all $x, a \in M$. Hence μ is an anti fuzzy $k-$ideal of an M. Let $x, y \in M$ and $x \leqslant y$. Suppose $\mu(x) > \mu(y) = t, (say)$. Then $y \in \mu_t$
and $x \notin \mu_t$. This is a contradiction to the fact μ_t is an ideal of ordered $\Gamma-$semiring M, for all $t \in [0, 1]$. Therefore $\mu(x) \leqslant \mu(y)$.
Hence μ is an anti fuzzy $k-$ideal of M. □

THEOREM 3.3. *Let μ be a fuzzy subset of an ordered $\Gamma-$semiring M. μ is an anti fuzzy $k-$ideal of M if and only if μ^c is a fuzzy $k-$ideal of M.*

PROOF. Let M be an ordered $\Gamma-$semiring and μ be an anti fuzzy $k-$ideal of M and $x, y \in M, \alpha \in \Gamma$.

$$\mu^c(x+y) = 1-\mu(x+y) \geqslant 1-\max\{\mu(x),\mu(y)\} = \min\{1-\mu(x), 1-\mu(y)\}$$
$$= \min\{\mu^c(x), \mu^c(y)\}$$
$$\mu^c(x\alpha y) = 1-\mu(x\alpha y) \geqslant \max\{\mu^c(x), \mu^c(y)\}.$$

Therefore μ^c is a fuzzy ideal of M.

$$\mu^c(x) = 1-\mu(x) \geqslant 1-\max\{\mu(x+y),\mu(y)\} = \min\{1-\mu(x+y), 1-\mu(y)\}$$
$$= \min\{\mu^c(x+y), \mu^c(y)\}.$$

Suppose $x \leqslant y$ and $x, y \in M$.

$$\Rightarrow \mu(x) \leqslant \mu(y)$$
$$\Rightarrow 1-\mu(x) \geqslant 1-\mu(y)$$
$$\Rightarrow \mu^c(x) \geqslant \mu^c(y).$$

Hence μ^c is a fuzzy $k-$ideal of M.
Conversely suppose that μ^c is a fuzzy $k-$ideal of M. Let $x, y \in M, \alpha \in \Gamma$.

$$\mu(x+y) = 1-\mu^c(x+y) \leqslant 1-\min\{\mu^c(x),\mu^c(y)\} = \max\{\mu(x),\mu(y)\}$$
$$\mu(x\alpha y) = 1-\mu^c(x\alpha y) \leqslant 1-\max\{\mu^c(x),\mu^c(y)\} = \min\{\mu(x),\mu(y)\}.$$

Therefore μ is an anti fuzzy ideal of M.

$$\mu(x) = 1-\mu^c(x) \leqslant 1-\min\{\mu^c(x+y),\mu^c(y)\} = \max\{1-\mu^c(x+y), 1-\mu^c(y)\}$$
$$= \max\{\mu(x+y), \mu(y)\}.$$

Let $x, y \in M$ and $x \leqslant y$.

$$\Rightarrow \mu^c(x) \geqslant \mu^c(y)$$
$$\Rightarrow \mu(x) \leqslant \mu(y).$$

Hence μ is an anti fuzzy $k-$ideal of M. □

LEMMA 3.1. *If μ is an anti fuzzy ideal of M and $\mu(x+y) = 1, x, y \in M$ then $\mu(x) = 1$ or $\mu(y) = 1$.*

PROOF. Suppose μ is an anti fuzzy ideal of $M, x, y \in M$ and $\mu(x+y) = 1$. $\mu(x+y) \leqslant \max\{\mu(x),\mu(y)\} \Rightarrow 1 \leqslant \max\{\mu(x),\mu(y)\} \Rightarrow \mu(x) = 1$ or $\mu(y) = 1$. □

THEOREM 3.4. *If f and g be anti fuzzy $k-$ideals of an ordered $\Gamma-$semiring M then $f \cap g$ is an anti fuzzy $k-$ideal of an ordered $\Gamma-$semiring M.*

PROOF. Suppose f and g are anti fuzzy $k-$ideals of M and $x, y \in M, \alpha \in \Gamma$.

$$f \cap g(x+y) = \min\{f(x+y), g(x+y)\}$$
$$\leqslant \min\{\max\{f(x), f(y)\}, \max\{g(x), g(y)\}\}$$
$$= \max\{\min\{f(x), g(x)\}, \min\{f(y), g(y)\}\}$$
$$= \max\{f \cap g(x), f \cap g(y)\}$$

$$\begin{aligned} f \cap g(x\alpha y) &= \min\{f(x\alpha y), g(x\alpha y)\} \\ &\leqslant \min\{\min\{f(x), f(y)\}, \min\{g(x), g(y)\}\} \\ &= \min\{\min\{f(x), g(x)\}, \min\{f(y), g(y)\}\} \\ &= \min\{f \cap g(x), f \cap g(y\} \end{aligned}$$

$$\begin{aligned} f \cap g(x) &= \min\{f(x), g(x)\} \\ &\leqslant \min\{\max\{f(x+y), f(y)\}, \max\{g(x+y), g(y)\}\} \\ &= \max\{\min\{f(x+y), g(x+y)\}, \min\{f(y), g(y)\}\} \\ &= \min\{f \cap g(x), f \cap g(y)\}. \end{aligned}$$

Suppose $x, y \in M$ and $x \leqslant y$.

$$\Rightarrow f(x) \leqslant f(y) \text{ and } g(x) \leqslant g(y).$$

$$\begin{aligned} f \cap g(x) &= \min\{f(x), g(x)\} \\ &\leqslant \min\{f(y), g(y)\} \\ &= f \cap g(y). \end{aligned}$$

Hence $f \cap g$ is an anti fuzzy $k-$ideal of M. □

THEOREM 3.5. *If f and g are anti fuzzy $k-$ideals of an ordered $\Gamma-$semiring M then $f \cup g$ is an anti fuzzy $k-$ideal of an ordered $\Gamma-$semiring M.*

PROOF. Let f and g be anti fuzzy $k-$ideals of M and $x, y \in M, \alpha \in \Gamma$.

$$\begin{aligned} f \cup g(x+y) &= \max\{f(x+y), g(x+y)\} \\ &\leqslant \max\{\max\{f(x), f(y)\}, \max\{g(x), g(y)\}\} \\ &= \max\{\max\{f(x), g(x)\}, \max\{f(y), g(y)\}\} \\ &= \max\{f \cup g(x), f \cup g(y)\} \end{aligned}$$

$$\begin{aligned} f \cup g(x\alpha y) &= \max\{f(x\alpha y), g(x\alpha y)\} \\ &\leqslant \max\{\min\{f(x), f(y)\}, \min\{g(x), g(y)\}\} \\ &= \min\{\max\{f(x), g(x)\}, \max\{f(y), g(y)\}\} \\ &= \min\{f \cup g(x), f \cup g(y)\} \end{aligned}$$

$$\begin{aligned} f\cup g(x) &= \max\{f(x), g(x)\} \\ &\geqslant \max\{\max\{f(x+y), f(y)\}, \max\{g(x+y), g(y)\}\} \\ &= \max\{\max\{f(x+y), g(x+y)\}, \max\{f(y), g(y)\}\} \\ &= \max\{f\cup g(x+y), f\cup g(y)\}. \end{aligned}$$

Suppose $x, y \in M$ and $x \leqslant y$.

$$\Rightarrow f(x) \leqslant f(y) \text{ and } g(x) \leqslant g(y).$$

$$\begin{aligned} f\cup g(x) &= \max\{f(x), g(x)\} \\ &\leqslant \max\{f(y), g(y)\} \\ &= f\cup g(y). \end{aligned}$$

Hence $f \cup g$ is an anti fuzzy $k-$ideal of M. □

Definition 3.5. An anti fuzzy $k-$ideal μ of an ordered $\Gamma-$semiring M is said to be a normal if $\mu(0) = 0$

Theorem 3.6. *Let μ be an anti fuzzy $k-$ideal of an ordered $\Gamma-$semiring M. If μ^+ be a fuzzy subset of M is defined by $\mu^+(x) = \mu(x) - \mu(0)$ for all $x \in M$ then μ^+ is a normal anti fuzzy $k-$ideal of M and $\mu^+ \subseteq \mu$.*

Proof. Let μ be the anti fuzzy $k-$ideal of M and μ^+ be a fuzzy subset of M, defined by $\mu^+(x) = \mu(x) - \mu(0)$ for all $x \in M$. For any $x, y \in M$ and $\alpha \in \Gamma$,

$$\begin{aligned} \mu^+(x+y) &= \mu(x+y) - \mu(0) \\ &\leqslant \max\{\mu(x), \mu(y)\} - \mu(0) \\ &= \max\{\mu(x) - \mu(0), \mu(y) - \mu(0)\} \\ &= \max\{\mu^+(x), \mu^+(y)\} \end{aligned}$$

$$\begin{aligned} \mu^+(x\alpha y) &= \mu(x\alpha y) - \mu(0) \\ &\leqslant \min\{\mu(x), \mu(y)\} - \mu(0) \\ &= \min\{\mu(x) - \mu(0), \mu(y) - \mu(0)\} \\ &= \min\{\mu^+(x), \mu^+(y)\} \end{aligned}$$

Suppose $x, y \in M$ and $x \leqslant y$.

$$\begin{aligned} &\Rightarrow \mu(x) \leqslant \mu(y), \text{ since } \mu \text{ is the anti fuzzy } k-\text{ideal} \\ &\Rightarrow \mu(x) - \mu(0) \leqslant \mu(y) - \mu(0) \\ &\Rightarrow \mu^+(x) \leqslant \mu^+(y). \end{aligned}$$

We have $\mu(x) \leqslant \max\{\mu(x+y), \mu(y)\}$

$$\begin{aligned} &\Rightarrow \mu(x) - \mu(0) \leqslant \max\{\mu(x+y) - \mu(0), \mu(y) - \mu(0)\} \\ &\Rightarrow \mu^+(x) \leqslant \max\{\mu^+(x+y), \mu^+(y)\}. \end{aligned}$$

$$\begin{aligned} \mu^+(x) &= \mu(x) - \mu(0) \\ &\Rightarrow \mu^+(0) = 0 \end{aligned}$$

Hence μ^+ is an anti fuzzy $k-$ideal of M.
Clearly μ contains μ^+ and μ^+ is a normal anti fuzzy $k-$ideal of M. □

Let μ be a fuzzy subset of M and $a \in M$. Then the set $\{b \in M \mid \mu(b) \leqslant \mu(a)\}$ is denoted by I_a.

THEOREM 3.7. *Let μ be an anti fuzzy $k-$left ideal of an ordered $\Gamma-$semiring M. If $a \in M$ then I_a is a left $k-$ideal of an ordered $\Gamma-$semiring M.*

PROOF. Let μ be the anti fuzzy $k-$ideal of M and $a \in M$.
We have $\mu(0) \leqslant \mu(x)$ for all $x \in M$. Therefore $0 \in I_a$. Let $b, c \in I_a$.
Then $\mu(b) \leqslant \mu(a)$ and $\mu(c) \leqslant \mu(a)$.

$$\begin{aligned}\mu(b+c) &\leqslant \max\{\mu(b), \mu(c)\}\\ &\leqslant \max\{\mu(a), \mu(a)\}\\ &= \mu(a).\end{aligned}$$

Then $b + c \in I_a$. Suppose $b \in I_a, c \in M, \alpha \in \Gamma$. Then $\mu(b) \leqslant \mu(a)$. Now $\mu(c\alpha b) \leqslant \mu(b) \leqslant \mu(a)$. Therefore $c\alpha b \in I_a$.
Suppose $x \in M, y \in I_a$ and $x \leqslant y$. Then $\mu(y) \leqslant \mu(a)$ and $\mu(x) \leqslant \mu(y) \Rightarrow \mu(x) \leqslant \mu(y) \leqslant \mu(a)$. Therefore $x \in I_a$.
Suppose $x \in I_a$ and $x + y \in I_a$.
$\Rightarrow \mu(x) \leqslant \mu(a), \mu(x + y) \leqslant \mu(a)$
$\Rightarrow \mu(y) \leqslant \max\{\mu(x + y), \mu(x)\} \leqslant \mu(a)$
$\Rightarrow y \in I_a$.
Hence I_a is a left $k-$ideal of M. □

COROLLARY 3.1. *Let μ be an anti fuzzy right $k-$ideal of an ordered $\Gamma-$semiring M and $a \in M$. Then I_a is a right $k-$ideal of an ordered $\Gamma-$semiring M.*

COROLLARY 3.2. *Let μ be an anti fuzzy $k-$ideal of an ordered $\Gamma-$semiring M and $a \in M$. Then I_a is a $k-$ideal of an ordered $\Gamma-$semiring M.*

DEFINITION 3.6. A family of fuzzy subsets $\{\mu_i \mid i \in I\}$ of an ordered $\Gamma-$semiring M, then $\bigvee_{i\in I} \mu_i$ is defined by $\bigvee_{i\in I} \mu_i(x) = \sup\{\mu_i(x) \mid i \in I\}$, for all $x \in M$.

THEOREM 3.8. *If $\{\mu_i \mid i \in I\}$ is a family of anti fuzzy ideals of an ordered $\Gamma-$semiring M then $\bigvee_{i\in I} \mu_i$ is an anti fuzzy ideal of an ordered $\Gamma-$semiring M.*

PROOF. Let $\{\mu_i \mid i \in I\}$ be a family of anti fuzzy ideals of M and $x, y \in M, \alpha \in \Gamma$. Then we have,

$$\begin{aligned}\left(\bigvee_{i\in I}\mu_i\right)(x+y) &= \sup\{\mu_i(x+y) \mid i \in I\}\\ &\leqslant \sup\{\max\{\mu_i(x),\ \mu_i(y)\} \mid i \in I\}\\ &= \max\{\sup_{i\in I}\mu_i(x), \sup_{i\in I}\mu_i(y)\}\\ &= \max\left\{\bigvee_{i\in I}\mu_i(x), \bigvee_{i\in I}\mu_i(y)\right\}\end{aligned}$$

$$\begin{aligned}\left(\bigvee_{i\in I}\mu_i\right)(x\alpha y) &= \sup\{\mu_i(x\alpha y) \mid i \in I\}\\ &\leqslant \sup\{\min\{\mu_i(x)\mu_i(y)\} \mid i \in I\}\\ &= \min\{\sup_{i\in I}\mu_i(x), \sup_{i\in I}\mu_i(y)\}\\ &= \min\left\{\bigvee_{i\in I}\mu_i(x), \bigvee_{i\in I}\mu_i(y)\right\}.\end{aligned}$$

Suppose $x, y \in M$ and $x \leqslant y$. Then

$$\begin{aligned}&\mu_i(x) \leqslant \mu_i(y), \text{ for all } i \in I\\ &\Rightarrow \bigvee_{i\in I}\mu_i(x) \leqslant \bigvee_{i\in I}\mu_i(y).\end{aligned}$$

Hence $\bigvee_{i\in I}\mu_i$ is an anti fuzzy ideal of M. □

DEFINITION 3.7. Let μ be a fuzzy subset of X and $\alpha \in [0, 1-\sup\{\mu(x) \mid x \in X\}]$ The mapping $\mu_\alpha^T : X \to [0,1]$ is called a fuzzy translation of μ if $\mu_\alpha^T(x) = \mu(x) + \alpha$.

DEFINITION 3.8. Let μ be a fuzzy subset of X and $\beta \in [0,1]$. Then mapping $\mu_\beta^M : X \to [0,1]$ is called a fuzzy multiplication of μ if $\mu_\beta^M(x) = \beta\mu(x)$.

DEFINITION 3.9. Let μ be a fuzzy subset of X and $\alpha \in [0, 1-\sup\{\mu(x) \mid x \in X\}]$, $\beta \in [0,1]$. Then mapping $\mu_{\beta,\alpha}^{MT} : X \to [0,1]$ is called a magnified translation of μ if $\mu_{\beta,\alpha}^{MT}(x) = \beta\mu(x) + \alpha$, for all $x \in X$.

THEOREM 3.9. *A fuzzy subset μ is an anti fuzzy $k-$ideal of an ordered $\Gamma-$semiring M if and only if μ_α^T is an anti fuzzy $k-$ideal of an ordered $\Gamma-$semiring M.*

Proof. Suppose μ is an anti fuzzy $k-$ideal of M and $x, y \in M, \gamma \in \Gamma$.

$$\begin{aligned}\mu_\alpha^T(x+y) &= \mu(x+y)+\alpha \\ &\leqslant \max\{\mu(x),\mu(y)\}+\alpha \\ &= \max\{\mu(x)+\alpha,\mu(y)+\alpha\} \\ &= \max\{\mu_\alpha^T(x),\mu_\alpha^T(y)\}\end{aligned}$$

$$\begin{aligned}\mu_\alpha^T(x\gamma y) &= \mu(x\gamma y)+\alpha \\ &\leqslant \min\{\mu(x),\mu(y)\}+\alpha \\ &= \min\{\mu(x)+\alpha,\mu(y)+\alpha\} \\ &= \min\{\mu_\alpha^T(x),\mu_\alpha^T(y)\}\end{aligned}$$

$$\begin{aligned}\mu_\alpha^T(x) &= \mu(x)+\alpha \\ &\leqslant \max\{\mu(x+y),\mu(y)\}+\alpha \\ &= \max\{\mu(x+y)+\alpha,\mu(y)+\alpha\} \\ &= \max\{\mu_\alpha^T(x+y),\mu_\alpha^T(y)\}.\end{aligned}$$

Let $x \leqslant y$.Then $\mu(x) \leqslant \mu(y)$

$\Rightarrow \mu(x)+\alpha \leqslant \mu(y)+\alpha$

$\Rightarrow \mu_\alpha^T(x) \leqslant \mu_\alpha^T(y)$.

Hence μ_α^T is an anti fuzzy $k-$ideal of M.
Conversely suppose that μ_α^T is an anti fuzzy $k-$ideal of M, $x, y \in M$ and $\gamma \in \Gamma$.

$$\begin{aligned}\mu(x+y)+\alpha &= \mu_\alpha^T(x+y) \\ &\leqslant \max\{\mu_\alpha^T(x),\mu_\alpha^T(y)\} \\ &= \max\{\mu(x)+\alpha,\mu(y)+\alpha\} \\ &= \max\{\mu(x),\mu(y)\}+\alpha\end{aligned}$$

Therefore $\mu(x+y) \leqslant \max\{\mu(x),\mu(y)\}$.

$$\begin{aligned}\mu(x\gamma y)+\alpha &= \mu_\alpha^T(x\gamma y) \\ &\leqslant \min\{\mu_\alpha^T(x),\mu_\alpha^T(y)\} \\ &= \min\{\mu(x)+\alpha,\mu(y)+\alpha\} \\ &= \min\{\mu(x),\mu(y)\}+\alpha\end{aligned}$$

Therefore $\mu(x\gamma y) \leqslant \min\{\mu(x),\mu(y)\}$.

$$\begin{aligned}
\mu(x)+\alpha &= \mu_\alpha^T(x) \\
&\leqslant \max\{\mu_\alpha^T(x+y), \mu_\alpha^T(y)\} \\
&= \max\{\mu(x+y)+\alpha, \mu(y)+\alpha\} \\
&= \max\{\mu(x+y), \mu(y)\}+\alpha \\
\text{Therefore } \mu(x) &\geqslant \min\{\mu(x+y), \mu(y)\}. \\
\text{Let } x \leqslant y. &\text{Then } \mu_\alpha^T(x) \leqslant \mu_\alpha^T(y). \\
&\Rightarrow \mu(x)+\alpha \leqslant \mu(y)+\alpha \\
&\Rightarrow \mu(x) \leqslant \mu(y).
\end{aligned}$$

Hence μ is an anti fuzzy $k-$ideal of M. □

THEOREM 3.10. *A fuzzy subset μ is an anti fuzzy $k-$ideal of an ordered $\Gamma-$semiring M if and only if μ_β^M is an anti fuzzy $k-$ideal of an ordered $\Gamma-$semiring M.*

PROOF. Suppose μ is an anti fuzzy $k-$ideal of M and $x, y \in M, \gamma \in \Gamma$. Then

$$\begin{aligned}
\mu_\beta^M(x+y) &= \beta\mu(x+y) \\
&\leqslant \beta\max\{\mu(x), \mu(y)\} \\
&= \max\{\beta\mu(x), \beta\mu(y)\} \\
&= \max\{\mu_\beta^M(x), \mu_\beta^M(y)\}.
\end{aligned}$$

$$\begin{aligned}
\mu_\beta^M(x\gamma y) &= \beta\mu(x\gamma y) \\
&\leqslant \beta\min\{\mu(x), \mu(y)\} \\
&= \min\{\beta\mu(x), \beta\mu(y)\} \\
&= \min\{\mu_\beta^M(x), \mu_\beta^M(y)\}. \\
\mu_\beta^M(x) &= \beta\mu(x) \\
&\leqslant \beta\max\{\mu(x+y), \mu(y)\} \\
&= \max\{\beta\mu(x+y), \beta\mu(y)\} \\
&= \max\{\mu_\beta^M(x+y), \mu_\beta^M(y)\}. \\
\text{Let } x \leqslant y. &\text{Then } \mu(x) \leqslant \mu(y) \\
&\Rightarrow \beta\mu(x) \leqslant \beta\mu(y) \\
&\Rightarrow \mu_\beta^M(x) \leqslant \beta_\beta^M\mu(y).
\end{aligned}$$

Hence μ_β^M is an anti fuzzy ideal of M.
Conversely, suppose that μ_β^M is an anti fuzzy ideal of M and $x, y \in M, \gamma \in \Gamma$. Then

$$\begin{aligned}
\mu_\beta^M(x+y) &\leqslant \max\{\mu_\beta^M(x), \mu_\beta^M(y)\}\\
\Rightarrow\ \beta\mu(x+y) &\leqslant \max\{\beta\mu(x), \beta\mu(y)\}\\
&= \beta\max\{\mu(x), \mu(y)\}\\
\text{Therefore } \mu(x+y) &\leqslant \max\{\mu(x), \mu(y)\}.\\
\mu_\beta^M(x\gamma y) &\leqslant \min\{\mu_\beta^M(x), \mu_\beta^M(y)\}\\
&= \beta\min\{\mu(x), \mu(y)\}\\
\beta\mu(x\gamma y) &= \beta\min\{\mu(x), \mu(y)\}\\
\text{Therefore } \mu(x\gamma y) &= \min\{\mu(x), \mu(y)\}.\\
\mu_\beta^M(x) &\leqslant \max\{\mu_\beta^M(x+y), \mu_\beta^M(y)\}\\
&= \max\{\beta\mu(x+y), \beta\mu(y)\}\\
&= \beta\max\{\mu(x+y), \mu(y)\}\\
\beta\mu(x) &= \beta\max\{\mu(x+y), \mu(y)\}\\
\text{Therefore } \mu(x) &\leqslant \max\{\mu(x+y), \mu(y)\}.
\end{aligned}$$

$$\begin{aligned}
&\text{Let } x \leqslant y.\text{Then } \mu_\beta^M(x) \leqslant \beta_\beta^M\mu(y)\\
&\Rightarrow \beta\mu(x) \leqslant \beta\mu(y)\\
&\Rightarrow \mu(x) \leqslant \mu(y).
\end{aligned}$$

Hence μ is an anti fuzzy $k-$ideal of M. □

THEOREM 3.11. *A fuzzy subset μ is an anti fuzzy $k-$ideal of an ordered $\Gamma-$semiring M if and only if $\mu_{\beta,\alpha}^{MT} : X \to [0,1]$ is an anti fuzzy $k-$ideal of an ordered $\Gamma-$semiring M.*

PROOF. Suppose μ is an anti fuzzy $k-$ideal of M.
$\Leftrightarrow$ μ_β^M is an anti fuzzy $k-$ideal of M, by Theorem 3.10
$\Leftrightarrow$ $\mu_{\beta,\alpha}^{MT}$ is an anti fuzzy $k-$ideal of M, by Theorem 3.9

□

4. Homomorphisms of ordered $\Gamma-$semirings

In this section the concept of an anti homomorphism of ordered $\Gamma-$semirings has been introduced. The properties of homomorphic, anti homomorphic image and pre-image of fuzzy ideals and anti fuzzy ideals of an ordered $\Gamma-$semiring are studied.

DEFINITION 4.1. A function $\phi : M \to N$ where M and N are ordered $\Gamma-$semirings is called a homomorphism(anti homomorphism) of ordered $\Gamma-$semirings if
$\phi(a+b) = \phi(a) + \phi(b), \phi(a\alpha b) = \phi(a)\alpha\phi(b)[\phi(b)\alpha\phi(a)]$
and $a \leqslant b \Rightarrow \phi(a) \leqslant \phi(b),$ for all $a, b \in M, \alpha \in \Gamma$.

Theorem 4.1. *Let $f : M \to N$ be a homomorphism of ordered $\Gamma-$semirings and η be an anti fuzzy ideal of N. If $\eta \circ f = \mu$ then μ is an anti fuzzy ideal of M.*

Proof. Let $f : M \to N$ be a homomorphism of ordered $\Gamma-$semirings, η be an anti fuzzy ideal of $N, \eta \circ f = \mu$ and $x, y \in M$.

$$\begin{aligned} \mu(x+y) &= \eta(f(x+y)) = \eta(f(x)+f(y)) \\ &\leqslant \max\{\eta(f(x)), \eta(f(y))\} \\ &= \max\{\mu(x), \mu(y)\} \\ \mu(x\alpha y) &= \eta(f(xy)) = \eta(f(x)f(y)) \\ &\leqslant \min\{\eta(f(x)), \eta(f(y))\} \\ &= \min\{\mu(x), \mu(y)\}. \end{aligned}$$

Suppose $x, y \in M$ and $x \leqslant y$. Since $f : M \to N$ be a homomorphism, we have

$$\begin{aligned} &f(x) \leqslant f(y) \\ \Rightarrow &\eta(f(x)) \geqslant \eta(f(y)) \\ \Rightarrow &\mu(x) \leqslant \mu(y) \end{aligned}$$

Hence μ is an anti fuzzy ideal of an ordered $\Gamma-$semiring M. □

Definition 4.2. Let M and N be two ordered $\Gamma-$semirings and f be a function from M into N. If μ is a fuzzy ideal of N then the pre-image of μ under f is the fuzzy subset of M, defined by $f^{-1}(\mu)(x) = \mu(f(x))$ for all $x \in M$.

Theorem 4.2. *Let $f : M \to N$ be an onto homomorphism of ordered $\Gamma-$semirings. If μ is an anti fuzzy $k-$ideal of N then $f^{-1}(\mu)$ is an anti fuzzy $k-$ideal of M.*

Proof. Suppose $f : M \to N$ is an onto homomorphism of ordered $\Gamma-$semirings and μ is an anti fuzzy $k-$ideal of N and $x_1, x_2 \in M, \alpha \in \Gamma$.

$$\begin{aligned} f^{-1}(\mu)(x_1+x_2) &= \mu(f(x_1+x_2) = \mu\big(f(x_1)+f(x_2)\big) \\ &\leqslant \max\{\mu(f(x_1)), \mu(f(x_2))\} \\ &= \max\{f^{-1}(\mu)(x_1)), f^{-1}(\mu)(x_2))\} \end{aligned}$$

$$\begin{aligned} f^{-1}(\mu)(x_1\alpha x_2) &= \mu(f(x_1\alpha x_2)) \leqslant \min\{\mu(f(x_1)), \mu(f(x_2))\} \\ &= \min\{f^{-1}(\mu)(x_1)), f^{-1}(\mu)(x_2))\} \\ f^{-1}(\mu)(x) &= \mu(f(x)) \leqslant \max\{\mu(f(x+y)), \mu(f(y))\} \\ &= \max\{f^{-1}(\mu(x+y)), f^{-1}(\mu(x+y))\}, \text{ for all } x, y \in M. \end{aligned}$$

Let $x_1, x_2 \in M$ and $x_1 \leqslant x_2$.

$$\begin{aligned} \Rightarrow &f(x_1) \leqslant f(x_2) \\ \Rightarrow &\mu(f(x_1)) \leqslant \mu(f(x_2)) \\ \Rightarrow &f^{-1}(\mu)(x_1) \leqslant f^{-1}(\mu)(x_2). \end{aligned}$$

Hence $f^{-1}(\mu)$ is an anti fuzzy $k-$ideal of M. □

THEOREM 4.3. *Let $f : M \to N$ be an onto anti homomorphism of ordered $\Gamma-$semirings.If η is a fuzzy left ideal of N and μ is the pre-image of η under f then μ is a fuzzy right ideal of M.*

PROOF. Let $f : M \to N$ be an onto anti homomorphism of ordered $\Gamma-$semirings. If η is a fuzzy left ideal of N, μ is the pre-image of η under f, $x, y \in N$ and $\alpha \in \Gamma$.

$$\begin{aligned}\mu(x+y) &= \eta\big(f(x+y)\big) = \eta\big(f(x)+f(y)\big) \geqslant \min\{\eta(f(x)), \eta(f(y))\}\\ &= \min\{\mu(x), \mu(y)\}\\ \mu(x\alpha y) &= \eta\big(f(x\alpha y)\big) = \eta\big(f(y)\alpha f(x)\big) \geqslant \eta(f(x)) = \mu(x).\end{aligned}$$

Let $x, y \in M$ and $x \leqslant y$.

$$\begin{aligned}&\Rightarrow f(x) \leqslant f(y)\\ &\Rightarrow \eta(f(x)) \geqslant \eta(f(y))\\ &\Rightarrow \mu(x) \geqslant \mu(y).\end{aligned}$$

Hence μ is an fuzzy right ideal of M. □

COROLLARY 4.1. *Let $f : M \to N$ be an onto anti homomorphism of ordered $\Gamma-$semirings. If η is a fuzzy right ideal of N and μ is the pre-image of η under f then μ is a fuzzy left ideal of M.*

COROLLARY 4.2. *Let $f : M \to N$ be an onto anti homomorphism of ordered $\Gamma-$semirings. If η is a fuzzy ideal of N and μ is the pre-image of η under f then μ is a fuzzy ideal of M.*

THEOREM 4.4. *Let $f : M \to N$ be an onto anti homomorphism of ordered $\Gamma-$semirings. If μ is an anti fuzzy left $k-$ideal of N then $f^{-1}(\mu)$ is an anti fuzzy right $k-$ideal of M.*

PROOF. Let $f : M \to N$ be an onto anti homomorphism of ordered $\Gamma-$semirings, μ be an anti fuzzy left $k-$ideal of N and $x_1, x_2, x, y \in M$ and $\alpha \in \Gamma$.

$$\begin{aligned}f^{-1}(\mu)(x_1+x_2) &= \mu(f(x_1+x_2)) = \mu\big(f(x_1)+f(x_2)\big)\\ &\geqslant \max\{\mu(f(x_1)), \mu(f(x_2))\}\\ &= \max\{f^{-1}(\mu)(x_1), f^{-1}(\mu)(x_2)\}\\ f^{-1}(\mu)(x_1\alpha x_2) &= \mu(f(x_1\alpha x_2)) = \mu(f(x_2)\alpha f(x_1))\\ &\geqslant \mu(f(x_1))\\ &= f^{-1}(\mu)(x_1)\\ f^{-1}(\mu)(x) &= \mu(f(x)) \leqslant \max\{\mu(f(x+y)), \mu(f(y))\}\\ &= \max\{f^{-1}(\mu)(x+y), f^{-1}(\mu)(y)\}.\end{aligned}$$

$$\begin{aligned}&\text{Let } x \leqslant y. \text{ Then } f(x) \leqslant f(y)\\ &\Rightarrow \mu(f(x)) \leqslant \mu(f(y))\\ &\Rightarrow f^{-1}(\mu)(x) \leqslant f^{-1}(\mu)(y).\end{aligned}$$

Hence $f^{-1}(\mu)$ is an anti fuzzy $k-$right ideal of M. □

Definition 4.3. Let $f : M \to N$ be a homomorphism of ordered $\Gamma-$semirings M and N, μ be a fuzzy subset of M. Then μ is said to be an invariant of f if $f(x) \leqslant f(y)$ and $\mu(x) \leqslant \mu(y)$ for all $x, y \in M$.

Theorem 4.5. *Let $f : M \to N$ be an onto anti homomorphism of ordered $\Gamma-$semirings. Then anti homomorphic image of an anti fuzzy left ideal μ, invariant f of M is an anti fuzzy right ideal of N.*

Proof. Let $f : M \to N$ be an onto anti homomorphism of ordered $\Gamma-$semirings, μ be an anti fuzzy left ideal of M and η be a fuzzy subset of N such that $\eta \circ f = \mu$. Let $a, b \in N, \alpha \in \Gamma$ then there exist $x, y \in M$ such that $f(x) = a$ and $f(y) = b$.

$$\begin{aligned}\eta(a+b) &= \eta(f(x)+f(y)) = \eta(f(x+y))\\ &= \mu(x+y) \leqslant \max\{\mu(x), \mu(y)\}\\ &= \max\{\eta(f(x)), \eta(f(y))\}.\\ &= \max\{\eta(a), \eta(b)\}.\end{aligned}$$

$$\begin{aligned}\eta(a\alpha b) &= \eta(f(x)\alpha f(y)) = \eta(f(y\alpha x))\\ &= \mu(y\alpha x) \leqslant \mu(x)\\ &= \eta(f(x)) = \eta(a).\end{aligned}$$

Let $a, b \in N$ and $a \leqslant b$. There exist $x, y \in M$,
such that $f(x) = a, f(y) = b$.
$\Rightarrow f(x) \leqslant f(y) \Rightarrow \mu(x) \leqslant \mu(y)$,
$\Rightarrow \eta \circ f(x) \leqslant \eta \circ f(y), \Rightarrow \eta(f(x)) \leqslant \eta(f(y))$.
Therefore $\eta(a) \leqslant \eta(b)$.

Hence η is an anti fuzzy right ideal of N. □

Theorem 4.6. *Let $f : M \to N$ be an onto anti homomorphism of the ordered $\Gamma-$semiring. If η is an anti fuzzy left $k-$ideal of N and μ is the pre-image of η under f then μ is an anti fuzzy right $k-$ideal of M.*

Proof. Let $f : M \to N$ be an onto anti homomorphism of an ordered $\Gamma-$semiring, η be an anti fuzzy left $k-$ideal of N, μ be the pre-image of η under f, $x, y \in M$ and $\alpha \in \Gamma$.

$$\begin{aligned}\mu(x+y) &= \eta(f(x+y)) = \eta(f(x)+f(y)) \leqslant \max\{\eta(f(x)), \eta(f(y))\}\\ &= \max\{\mu(x), \mu(y)\}\\ \mu(x\alpha y) &= \eta(f(x\alpha y)) = \eta(f(y)\alpha f(x)) \leqslant \eta(f(x)) = \mu(x)\\ \mu(x) &= \eta(f(x)) \leqslant \max\{\eta(f(x+y)), \eta(f(x))\} = \max\{\mu(x+y), \mu(y)\}.\end{aligned}$$

Let $x, y \in M$ and $x \leqslant y$. Then $f(x) \leqslant f(y)$, since f is an anti homomorphism.
$\Rightarrow \eta(f(x)) \leqslant \eta(f(y))$, since η is an anti fuzzy left $k-$ideal, $\Rightarrow \mu(x) \leqslant \mu(y)$.
Hence μ is an anti fuzzy right $k-$ideal of M. □

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M. Murali Krishna Rao, Department of Mathematics, Sankethika Institute of Tech. and Management, Visakhapatnam-530 041, India
Email address: `mmarapureddy@gmail.com`

Noorbhasha Rafi, Department of Mathematics, Bapatla Engineering College, Bapatla, Andhra Pradesh-522 101, India
Email address: `rafimathsy@gmail.com`

Rajendra Kumar Kona, Department of Mathematics, GIS, GITAM (Deemed to be University), Visakhapatnam- 530 045, India
Email address: `rkkona1972@gmail.com`