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BIPOLAR FUZZY IMPLICATIVE SBG-IDEALS AND THEIR NORMALITY

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Abstract. This paper introduces bipolar fuzzy implicative SBG-ideals in the context of SBG-algebras, establishing that these ideals are also bipolar fuzzy SBG-ideals. We explore their key properties under homomorphisms and (α, β)-translations, and demonstrate how they are preserved through these operations. The relationship between implicative SBG-ideals and their cuts is examined, showing that nonempty cuts form implicative ideals. Additionally, we investigate bipolar fuzzy sub-implicative SBG-ideals, particularly in medial SBG-algebras, where these ideals act as both sub-implicative and implicative ideals. We also define the concept of normality for bipolar-valued fuzzy SBG-ideals, characterizing the conditions under which these ideals are normal, and explore their maximality in partially ordered sets. This research sets the stage for further studies on complex algebraic structures and computational applications in fields like artificial intelligence.

1. Introduction

In mathematical modeling, reducing complexity by minimizing the number of axioms or operations plays a crucial role. Tarski [**23**] and Sheffer [**22**] exemplified this by using minimal operations to define Abelian groups and Boolean functions, respectively. Expanding on this approach, McCune et al. [**10**] axiomatized Boolean algebras using only the Sheffer stroke.

Logical algebraic structures find broad applications in fields such as artificial intelligence, computer science, quantum logics, and probability theory. Sheffer

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stroke basic algebras, introduced by Oner and Senturk [**15**], offer a streamlined version of basic algebras using a single operation, thereby providing computational efficiency in technological applications. This concept was later extended to Sheffer stroke BG-algebras in [**17**].

The development of algebraic structures progressed with Imai and Iséki's introduction of BCK- and BCI-algebras [**2**], followed by Neggers and Kim's B-algebras [**14**], and Kim and Kim's generalization, BG-algebras [**7**].

Fuzzy set theory, introduced by Zadeh in 1965 [**25**], has undergone various extensions, including intuitionistic, interval-valued, vague, and bipolar-valued fuzzy sets. In particular, bipolar-valued fuzzy sets, introduced by Lee in 2000 [**8**], extend traditional fuzzy sets by allowing a membership degree range of [-1, 1]. Subsequent studies explored bipolar fuzzy structures in algebraic frameworks such as BCH-algebras [**4**, **5**], BCI-algebras [**6**, **9**], KU-algebras [**11**], and BG-algebras [**20**]. More recent research has further expanded these ideas into bipolar fuzzy translations in various algebraic structures [**3**, **18**, **19**].

Bipolar-valued fuzzy sets play a pivotal role in numerous applications, contributing to the development of advanced algorithms for solving complex problems. In the framework of BG-algebras, Ahn and Lee [**1**] examined fuzzy subalgebras, while Muthuraj et al. [**12**, **13**] focused on fuzzy ideals and multi-fuzzy subalgebras. Building on these studies, Oner, Senturk, and Rezaei explored bipolar-valued fuzzy translations in Sheffer stroke MTL-algebras [**16**].

In this paper, we delve into the concepts of bipolar fuzzy implicative SBG-ideals and normal bipolar-valued fuzzy SBG-ideals within the framework of SBG-algebras. We begin by defining bipolar fuzzy implicative SBG-ideals, demonstrating that they are preserved under homomorphisms and (α, β) - translations, and establishing that every bipolar fuzzy implicative SBG-ideal is also a bipolar fuzzy SBG-ideal. We further explore the relationship between these ideals and their cuts, showing that nonempty cuts form implicative SBG-ideals. Additionally, we examine how bipolar fuzzy sub-implicative SBG-ideals function in medial SBG-algebras and prove that these implicative ideals are maintained under surjective homomorphisms. The second part of the paper focuses on normal bipolar-valued fuzzy SBG-ideals, providing a characterization of their normality through conditions on membership functions. We also explore the relationship between normal and larger ideals using a novel inclusion-based construction, and we conclude by proving maximality conditions, describing the behavior of these ideals within a partially ordered set.

2. Preliminaries

During this section, we present the essential definitions, lemmas,theorem and propositions related to Sheffer stroke BG-algebras, their corresponding ideals, subalgebras, bipolar fuzzy sets, bipolar fuzzy SBG-ideals. These foundational concepts will be utilized throughout the paper.

Definition 2.1. [**22**] Let $G = \langle G, | \rangle$ be a groupoid. The operation $|$ is said to be a Sheffer stroke operation if it satisfies the following conditions:

$(S1)$ $\alpha|\beta = \beta|\alpha$

$(S2)$ $(\alpha|\alpha)|(\alpha|\beta) = \alpha$
$(S3)$ $\alpha|((\beta|\gamma)|(\beta|\gamma)) = ((\alpha|\beta)|(\alpha|\beta))|\gamma$
$(S4)$ $(\alpha|((\alpha|\alpha)|(\beta|\beta)))|(\alpha|((\alpha|\alpha)|(\beta|\beta))) = \alpha.$

Definition 2.2. [**17**] A Sheffer stroke BG-algebra (briefly, SBG-algebra) is a structure $\langle A, |\rangle$ of type (2) where 0 is the fixed element in A, and the following conditions are satisfied for all $\alpha, \beta, \gamma \in A$:
$(SBG-1)$ $(\alpha|(\alpha|\alpha))|(\alpha|(\alpha|\alpha)) = 0,$
$(SBG-2)$ $(0|(\beta|\beta))|(\alpha|(\beta|\beta))|(\alpha|(\beta|\beta)) = \alpha|\alpha.$

Proposition 2.1. *[**17**] Let $\langle A, |\rangle$ be an SBG-algebra. Then the binary relation $\alpha \leqslant \beta$ if and only if $(\beta|(\alpha|\alpha))|(\beta|(\alpha|\alpha)) = 0$ is a partial order on A.*

Definition 2.3. [**17**] A nonempty subset G of a Sheffer stroke BG-algebra H is called an SBG-subalgebra of H if $(\alpha|(\beta|\beta))|(\alpha|(\beta|\beta)) \in G$ for all $\alpha, \beta \in G$.

Definition 2.4. [**17**] A nonempty subset G of a Sheffer stroke BG-algebra H is called an SBG-ideal of H if for all $\alpha, \beta \in G$:

(1) $0 \in G$,
(2) $(\alpha|(\beta|\beta))|(\alpha|(\beta|\beta)) \in G$ and $\beta \in G \Rightarrow \alpha \in G$.

Lemma 2.1. *[**17**] Let A be a Sheffer stroke BG-algebra. Then the following properties hold for all $\alpha_1, \alpha_2, \alpha_3 \in A$:*

(1) $(0|0)|(\alpha_1|\alpha_1) = \alpha_1,$
(2) $(\alpha_1|(0|0))|(\alpha_1|(0|0)) = \alpha_1,$
(3) $(\alpha_1|(\alpha_2|\alpha_2))|(\alpha_1|(\alpha_2|\alpha_2)) = (\alpha_3|(\alpha_2|\alpha_2))|(\alpha_3|(\alpha_2|\alpha_2))$ *implies* $\alpha_1 = \alpha_3,$
(4) $(0|(0|(\alpha_1|\alpha_1))) = \alpha_1|\alpha_1,$
(5) *If* $(\alpha_1|(\alpha_2|\alpha_2))|(\alpha_1|(\alpha_2|\alpha_2)) = 0$*, then* $\alpha_1 = \alpha_2,$
(6) *If* $(0|(\alpha_1|\alpha_1)) = (0|(\alpha_2|\alpha_2))$*, then* $\alpha_1 = \alpha_2,$
(7) $(((\alpha_1|(0|(\alpha_1|\alpha_1)))|(\alpha_1|(0|(\alpha_1|\alpha_1))))|(\alpha_1|\alpha_1)) = \alpha_1|\alpha_1,$
(8) $(\alpha_1|(\alpha_1|\alpha_1))|(\alpha_1|\alpha_1) = \alpha_1.$

Definition 2.5. [**17**] An SBG-algebra is called SBG-implicative if it satisfies the condition:

$$x|x^y = y|y^x$$

for all $x, y \in L$.

Definition 2.6. [**17**] An SBG-algebra L is called medial if it satisfies the condition:

$$x|x^y = y|y$$

for all $x, y \in L$.

Definition 2.7. [**26**] Let X be a nonempty set. A bipolar fuzzy set B in X is an object of the form

$$B = \{(\xi, f^-(\xi), f^+(\xi)) \mid \xi \in X\},$$

where $f^+ : X \to [0, 1]$ and $f^- : X \to [-1, 0]$ are mappings.

The positive membership degree $f^+(\xi)$ denotes the satisfaction degree of an element ξ to the property corresponding to a bipolar fuzzy set B, while the negative membership degree $f^-(\xi)$ represents the satisfaction degree of an element ξ to some implicit counter-property of B.

If $f^+(\xi) \neq 0$ and $f^-(\xi) = 0$, then ξ has only positive satisfaction for B. If $f^+(\xi) = 0$ and $f^-(\xi) \neq 0$, then ξ does not satisfy the property of B but somewhat satisfies its counter-property.

It is also possible for an element ξ to satisfy both $f^+(\xi) = 0$ and $f^-(\xi) = 0$ if the membership functions of the property and its counter-property overlap in some portion of X.

For simplicity, we use the notation $f = (f^+, f^-)$ to represent the bipolar fuzzy set $B = \{(\xi, f^-(\xi), f^+(\xi)) \mid \xi \in X\}$.

Lemma 2.2. *[**24**] Let $\alpha, \beta, \gamma \in \mathbb{R}$. Then the following statements hold:*

(1) $\alpha - \min\{\beta, \gamma\} = \max\{\alpha - \beta, \alpha - \gamma\}$,
(2) $\alpha - \max\{\beta, \gamma\} = \min\{\alpha - \beta, \alpha - \gamma\}$.

Definition 2.8. [**21**] A bipolar-valued fuzzy set $f = (L; f^-, f^+)$ in L is called a bipolar-valued fuzzy SBG-subalgebra of $\mathcal{L} = (L; |)$ if it satisfies the following condition for all $x, y \in L$:

$$(\forall x, y \in L) \left(\begin{array}{c} f^-(x^y|x^y) \leqslant \max\{f^-(x), f^-(y)\} \\ f^+(x^y|x^y) \geqslant \min\{f^+(x), f^+(y)\} \end{array} \right). \tag{2.1}$$

Theorem 2.1. *[**21**] A bipolar-valued fuzzy set $f = (L; f^-, f^+)$ in L is a bipolar-valued fuzzy SBG-subalgebra of $\mathcal{L} = (L; |)$ if and only if its negative s-cut and positive t-cut are SBG-subalgebras of $\mathcal{L} = (L; |)$ whenever they are nonempty for all $(s, t) \in [-1, 0] \times [0, 1]$.*

3. Structural analysis and preservation of bipolar fuzzy implicative SBG-ideals

In this section, we introduced the concept of bipolar fuzzy implicative SBG-ideals within the framework of SBG-algebras. We began by defining a bipolar fuzzy implicative SBG-ideal and presented conditions involving both the negative and positive components of bipolar fuzzy sets. Following this, we demonstrated that every bipolar fuzzy implicative SBG-ideal is, in fact, a bipolar fuzzy SBG-ideal. We explored several key propositions and theorems, proving that these implicative ideals are preserved under certain conditions such as homomorphisms and (α, β)-translations. Additionally, we examined the relationship between these ideals and cuts (negative s-cuts and positive t-cuts), establishing that if the cuts are nonempty, they form implicative SBG-ideals. This section also provided a thorough discussion of how bipolar fuzzy sub-implicative SBG-ideals relate to implicative ones, emphasizing that in medial SBG-algebras, bipolar fuzzy implicative SBG-ideals also function as bipolar fuzzy sub-implicative ideals. Finally, we proved that these implicative ideals are maintained under surjective homomorphisms, further solidifying the structure and behavior of bipolar fuzzy implicative SBG-ideals in various algebraic settings.

DEFINITION 3.1. A bipolar fuzzy set $f = (L; f^-, f^+)$ on an SBG-algebra L is called a bipolar fuzzy implicative SBG-ideal of L if, for all $x, y \in L$,

(i) $f^-(0) \leqslant f^-(x) \leqslant \max\left\{f^-((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f^-(z)\right\}$,

(ii) $f^+(0) \geqslant f^+(x) \geqslant \min\left\{f^+((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f^+(z)\right\}$.

PROPOSITION 3.1. *Each bipolar fuzzy implicative SBG-ideal of an SBG-algebra L is a bipolar fuzzy SBG-ideal.*

PROOF. Let f be a bipolar fuzzy implicative SBG-ideal of an SBG-algebra L. Then, we have $f^-(0) \leqslant f^-(x)$ and $f^+(0) \geqslant f^+(x)$. Additionally, the following holds:

$$\begin{aligned} f^-(x) &\leqslant \max\{f^-(y), f^-(((x|(x^x))(x|(x^x)))^y|((x|(x^x))(x|(x^x)))^y)\} \\ &= \max\{f^-(y), f^-((x^0|x^0)^y|(x^0|x^0)^y)\} \\ &= \max\{f^-(y), f^-(x^y|x^y)\}, \end{aligned}$$

For the positive component:

$$\begin{aligned} f^+(x) &\geqslant \min\{f^+(y), f^+(((x|(x^x))(x|(x^x)))^y|((x|(x^x))(x|(x^x)))^y)\} \\ &= \min\{f^+(y), f^+((x^0|x^0)^y|(x^0|x^0)^y)\} \\ &= \min\{f^+(y), f^+(x^y|x^y)\}. \end{aligned}$$

Thus, f is a bipolar fuzzy SBG-ideal of L. □

THEOREM 3.1. *A bipolar-valued fuzzy set $f = (L; f^-, f^+)$ on L is a bipolar-valued fuzzy implicative SBG-ideal of $\mathcal{L} = (L; |)$ if and only if its negative s-cut and positive t-cut are implicative SBG-ideals of $\mathcal{L}$, provided they are nonempty for all $(s, t) \in [-1, 0] \times [0, 1]$.*

PROOF. Assume that $f = (L; f^-, f^+)$ is a bipolar-valued fuzzy implicative SBG-ideal of $\mathcal{L} = (L; |)$ and $L(f^-, s) \neq \emptyset \neq U(f^+, t)$ for all $(s, t) \in [-1, 0] \times [0, 1]$. Let $x, a \in L$ be such that $(x, a) \in L(f^-, s) \times U(f^+, t)$. Then $f^-(x) \leqslant s$ and $f^+(a) \geqslant t$, and therefore $f^-(0) \leqslant f^-(x) \leqslant s$ and $f^+(0) \geqslant f^+(a) \geqslant t$. Thus, $0 \in L(f^-, s) \times U(f^+, t)$.

Let $x, y, z, a, b, c \in L$ be such that

$$((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z), (((a|b^a)|(a|b^a))^c)|(((a|b^a)|(a|b^a))^c)) \in L(f^-, s) \times U(f^+, t),$$

and $(z, c) \in L(f^-, s) \times U(f^+, t)$. Then, we attain

$$f^-((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)) \leqslant s, \quad f^-(z) \leqslant s,$$

and

$$f^+((((a|b^a)|(a|b^a))^c)|(((a|b^a)|(a|b^a))^c)) \geqslant t, \quad f^+(c) \geqslant t.$$

Thus, we get

$$f^-(x) \leqslant \max\{f^-((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f^-(z)\} \leqslant s$$

and

$$f^+(a) \geqslant \min\{f^+((((a|b^a)|(a|b^a))^c)|(((a|b^a)|(a|b^a))^c)), f^+(c)\} \geqslant t,$$

so $(x, a) \in L(f^-, s) \times U(f^+, t)$. Therefore, $L(f^-, s)$ and $U(f^+, t)$ are implicative SBG-ideals of $\mathcal{L} = (L; |)$.

Conversely, let $f = (L; f^-, f^+)$ be a bipolar-valued fuzzy set on L for which its negative s-cut and positive t-cut are implicative SBG-ideals of $\mathcal{L}$ whenever they are nonempty for all $(s,t) \in [-1,0] \times [0,1]$. Suppose $f^-(0) > f^-(a)$ for some $a \in L$. Then $a \in L(f^-, f^-(b))$ but $0 \notin L(f^-, f^-(a))$, a contradiction. Hence, $f^-(0) \leqslant f^-(x)$ for all $x \in L$. Similarly, suppose $f^+(0) < f^+(x)$ for some $x \in L$. Then $x \in U(f^+, f^+(y))$ but $0 \notin U(f^+, f^+(y))$, a contradiction. Hence, $f^+(0) \geqslant f^+(a)$ for all $a \in L$.

Now, suppose

$$f^-(a) > \max\{f^-((((a|b^a)|(a|b^a))^c)|(((a|b^a)|(a|b^a))^c)), f^-(c)\}$$

or

$$f^+(x) < \min\{f^+((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f^+(z)\}$$

for some $a, b, c, x, y, z \in L$. Then, we have

$$(((a|b^a)|(a|b^a))^c)|(((a|b^a)|(a|b^a))^c), c \in L(f^-, s)$$

or

$$(((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z), z \in U(f^+, t),$$

where

$$s = \max\{f^-((((a|b^a)|(a|b^a))^c)|(((a|b^a)|(a|b^a))^c)), f^-(c)\}$$

and

$$t = \min\{f^+((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f^+(z)\}.$$

But $a \notin L(f^-, s)$ or $x \notin U(f^+, t)$, a contradiction. Therefore, we achieve

$$f^-(x) \leqslant \max\{f^-((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f^-(z)\}$$

and

$$f^+(x) \geqslant \min\{f^+((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f^+(z)\}$$

for all $x, y, z \in L$. Consequently, $f = (L; f^-, f^+)$ is a bipolar-valued fuzzy implicative SBG-ideal of $\mathcal{L} = (L; |)$. □

THEOREM 3.2. *A bipolar-valued fuzzy set $f = (L; f^-, f^+)$ on L is a bipolar-valued fuzzy implicative SBG-ideal of $\mathcal{L} = (L; |)$ if and only if the fuzzy sets f_c^- and f^+ are fuzzy implicative SBG-ideals of $\mathcal{L}$, where $f_c^- : L \to [0,1]$ is defined by $f_c^-(x) = -f^-(x)$ for all $x \in L$.*

PROOF. Assume that $f = (L; f^-, f^+)$ is a bipolar-valued fuzzy implicative SBG-ideal of $\mathcal{L} = (L; |)$. It is clear that f is a fuzzy implicative SBG-ideal of $\mathcal{L}$.

For every $x, y \in L$, we have

$$\begin{aligned} f_c^-(0) &= -f^-(0) \\ &\geqslant -f^-(x) \\ &= f_c^-(x), \end{aligned}$$

and

$$\begin{aligned} f_c^-(x) &= -f^-(x) \\ &\geqslant -\max\{f^-((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f^-(z)\} \\ &= \min\{-f^-((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), -f^-(z)\} \\ &= \min\{f_c^-((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f_c^-(z)\}. \end{aligned}$$

Hence, f_c^- is a fuzzy implicative SBG-ideal of $\mathcal{L} = (L; |)$.

Conversely, let $f = (L; f^-, f^+)$ be a bipolar-valued fuzzy set of $\mathcal{L}$ for which f_c^- and f^+ are fuzzy implicative SBG-ideals of $\mathcal{L}$. Let $x, y, z \in L$. Then, we get

$$\begin{array}{rcl} -f^-(0) & = & f_c^-(0) \\ & \geqslant & f_c^-(x) \\ & = & -f^-(x), \end{array}$$

and

$$f^-(0) \quad \leqslant \quad f^-(x),$$

and also

$$\begin{array}{rcl} -f^-(x) & = & f_c^-(x) \\ & \geqslant & \min\{f_c^-((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f_c^-(z)\} \\ & = & \min\{-f^-((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), -f^-(z)\} \\ & = & -\max\{f^-((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f^-(z)\}. \end{array}$$

Thus, we attain

$$f^-(x) \quad \leqslant \quad \max\{f^-((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f^-(z)\}.$$

Hence, $f = (L; f^-, f^+)$ is a bipolar-valued fuzzy implicative SBG-ideal of $\mathcal{L} = (L; |)$. □

THEOREM 3.3. *Given a nonempty subset F of L, let $f_F = (L; f_F^-, f_F^+)$ be a bipolar-valued fuzzy set on L, defined as follows:*

$$f_F^- : L \to [-1, 0], \quad a \mapsto \begin{cases} s^- & \text{if } a \in F, \\ t^- & \text{otherwise}, \end{cases}$$

and

$$f_F^+ : L \to [0, 1], \quad x \mapsto \begin{cases} s^+ & \text{if } x \in F, \\ t^+ & \text{otherwise}, \end{cases}$$

where $s^- < t^-$ in $[-1, 0]$ and $s^+ > t^+$ in $[0, 1]$. Then $f_F = (L; f_F^-, f_F^+)$ is a bipolar-valued fuzzy implicative SBG-ideal of $\mathcal{L} = (L; |)$ if and only if F is an implicative SBG-ideal of $\mathcal{L} = (L; |)$.

PROOF. Suppose that $f_F = (L; f_F^-, f_F^+)$ is a bipolar-valued fuzzy implicative SBG-ideal in $\mathcal{L} = (L, |)$. Take $x, y, z \in L$ where $x, y \in F$. Then, we have:

$$f_F^-(0) \leqslant f_F^-(x) = s^-, \quad f_F^+(0) \geqslant f_F^+(x) = s^+,$$

which implies $f_F^-(0) = s^-$ and $f_F^+(0) = s^+$. Thus, $0 \in F$. Additionally, since

$$f_F^-(x) \leqslant \max\{f_F^-((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f_F^-(y)\} = s^-,$$

and

$$f_F^+(x) \geqslant \min\{f_F^+((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f_F^+(y)\} = s^+,$$

it follows that $f_F^-(x) = s^-$ and $f_F^+(x) = s^+$. Hence, $x \in F$. This demonstrates that F is an implicative SBG-ideal in $\mathcal{L} = (L; |)$.

Now, assume that F is an implicative SBG-ideal in $\mathcal{L}$. For any $x, y \in L$, if $x \in F$, then $0 \in F$. Consequently:

$$f_F^-(0) = s^-, \quad f_F^-(x) = s^- \quad \text{and} \quad f_F^+(0) = s^+, \quad f_F^+(x) = s^+.$$

If $0 \notin F$, we instead have:

$$f_F^-(0) = t^- > f_F^-(x), \quad f_F^+(0) = t^+ < f_F^+(x).$$

Consider $x, y, z \in L$ where:

$$(((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z) \in F.$$

Since $x \in F$, we get:

$$f_F^-(x) = s^- = \max\{f_F^-((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f_F^-(z)\}$$

and

$$f_F^+(x) = s^+ = \min\{f_F^+((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f_F^+(z)\}.$$

If instead

$$(((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z) \notin F \text{ or } z \notin F,$$

then we obtain

$$f_F^-(x) \leqslant t^- = \max\{f_F^-((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f_F^-(z)\}$$

and

$$f_F^+(x) \geqslant t^+ = \min\{f_F^+((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f_F^+(z)\}.$$

Thus, $f_F = (L; f_F^-, f_F^+)$ is a bipolar-valued fuzzy implicative SBG-ideal of $\mathcal{L} = (L; |)$. □

Proposition 3.2. *If $f_i = \{(f_i^+, f_i^-) : i \in \Delta\}$ is a family of bipolar fuzzy implicative SBG-ideals of an SBG-algebra H, then $\bigwedge_{i\in\Delta} f_i$ is also a bipolar fuzzy implicative SBG-ideal of H.*

Proof. Let $f_i = \{(f_i^+, f_i^-) : i \in \Delta\}$ be a family of bipolar fuzzy implicative SBG-ideals of the SBG-algebra H.

For any $x, y \in H$, we have:

$$(\bigwedge_{i\in\Delta} f_i^+)(0) = \inf_{i\in\Delta}\{f_i^+(0)\} \geqslant \inf_{i\in\Delta}\{f_i^+(x)\} = (\bigwedge_{i\in\Delta} f_i^+)(x),$$

and

$$(\bigwedge_{i\in\Delta} f_i^-)(0) = \sup_{i\in\Delta}\{f_i^-(0)\} \leqslant \sup_{i\in\Delta}\{f_i^-(x)\} = (\bigwedge_{i\in\Delta} f_i^-)(x).$$

Then, for all $x, y \in H$:

$$\begin{aligned}
(\bigwedge_{i\in\Delta} f_i^+)(x) &= \inf_{i\in\Delta}\{f_i^+(x)\} \\
&\geqslant \inf_{i\in\Delta}\left\{\min\left(f_i^+((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f_i^+(z)\right)\right\} \\
&= \min\left\{\inf_{i\in\Delta} f_i^+((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), \inf_{i\in\Delta} f_i^+(z)\right\} \\
&= \min\left\{(\bigwedge_{i\in\Delta} f_i^+)((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), (\bigwedge_{i\in\Delta} f_i^+)(z)\right\}.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}(\bigwedge_{i\in\Delta} f_i^-)(x) &= \sup_{i\in\Delta}\{f_i^-(x)\}\\ &\leqslant \sup_{i\in\Delta}\left\{\max\left(f_i^-((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f_i^-(z)\right)\right\}\\ &= \max\left\{\sup_{i\in\Delta} f_i^-((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), \sup_{i\in\Delta} f_i^-(z)\right\}\\ &= \max\left\{(\bigwedge_{i\in\Delta} f_i^-)((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), (\bigwedge_{i\in\Delta} f_i^-)(z)\right\}.\end{aligned}$$

Thus, $\bigwedge_{i\in\Delta} f_i$ is a bipolar fuzzy implicative SBG-ideal of the SBG-algebra H. □

Definition 3.2. A bipolar fuzzy subset $f = (L; f^-, f^+)$ of an SBG-algebra L is called a bipolar fuzzy sub-implicative SBG-ideal of L if the following conditions hold for all $x, y, z \in L$:
(i) $f^-(0) \leqslant f^-(x)$,
(ii) $f^+(0) \geqslant f^+(x)$,
(iii) $f^-((y|y^x)|(y|y^x)) \leqslant \max\{f^-((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f^-(z)\}$,
(iv) $f^+((y|y^x)|(y|y^x)) \geqslant \min\{f^+((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f^+(z)\}$.

Proposition 3.3. *Let L be an SBG-algebra. Every bipolar fuzzy subimplicative SBG-ideal of L is also a bipolar fuzzy SBG-ideal of L.*

Proof. Let $f = (L; f^-, f^+)$ be a bipolar fuzzy sub-implicative SBG-ideal of L. We know that $f^-(0) \leqslant f^-(x)$ and $f^+(0) \geqslant f^+(x)$. Now, for any $x, z \in L$, we have

$$f^-(x) = f^-(x^0|x^0) = f^-((x|x^x)|(x|(x^x)).$$

Thus, we get

$$\begin{aligned} f^-(x) &\leqslant \max\left\{f^-(((x|x^x)|(x|(x^x)))^z|((x|x^x)|(x|(x^x)))^z), f^-(z)\right\}.\\ &= \max\left\{f^-((x^0|x^0)^z|(x^0|x^0)^z), f^-(z)\right\}\end{aligned}$$

This simplifies to:

$$f^-(x) \leqslant \max\left\{f^-(x^z|x^z), f^-(z)\right\}.$$

Similarly, for the positive part, we obtain the follows:

$$f^+(x) = f^+(x^0|x^0) = f^+((x|x^x)|(x|x^x)).$$

Then, we have:

$$\begin{aligned} f^+(x) &\geqslant \min\left\{f^+(((x|x^x)|(x|(x^x)))^z|((x|x^x)|(x|(x^x)))^z), f^+(z)\right\},\\ &= \min\left\{f^+((x^0|x^0)^z|(x^0|x^0)^z), f^+(z)\right\}\end{aligned}$$

which reduces to

$$f^+(x) \geqslant \min\left\{f^+(x^z|x^z), f^+(z)\right\}.$$

Thus, $f = (L; f^-, f^+)$ satisfies the conditions of a bipolar fuzzy SBG-ideal of L. □

Theorem 3.4. *Let $f = (L; f^-, f^+)$ be a bipolar fuzzy SBG-ideal of an SBG-algebra L. Then $f = (L; f^-, f^+)$ is a bipolar fuzzy sub-implicative SBG-ideal of L if and only if the following conditions hold:*

(*i*) $f^{-}((y|y^{x})|(y|y^{x})) \leqslant f^{-}((x|x^{y})|(x|x^{y}))$,

(*ii*) $f^{+}((y|y^{x})|(y|y^{x})) \geqslant f^{+}((x|x^{y})|(x|x^{y}))$.

PROOF. Let $f = (L; f^{-}, f^{+})$ be a bipolar fuzzy sub-implicative SBG-ideal of L. First, we have:

$$\begin{aligned} f^{-}((y|y^{x})|(y|y^{x})) &\leqslant \max\left\{f^{-}(((x|x^{y})|(x|x^{y}))^{0}|((x|x^{y})|(x|x^{y}))^{0}), f^{-}(0)\right\} \\ &= \max\left\{f^{-}(((x|x^{y})|(x|x^{y}))|((x|x^{y})|(x|x^{y}))), f^{-}(0)\right\} \\ &= f^{-}\left((x|x^{y})|(x|x^{y})\right). \end{aligned}$$

Similarly, for the positive component, we attain:

$$\begin{aligned} f^{+}((y|y^{x})|(y|y^{x})) &\leqslant \min\left\{f^{+}(((x|x^{y})|(x|x^{y}))^{0}|((x|x^{y})|(x|x^{y}))^{0}), f^{+}(0)\right\} \\ &= \min\left\{f^{+}(((x|x^{y})|(x|x^{y}))|((x|x^{y})|(x|x^{y}))), f^{+}(0)\right\} \\ &= f^{+}\left((x|x^{y})|(x|x^{y})\right). \end{aligned}$$

Now, conversely, since f is a bipolar fuzzy SBG-ideal, we have $f^{-}(0) \leqslant f^{-}(x)$ and $f^{+}(0) \geqslant f^{+}(x)$. Therefore, we get

$$\begin{aligned} f^{-}((y|y^{x})|(y|y^{x})) &\leqslant f^{-}((x|x^{y})|(x|x^{y})) \\ &\leqslant \max\left\{f^{-}((((x|y^{x})|(x|y^{x}))^{z})|(((x|y^{x})|(x|y^{x}))^{z})), f^{-}(z)\right\} \end{aligned}$$

and also

$$\begin{aligned} f^{+}((y|y^{x})|(y|y^{x})) &\geqslant f^{+}((x|x^{y})|(x|x^{y})) \\ &\geqslant \min\left\{f^{+}((((x|y^{x})|(x|y^{x}))^{z})|(((x|y^{x})|(x|y^{x}))^{z})), f^{+}(z)\right\} \end{aligned}$$

Thus, f satisfies the conditions of a bipolar fuzzy sub-implicative SBG-ideal of L. □

THEOREM 3.5. *Let L be an implicative SBG-algebra. Then every fuzzy SBG-ideal of L is also a bipolar fuzzy sub-implicative SBG-ideal of L.*

PROOF. Let f be a bipolar fuzzy SBG-ideal of L. We know that $f^{-}(0) \leqslant f^{-}(x)$ and $f^{+}(0) \geqslant f^{+}(x)$. Now consider:

$$\begin{aligned} f^{-}((y|(y|(x|x)))|(y|(y|(x|x)))) &\leqslant \max\left\{f^{-}(((y|y^{x})|(y|y^{x}))^{z}|(((y|y^{x})|(y|y^{x}))^{z})), f^{-}(z)\right\} \\ &= \max\left\{f^{-}((((x|y^{x})|(x|y^{x}))^{z})|(((x|y^{x})|(x|y^{x}))^{z})), f^{-}(z)\right\}. \end{aligned}$$

Similarly, for the positive part:

$$\begin{aligned} f^{+}((y|(y|(x|x)))|(y|(y|(x|x)))) &\geqslant \min\left\{f^{+}(((y|y^{x})|(y|y^{x}))^{z}|(((y|y^{x})|(y|y^{x}))^{z})), f^{+}(z)\right\} \\ &= \min\left\{f^{+}((((x|y^{x})|(x|y^{x}))^{z})|(((x|y^{x})|(x|y^{x}))^{z})), f^{+}(z)\right\}. \end{aligned}$$

Thus, f satisfies the conditions for being a bipolar fuzzy sub-implicative SBG-ideal of L. □

LEMMA 3.1. *In an SBG-algebra L, the following property holds for all $x, y \in L$:*

$$((x|x^{y})|(x|x^{y}))|y^{x} = x|x^{y}.$$

THEOREM 3.6. *In a medial SBG-algebra L, every bipolar fuzzy SBG-ideal is also a bipolar fuzzy sub-implicative SBG-ideal.*

PROOF. Let $f = (L; f^-, f^+)$ be a bipolar fuzzy SBG-ideal in a medial SBG-algebra L. Since f is a bipolar fuzzy SBG-ideal, we have:

$$f^-(0) \leqslant f^-(x), \quad f^+(0) \geqslant f^+(x).$$

Then, we obtain the following inequalities:

$$\begin{aligned}
f^-\left((y|y^x)|(y|y^x)\right) &= f^-(x) \\
&\leqslant \max\left\{f^-\left(x^z|x^z\right), f^-(z)\right\} \\
&= \max\left\{f^-\left(((y|y^x)|(y|y^x))^z|(((y|y^x)|(y|y^x))^z)\right), f^-(z)\right\} \\
&= \max\{f^-(((((x|x^y)|(x|x^y))|y^x)|(((x|x^y)|(x|x^y))|y^x))^z \\
&\quad |((((x|x^y)|(x|x^y))|y^x)|(((x|x^y)|(x|x^y))|y^x))^z), f^-(z)\} \\
&= \max\left\{f^+((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f^+(z)\right\},
\end{aligned}$$

and

$$\begin{aligned}
f^+\left((y|y^x)|(y|y^x)\right) &= f^+(x) \\
&\geqslant \min\left\{f^+\left(x^z|x^z\right), f^+(z)\right\} \\
&= \min\left\{f^+\left(((y|y^x)|(y|y^x))^z|(((y|y^x)|(y|y^x))^z)\right), f^+(z)\right\} \\
&= \min\{f^+(((((x|x^y)|(x|x^y))|y^x)|(((x|x^y)|(x|x^y))|y^x))^z \\
&\quad |((((x|x^y)|(x|x^y))|y^x)|(((x|x^y)|(x|x^y))|y^x))^z), f^+(z)\} \\
&= \min\left\{f^+((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f^+(z)\right\}.
\end{aligned}$$

Hence, f satisfies the conditions of a bipolar fuzzy sub-implicative SBG-ideal of L. $\square$

THEOREM 3.7. *Let L be an SBG-algebra that satisfies the following condition for all $x, y, z \in L$:*

$$f^-\left(y^z|y^z\right) \leqslant f^-\left(((x|x^y)|(x|(x^y)))^z|((x|x^y)|(x|(x^y)))^z\right),$$
$$f^+\left(y^z|y^z\right) \geqslant f^+\left(((x|x^y)|(x|(x^y)))^z|((x|x^y)|(x|(x^y)))^z\right).$$

Then every bipolar fuzzy SBG-ideal of L is also a bipolar fuzzy sub-implicative SBG-ideal of L.

PROOF. We obtain the following inequalities:

$$\begin{aligned}
f^-\left((y|y^x)|(y|y^x)\right) &\leqslant f^-\left((((x|x^y)|(x|x^y))|y^x)|(((x|x^y)|(x|x^y))|y^x)\right) \\
&= f^-\left((x|x^y)|(x|x^y)\right).
\end{aligned}$$

and

$$\begin{aligned}
f^+\left((y|y^x)|(y|y^x)\right) &\geqslant f^+\left((((x|x^y)|(x|x^y))|y^x)|(((x|x^y)|(x|x^y))|y^x)\right) \\
&= f^-\left((x|x^y)|(x|x^y)\right).
\end{aligned}$$

Therefore, f satisfies the conditions to be a bipolar fuzzy sub-implicative SBG-ideal of L. $\square$

Theorem 3.8. *Every medial SBG-algebra is an implicative SBG-algebra.*

Theorem 3.9. *Let f be a bipolar fuzzy SBG-ideal of an SBG-algebra L. Then f is a bipolar fuzzy implicative SBG-ideal of L if and only if it satisfies the following condition for all $x, y \in L$:*

$$\begin{cases} f^-(x) & \leqslant f^-((x|y^x)|(x|y^x)), \\ f^+(x) & \geqslant f^+((x|y^x)|(x|y^x)). \end{cases} \tag{3.1}$$

Proof. Let f be a bipolar fuzzy implicative SBG-ideal of L. Then, we get the following inequalities:

$$\begin{aligned} f^-(x) &\leqslant \max\{f^-(0), f^-(((x|x^y)|(x|x^y))^0|((x|x^y)|(x|x^y))^0)\} \\ &= \max\{f^-(0), f^-((x|y^x)|(x|y^x))\} \\ &= f^-((x|y^x)|(x|y^x)), \end{aligned}$$

and

$$\begin{aligned} f^+(x) &\geqslant \min\{f^+(0), f^+(((x|x^y)|(x|x^y))^0|((x|x^y)|(x|x^y))^0)\} \\ &= \min\{f^+(0), f^+((x|y^x)|(x|y^x))\} \\ &= f^+((x|y^x)|(x|y^x)), \end{aligned}$$

for all $x, y \in L$.

Conversely, let f be a bipolar fuzzy SBG-ideal of L satisfying the inequalities in (3.1). Then it is clear that $f^-(0) \leqslant f^-(x)$ and $f^+(0) \geqslant f^+(x)$ for all $x \in L$. Since, the following inequalities

$$\begin{aligned} f^-(x) &\leqslant f^-((x|y^x)|(x|y^x)) \\ &\leqslant \max\{f^-(((x|y^x)|(x|y^x))^z|((x|y^x)|(x|y^x))^z), f^-(z)\}, \end{aligned}$$

and

$$\begin{aligned} f^+(x) &\geqslant f^+((x|y^x)|(x|y^x)) \\ &\geqslant \min\{f^+(((x|y^x)|(x|y^x))^z|((x|y^x)|(x|y^x))^z), f^+(z)\}, \end{aligned}$$

are verified for all $x, y, z \in L$, we have f is a bipolar fuzzy implicative SBG-ideal of L. □

Theorem 3.10. *Let L be a medial SBG-algebra. If the following inequalities*

$$\begin{cases} f^-((x|x^y)|(x|x^y)) & \leqslant f^-((x|y^x)|(x|y^x)) \\ f^+((x|x^y)|(x|x^y)) & \geqslant f^+((x|y^x)|(x|y^x)) \end{cases} \tag{3.2}$$

are satisfying for each $x, y \in L$, then every bipolar fuzzy sub-implicative SBG-ideal of L is a bipolar fuzzy implicative ideal of L.

Proof. Let f be a bipolar fuzzy sub-implicative SBG-ideal of a medial SBG-algebra L satisfying the inequalities in (3.2). Then we obtain the following cases:

$$\begin{aligned} f^-(x) &= f^-((y|y^x)|(y|y^x)) \\ &\leqslant \max\{f^-(((x|x^y)|(x|x^y))^0|((x|x^y)|(x|x^y))^0), f^-(0)\} \\ &= \max\{f^-((x|x^y)|(x|x^y)), f^-(0)\} \\ &= f^-((x|x^y)|(x|x^y)) \\ &\leqslant f^-((x|y^x)|(x|y^x)), \end{aligned}$$

and

$$\begin{array}{rcl} f^+(x) & = & f^+((y|y^x)|(y|y^x)) \\ & \geqslant & \min\{f^+(((x|x^y)|(x|x^y))^0|((x|x^y)|(x|x^y))^0), f^+(0)\} \\ & = & \min\{f^+((x|x^y)|(x|x^y)), f^+(0)\} \\ & = & f^+((x|x^y)|(x|x^y)) \\ & \geqslant & f^+((x|y^x)|(x|y^x)). \end{array}$$

As a result, we conclude that f is a bipolar fuzzy implicative SBG-ideal of L. □

THEOREM 3.11. *In an implicative SBG-algebra L, every bipolar fuzzy implicative SBG-ideal is also a bipolar fuzzy sub-implicative SBG-ideal of L.*

PROOF. Let f be a bipolar fuzzy implicative SBG-ideal in an implicative SBG-algebra L. Since f is also a bipolar fuzzy SBG-ideal, it follows that $f^-(0) \leqslant f^-(x)$ and $f^+(0) \geqslant f^+(x)$ for all $x \in L$. Now, consider:

$$\begin{array}{rcl} f^-((y|y^x)|(y|y^x)) & = & f^-((x|x^y)|(x|x^y)) \\ & \leqslant & \max\{f^-(((x|x^y)|(x|x^y))^z|((x|x^y)|(x|x^y))^z), f^-(0)\} \end{array}$$

Similarly, for the positive component:

$$\begin{array}{rcl} f^+((y|y^x)|(y|y^x)) & = & f^+((x|x^y)|(x|x^y)) \\ & \geqslant & \min\{f^-(((x|x^y)|(x|x^y))^z|((x|x^y)|(x|x^y))^z), f^-(0)\} \end{array}$$

Thus, f satisfies the conditions for being a bipolar fuzzy sub-implicative SBG-ideal of L. □

COROLLARY 3.1. *In a medial SBG-algebra L, every bipolar fuzzy implicative SBG-ideal is also a bipolar fuzzy sub-implicative SBG-ideal.*

DEFINITION 3.3. A bipolar fuzzy SBG-ideal $f = (L; f^+, f^-)$ of an SBG-algebra L is called bipolar fuzzy closed if, for every $x \in L$, the following conditions hold:

$$f^-(0^x|0^x) \leqslant f^-(x), \quad f^+(0^x|0^x) \geqslant f^+(x).$$

DEFINITION 3.4. Let $f = (L; f^+, f^-)$ be a bipolar fuzzy SBG-ideal of an SBG-algebra L. The ideal f is called a bipolar fuzzy completely closed SBG-ideal of L if, for all $x, y \in L$, the following conditions hold:

$$f^-(x^y|x^y) \leqslant \max\{f^-(x), f^-(y)\}$$

and

$$f^+(x^y|x^y) \geqslant \min\{f^+(x), f^+(y)\}.$$

THEOREM 3.12. *Let L be a SBG-algebra. If the following equality*

$$(((x^y|x^y)|x^z)|((x^y|x^y)|x^z))|z^y = 0|0 \tag{3.3}$$

is satisfied for each $x, y, z \in L$, then L is implicative if and only if every bipolar fuzzy closed SBG-ideal of L is a bipolar fuzzy implicative SBG-ideal of L.

PROOF. Let L be an SBG-algebra that satisfies the equality in (3.3). Assume L is implicative and f is a bipolar fuzzy closed SBG-ideal of L. Since f is a bipolar fuzzy SBG-ideal of L, it follows that $f^-(0) \leqslant f^-(x)$ and $f^+(0) \geqslant f^+(x)$ for all $x \in L$. Additionally, we have:

$$\begin{aligned} f^-(x) &\leqslant \max\{f^-(z), f^-(x^z|x^z)\} \\ &= \max\{f^-(z), f^-\left((((x|x)^x|(x|x)|y))^z\right)\} \\ &= \max\{f^-(z), f^-\left(((x|y^x)|(x|y^x))^z\right)\}. \end{aligned}$$

Similarly, for the positive component:

$$\begin{aligned} f^+(x) &\geqslant \min\{f^+(z), f^+(x^z|x^z)\} \\ &= \min\{f^+(z), f^+\left((((x|x)^x|(x|x)|y))^z\right)\} \\ &= \min\{f^+(z), f^+\left(((x|y^x)|(x|y^x))^z\right)\}. \end{aligned}$$

This shows that f is a bipolar fuzzy implicative SBG-ideal of L.

Conversely, assume that every bipolar fuzzy closed SBG-ideal of L is also a bipolar fuzzy implicative SBG-ideal. From the equality in (3.3), we have $z^y = x^z|(x^y|x^y)$. Since $z^y = x^z|(x^y|x^y) = ((x|x^z)|(x|x^z))^y$ from conditions Definition 2.1 $(S1)$ and $(S3)$, we use Definition 2.1 $(S2)$ and Lemma 2.1 (3) to obtain $z = (x|x^z)|(x|x^z)$. Hence, from Definition 2.1 $(S1)-(S3)$ and Lemma 2.1 (8), we derive:

$$\begin{aligned} x|x^y &= ((y|y^x)|(y|y^x))|((y|y^x)|(y|y^x))^y \\ &= ((y|y^x)|(y|y^x))|y^y \\ &= ((y^y|(y|y)^y)|(y^y|(y|y)^y))|y^x \\ &= y|y^x, \end{aligned}$$

for all $x, y \in L$. This confirms that L is implicative. $\square$

PROPOSITION 3.4. *Let L be an implicative SBG-algebra that satisfies the equality in (3.3). Then every bipolar fuzzy completely closed SBG-ideal of L is also a bipolar fuzzy implicative SBG-ideal of L.*

PROOF. Let f be a bipolar fuzzy completely closed SBG-ideal of an implicative SBG-algebra L. Since f is a bipolar fuzzy SBG-ideal of L, we have the following:

$$f^-(0^y|0^y) \leqslant \max\{f^-(0), f^-(y)\} = f^-(y),$$

and

$$f^+(0^y|0^y) \geqslant \min\{f^+(0), f^+(y)\} = f^+(y).$$

This shows that f satisfies the conditions for being a bipolar fuzzy closed SBG-ideal of L. Therefore, by definition, f is also a bipolar fuzzy implicative SBG-ideal of L. $\square$

COROLLARY 3.2. *Let L be a medial SBG-algebra that satisfies the equality in (3.3). Then every bipolar fuzzy completely closed SBG-ideal of L is also a bipolar fuzzy implicative SBG-ideal.*

DEFINITION 3.5. A bipolar fuzzy set f in an SBG-algebra L is called a bipolar fuzzy p-ideal of L if it satisfies the following conditions for all $x, y, z \in L$:

(*i*) $f^-(0) \leqslant f^-(x)$,
(*ii*) $f^+(0) \geqslant f^+(x)$,
(*iii*) $f^-(x) \leqslant \max\{f^-\left(((x^z|x^z)|y^z)|((x^z|x^z)|y^z)\right), f^-(y)\}$,
(*iv*) $f^+(x) \geqslant \min\{f^+\left(((x^z|x^z)|y^z)|((x^z|x^z)|y^z)\right), f^+(y)\}$.

DEFINITION 3.6. Let L be a SBG-algebra. Then the set

$$A^+ = \{x \in A : 0^x|0^x = 0\}$$

is called the BCA-part of L.

THEOREM 3.13. *Let $A = A^+$ be an SBG-algebra. Then every bipolar fuzzy p-ideal of L is also a bipolar fuzzy implicative SBG-ideal of L.*

PROOF. Let f be a bipolar fuzzy p-ideal of L. We can derive the following for the negative component:

$$\begin{aligned} f^-(x) &\leqslant \max\{f^-(((((x|y^x)|(x|y^x))|(0|y^x))|((x|y^x)|(x|y^x))|(0|y^x))), f^-(0)\} \\ &= \max\{f^-((((x|y^x)|(x|y^x))^0)|(((x|y^x)|(x|y^x))^0)), f^-(0)\} \\ &= \max\{f^-((x|y^x)|(x|y^x)), f^-(0)\} \\ &= f^-((x|y^x)|(x|y^x)), \end{aligned}$$

Similarly, for the positive component:

$$\begin{aligned} f^+(x) &\geqslant \max\{f^+(((((x|y^x)|(x|y^x))|(0|y^x))|((x|y^x)|(x|y^x))|(0|y^x))), f^+(0)\} \\ &= \min\{f^+((((x|y^x)|(x|y^x))^0)|(((x|y^x)|(x|y^x))^0)), f^+(0)\} \\ &= \min\{f^+((x|y^x)|(x|y^x)), f^+(0)\} \\ &= f^+((x|y^x)|(x|y^x)), \end{aligned}$$

Thus, f satisfies the conditions to be a bipolar fuzzy implicative SBG-ideal of L. $\square$

THEOREM 3.14. *Let $f = (A; f^-, f^+)$ be a bipolar fuzzy implicative SBG-ideal in A. Then, for all $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$, the bipolar fuzzy (α, β)-translation $f^{T_2}_{(\alpha,\beta)} = (A; f^-_{(\alpha,T_2)}, f^+_{(\beta,T_2)})$ of f is also a bipolar fuzzy implicative SBG-ideal of A.*

PROOF. Assume that $f = (A; f^-, f^+)$ is a bipolar fuzzy implicative SBG-ideal of A. For any $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$ and for all $x \in A$, we have:

$$f^-(0) \leqslant f^-(x), \quad f^+(0) \geqslant f^+(x).$$

Therefore, we can deduce:

$$f^-_{(\alpha,T_2)}(0) = f^-(0) - \alpha \leqslant f^-(x) - \alpha = f^-_{(\alpha,T_2)}(x),$$

and similarly for the positive component:

$$f^+_{(\beta,T_2)}(0) = f^+(0) - \beta \geqslant f^+(x) - \beta = f^+_{(\beta,T_2)}(x).$$

Moreover, let $x, y, z \in A$. Since f is a bipolar fuzzy implicative SBG-ideal, we have:

$$f^-(x) \leqslant \max\left\{f^-\left(((x|y^x)|(x|y^x))^z\right), f^-(z)\right\},$$

and:

$$f^+(x) \geqslant \min\left\{f^+\left(((x|y^x)|(x|y^x))^z\right), f^+(z)\right\}.$$

Thus, for the translated fuzzy set, we get:

$$\begin{array}{rcl} f^-_{(\alpha,T_2)}(x) & = & f^-(x)-\alpha \\ & \leqslant & \max\{f^-((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f^-(z)\}-\alpha \\ & = & \max\{f^-((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z))-\alpha, f^-(z)-\alpha\} \\ & = & \max\{f^-_{(\alpha,T_2)}((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f^-_{(\alpha,T_2)}(z)\}, \end{array}$$

For the positive component:

$$\begin{array}{rcl} f^+_{(\beta,T_2)}(x) & = & f^+(x)-\beta \\ & \geqslant & \min\{f^+((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f^+(z)\}-\beta \\ & = & \min\{f^+((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z))-\beta, f^+(z)-\beta\} \\ & = & \min\{f^+_{(\beta,T_2)}((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f^+_{(\beta,T_2)}(z)\}. \end{array}$$

Hence, $f^{T_2}_{(\alpha,\beta)} = (A; f^-_{(\alpha,T_2)}, f^+_{(\beta,T_2)})$ is a bipolar fuzzy implicative SBG-ideal of A. □

THEOREM 3.15. *If there exist $(\alpha,\beta) \in [\pm,0]\times[0,\mp]$ such that the bipolar fuzzy (α,β)-translation $f^{T_2}_{(\alpha,\beta)} = (A; f^-_{(\alpha,T_2)}, f^+_{(\beta,T_2)})$ of $f = (A; f^-, f^+)$ is a bipolar fuzzy implicative SBG-ideal of A, then $f = (A; f^-, f^+)$ itself is a bipolar fuzzy implicative SBG-ideal of A.*

PROOF. Assume that $f^{T_2}_{(\alpha,\beta)} = (A; f^-_{(\alpha,T_2)}, f^+_{(\beta,T_2)})$ is a bipolar fuzzy implicative SBG-ideal of A for some $(\alpha,\beta) \in [\pm,0]\times[0,\mp]$. For all $x \in A$, we know that:

$$f^-(0) \leqslant f^-(x), \quad f^+(0) \geqslant f^+(x).$$

Thus, we have:

$$f^-(0)-\alpha = f^-_{(\alpha,T_2)}(0) \leqslant f^-_{(\alpha,T_2)}(x) = f^-(x)-\alpha,$$

and:

$$f^+(0)-\beta = f^+_{(\beta,T_2)}(0) \geqslant f^+_{(\beta,T_2)}(x) = f^+(x)-\beta.$$

Now, let $x,y,z \in A$. Since $f^{T_2}_{(\alpha,\beta)}$ is a bipolar fuzzy implicative SBG-ideal, we have:

$$\begin{array}{rcl} f^-(x)-\alpha & = & f^-_{(\alpha,T_2)}(x) \\ & \leqslant & \max\{f^-_{(\alpha,T_2)}((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f^-_{(\alpha,T_2)}(y)\} \\ & = & \max\{f^-((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z))-\alpha, f^-(y)-\alpha\} \\ & = & \max\{f^-((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f^-(y)\}-\alpha, \end{array}$$

For the positive component, we get:

$$\begin{array}{rcl} f^+(x)-\beta & = & f^+_{(\beta,T_2)}(x) \\ & \geqslant & \min\{f^+_{(\beta,T_2)}((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f^+_{(\beta,T_2)}(z)\} \\ & = & \min\{f^+((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z))-\beta, f^+(z)-\beta\} \\ & = & \min\{f^+((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f^+(z)\}-\beta, \end{array}$$

Therefore, we conclude:

$$f^-(x) \leqslant \max\left\{f^-\left(((x|y^x)|(x|y^x))^z\right), f^-(z)\right\},$$

and

$$f^+(x) \geqslant \min\left\{f^+\left(((x|y^x)|(x|y^x))^z\right), f^+(z)\right\}.$$

Thus, $f = (A; f^-, f^+)$ is a bipolar fuzzy implicative SBG-ideal of A. □

THEOREM 3.16. *Let $\overline{f} = (A; \overline{f^-}, \overline{f^+})$ be a bipolar fuzzy set in A. Then $f = (A; f^-, f^+)$ is a bipolar fuzzy implicative SBG-ideal of A if and only if, for all $(t^-, t^+) \in [-1, 0] \times [0, 1]$, the sets $N_U(f, t^-)$ and $P_L(f, t^+)$ are implicative SBG-ideals of A, provided that $N_U(f, t^-)$ and $P_L(f, t^+)$ are nonempty.*

PROOF. Assume that $\overline{f} = (A; \overline{f^-}, \overline{f^+})$ is a bipolar fuzzy implicative SBG-ideal of A. Let $(t^-, t+) \in [-1, 0] \times [0, 1]$ be such that $N_U(f, t^-)$ and $P_L(f, t^+)$ are nonempty. Let $x \in A$ be such that $x \in N_U(f, t^-)$. Then $f^-(y) \geqslant t^-$. Since $f = (A; f^-, f^+)$ is a bipolar fuzzy implicative SBG-ideal of A, we have

$$\begin{aligned} \overline{f^-}(0) &\leqslant \overline{f^-}(x) \\ &\Rightarrow 1 - f^+(0) \leqslant 1 - f^+(x) \\ &\Rightarrow f^+(0) \geqslant f^+(x) \geqslant t^-. \end{aligned}$$

Hence, we get $0 \in N_U(f, t^-)$. Let $x, y, z \in A$ be such that

$$(((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z), z \in N_U(f, t^-).$$

Then we attain $\overline{f^-((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z))} \geqslant t^-$ and $f^-(z) \geqslant t^-$. Since $\overline{f} = (A, \overline{f^-}, \overline{f^+})$ is a bipolar fuzzy implicative SBG-ideal of A, we have

$$\begin{aligned} &\overline{f^-}(x) \leqslant \max\{\overline{f^-}((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), \overline{f^-}(z)\}, \\ \Rightarrow\ & 1 - f^-(x) \leqslant \max\{1 - f^-((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), 1 - f^-(z)\}, \\ \Rightarrow\ & 1 - f^-(x) \leqslant 1 - \min\{f^-((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f^-(z)\}, \\ \Rightarrow\ & f^-(x) \geqslant \min\{f^-((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f^-(z)\} \geqslant t^-. \end{aligned}$$

Hence, $x \in N_U(f, t^-)$. Therefore, $N_U(f, t^+)$ is an implicative SBG-ideal of A.

Let $x \in A$ be such that $x \in P_L(f, t^+)$. Then $f^+(x) \geqslant t^+$. Since $f = (A; f^-, f^+)$ is a bipolar fuzzy implicative SBG-ideal of A, we have

$$\begin{aligned} \overline{f^+}(0) \geqslant \overline{f^+}(x) &\Rightarrow 1 - f^+(0) \geqslant 1 - f^+(x) \\ &\Rightarrow f^+(0) \leqslant f^+(x) \leqslant t^-. \end{aligned}$$

Hence, we conclude that $0 \in P_L(f, t^-)$. Moreover, let $x, y, z \in A$ be such that

$$(((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z), z \in P_L(f, t^+).$$

Then, we have $\overline{f^+((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z))} \geqslant t^+$ and $f^+(z) \geqslant t^+$. Since $\overline{f} = (A; \overline{f^-}, \overline{f^+})$ is a bipolar fuzzy implicative SBG-ideal of A, we have

$$\begin{aligned} &\overline{f^+}(x) \leqslant \max\{\overline{f^+}((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), \overline{f^+}(z)\} \\ \Rightarrow\ & 1 - f^+(x) \leqslant \max\{1 - f^+((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), 1 - f^+(z)\} \\ \Rightarrow\ & 1 - f^+(x) \leqslant 1 - \min\{f^+((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f^+(z)\} \\ \Rightarrow\ & f^+(x) \geqslant \min\{f^+((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f^+(z)\} \geqslant t^+. \end{aligned}$$

Hence, $x \in P_L(f, t^+)$. Therefore, $P_L(f, t^+)$ is an implicative SBG-ideal of A.

Conversely, assume that for all $(t^-, t^+) \in [-1, 0] \times [0, 1]$, $N_U(f, t^-)$ and $P_L(f, t^+)$ are implicative SBG-ideals of A if $N_U(f, t^-)$ and $P_L(f, t^+)$ are nonempty. Let $x \in A$. Then $f^-(x) \in [-1, 0]$. Choose $t^- = f^-(x)$. Thus $f^-(0) \geqslant f^-(x) = t^-$, so

$0 \in N_U(f,t^-) \neq \emptyset$. By assumption, we have $N_U(f,t^-)$ is an implicative SBG-ideal of A and so $0 \in N_U(f,t^-)$. Thus $f^-(0) \geqslant t^- = f^-(x)$ and so $\overline{f^-}(0) = -1-f^-(0) \leqslant -1-f^-(x) = \overline{f^-}(x)$. Let $x,y,z \in A$. Then, we have

$$f^-((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f^-(z) \in [-1,0].$$

Choose $t^- = \min\{f^-((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f^-(z)\}$. Thus, we get $f^-((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)) \geqslant t^-$ and $f^-(z) \geqslant t^-$, so, we attain

$$(((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z), z \in N_U(f,t^-) \neq \emptyset.$$

By assumption, we have $N_U(f,t^-)$ is an implicative SBG-ideal of A and so $x \in N_U(f,t^-)$. Thus, we get

$$f^-(x) \geqslant t^- = \min\{f^-((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f^-(z)\}.$$

By Lemma 2.2 1, we have

$$\begin{aligned}
\overline{f^-}(x) &= -1-f^-(x)\\
&\leqslant -1-\min\{f^-((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f^-(z)\}\\
&= \max\{-1-f^-((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), -1-f^-(z)\}\\
&= \max\{\overline{f^-}((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), \overline{f^-}(z)\}.
\end{aligned}$$

Let $x,y \in A$. Then $f^+(x) \in [0,1]$. Choose $t^+ = f^+(x)$. Thus $f^+(0) \leqslant f^+(x) = t^+$, so $x \in P_L(f,t^+) \neq \emptyset$. By assumption, we have $P_L(f,t^+)$ is an implicative SBG-ideal of A and so $0 \in P_L(f,t^+)$. Thus $f^+(0) \geqslant t^+ = f^+(x)$ and so $\overline{f^+}(0) = 1-f^+(0) \geqslant 1-f^+(x) = \overline{f^+}(x)$. Let $x,y,z \in A$. Then $f^+((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z))$, $f^+(z) \in [0,1]$. Choose $t^+ = \max\{f^+((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f^+(z)\}$. Thus $f^+((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)) \leqslant t^+$ and $f^+(z) \leqslant t^+$, so we get

$$(((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z), z \in P_L(f,t^+) \neq \emptyset.$$

By assumption, $P_L(f,t^+)$ is an implicative SBG-ideal of A and so $x \in P_L(f,t^+)$. Thus, we attain

$$f^+(x) \geqslant t^+ = \max\{f^+((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f^+(z)\}.$$

By Lemma 2.2 1, we have

$$\begin{aligned}
\overline{f^+}(x) &= 1-f^+(x)\\
&\leqslant 1-\max\{f^+((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), f^+(z)\}\\
&= \min\{1-f^+((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), 1-f^+(z)\}\\
&= \min\{\overline{f^+}((((x|y^x)|(x|y^x))^z)|(((x|y^x)|(x|y^x))^z)), \overline{f^+}(z)\}.
\end{aligned}$$

Hence $\overline{f} = (A, \overline{f^-}, \overline{f^+})$ is a bipolar fuzzy implicative SBG-ideal of A. □

THEOREM 3.17. *Let $\langle A; |_A, 0_A\rangle$ and $\langle B; |_B, 0_B\rangle$ be SBG-algebras, and let $\alpha : A \to B$ be a surjective homomorphism. Suppose f is a bipolar fuzzy set on B. Then f is a bipolar fuzzy implicative SBG-ideal of B if and only if its preimage f^α, defined as $f^\alpha(x^y) = f(\alpha(x)^{\alpha(y)})$ for all $x,y \in A$, is a bipolar fuzzy implicative SBG-ideal of A.*

Proof. Let $\langle A, |_A, 0_A\rangle$ and $\langle B, |_B, 0_B\rangle$ be SBG-algebras, $f : A \to B$ be a surjective homomorphism and f be a bipolar fuzzy implicative SBG-ideal of B. Let $x_1, x_2, x_3 \in A$. Then, we have

$$f^{-\alpha}(0_A) = f^-(\alpha(0_A)) \leqslant f^-(\alpha(x_1)) = f^{-\alpha}(x_1),$$

and also

$$\begin{array}{rl} & f^{-\alpha}(x_1) = f^-(\alpha(x_1)) \\ \leqslant & \max\{f^-(\alpha(x_3)), f^-(((\alpha(x_1)|_B\alpha(x_2)^{\alpha(x_1)})|_B(\alpha(x_1)|_B\alpha(x_2)^{\alpha(x_1)}))^{\alpha(x_3)} \\ & |_B((\alpha(x_1)|_B\alpha(x_2)^{\alpha(x_1)})|_B(\alpha(x_1)|_B\alpha(x_2)^{\alpha(x_1)}))^{\alpha(x_3)})\} \\ = & \max\{f^-(\alpha(x_3)), f^-(\alpha(((x_1|_Ax_2^{x_1})|_A(x_1|_Ax_2^{x_1}))^{x_3}|_A((x_1|_Ax_2^{x_1})|_A(x_1|_Ax_2^{x_1}))^{x_3}))\} \\ = & \max\{f^{-\alpha}(x_3), f^{-\alpha}(((x_1|_Ax_2^{x_1})|_A(x_1|_Ax_2^{x_1}))^{x_3}|_A((x_1|_Ax_2^{x_1})|_A(x_1|_Ax_2^{x_1}))^{x_3})\} \end{array}$$

and

$$f^{+\alpha}(0_A) = f^+(\alpha(0_A)) \geqslant f^+(\alpha(x_1)) = f^{+\alpha}(x_1),$$

also

$$\begin{array}{rl} & f^{+\alpha}(x_1) = f^+(\alpha(x_1)) \\ \geqslant & \min\{f^+(\alpha(x_3)), f^+(((\alpha(x_1)|_B\alpha(x_2)^{\alpha(x_1)})|_B(\alpha(x_1)|_B\alpha(x_2)^{\alpha(x_1)}))^{\alpha(x_3)} \\ & |_B((\alpha(x_1)|_B\alpha(x_2)^{\alpha(x_1)})|_B(\alpha(x_1)|_B\alpha(x_2)^{\alpha(x_1)}))^{\alpha(x_3)})\} \\ = & \min\{f^+(\alpha(x_3)), f^+(\alpha(((x_1|_Ax_2^{x_1})|_A(x_1|_Ax_2^{x_1}))^{x_3}|_A((x_1|_Ax_2^{x_1})|_A(x_1|_Ax_2^{x_1}))^{x_3}))\} \\ = & \min\{f^{+\alpha}(x_3), f^{+\alpha}(((x_1|_Ax_2^{x_1})|_A(x_1|_Ax_2^{x_1}))^{x_3}|_A((x_1|_Ax_2^{x_1})|_A(x_1|_Ax_2^{x_1}))^{x_3})\} \end{array}$$

Hence f^α is a bipolar fuzzy implicative SBG-ideal of A.

Conversely, let f^α be a bipolar fuzzy implicative SBG-ideal of A. Let $y_1, y_2, y_3 \in B$ such that $\alpha(x_1) = y_1$, $\alpha(x_2) = y_2$ and $\alpha(x_3) = y_3$ for $x_1, x_2, x_3 \in A$. Then, we have

$$f^-(0_B) = f^-(\alpha(0_A)) \leqslant f^{-\alpha}(0_A) = f^-(\alpha(x_1)) = f^-(y_1),$$

and

$$\begin{array}{rl} & f^-(y_1) = f^-(\alpha(x_1)) \\ = & f^{-\alpha}(x_1) \\ \leqslant & \max\{f^{-\alpha}(x_3), f^{-\alpha}(((x_1|_Ax_2^{x_1})|_A(x_1|_Ax_2^{x_1}))^{x_3}|_A((x_1|_Ax_2^{x_1})|_A(x_1|_Ax_2^{x_1}))^{x_3})\} \\ = & \max\{f^-(\alpha(x_3)), f^-(\alpha(((x_1|_Ax_2^{x_1})|_A(x_1|_Ax_2^{x_1}))^{x_3}|_A((x_1|_Ax_2^{x_1})|_A(x_1|_Ax_2^{x_1}))^{x_3}))\} \\ = & \max\{f^-(\alpha(x_3)), f^-(\alpha(((x_1|_Ax_2^{x_1})|_A(x_1|_Ax_2^{x_1}))^{x_3}|_A((x_1|_Ax_2^{x_1})|_A(x_1|_Ax_2^{x_1}))^{x_3}))\} \\ = & \max\{f^-(\alpha(y_3)), f^-(((\alpha(y_1)|_B\alpha(y_2)^{\alpha(y_1)})|_B(\alpha(y_1)|_B\alpha(y_2)^{\alpha(y_1)}))^{\alpha(y_3)} \\ & |_B((\alpha(y_1)|_B\alpha(y_2)^{\alpha(y_1)})|_B(\alpha(y_1)|_B\alpha(y_2)^{\alpha(y_1)}))^{\alpha(y_3)})\} \end{array}$$

Moreover, we have

$$f^+(0_B) = f^+(\alpha(0_A)) \geqslant f^{+\alpha}(x_1) = f^+(\alpha(x_1)) = f^+(y_1),$$

and

$$\begin{array}{rl} & f^+(y_1) = f^+(\alpha(x_1)) = f^{+\alpha}(x_1) \\ \geqslant & \min\{f^{+\alpha}(x_3), f^{+\alpha}(((x_1|_Ax_2^{x_1})|_A(x_1|_Ax_2^{x_1}))^{x_3}|_A((x_1|_Ax_2^{x_1})|_A(x_1|_Ax_2^{x_1}))^{x_3})\} \\ = & \min\{f^+(\alpha(x_3)), f^+(\alpha(((x_1|_Ax_2^{x_1})|_A(x_1|_Ax_2^{x_1}))^{x_3}|_A((x_1|_Ax_2^{x_1})|_A(x_1|_Ax_2^{x_1}))^{x_3}))\} \\ = & \min\{f^+(\alpha(x_3)), f^+(\alpha(((x_1|_Ax_2^{x_1})|_A(x_1|_Ax_2^{x_1}))^{x_3}|_A((x_1|_Ax_2^{x_1})|_A(x_1|_Ax_2^{x_1}))^{x_3}))\} \\ = & \min\{f^+(\alpha(y_3)), f^+(((\alpha(y_1)|_B\alpha(y_2)^{\alpha(y_1)})|_B(\alpha(y_1)|_B\alpha(y_2)^{\alpha(y_1)}))^{\alpha(y_3)} \\ & |_B((\alpha(y_1)|_B\alpha(y_2)^{\alpha(y_1)})|_B(\alpha(y_1)|_B\alpha(y_2)^{\alpha(y_1)}))^{\alpha(y_3)})\} \end{array}$$

Hence f is a bipolar fuzzy implicative SBG-ideal of B. $\square$

4. Normality of bipolar-valued fuzzy SBG-ideals

In this section, we defined the concept of normality for bipolar-valued fuzzy SBG-ideals in SBG-algebras. Specifically, we introduced the definition of a normal bipolar-valued fuzzy SBG-ideal and provided several theorems to characterize the conditions under which a bipolar-valued fuzzy SBG-ideal can be considered normal. The theorems established the equivalence between the normality of these ideals and specific conditions on their membership functions. Additionally, we explored the relationship between normal bipolar-valued fuzzy SBG-ideals and their larger counterparts, using a novel construction that relates bipolar-valued fuzzy sets via inclusion. Finally, we concluded by proving the maximality conditions for normal bipolar-valued fuzzy SBG-ideals, describing how such ideals behave within a partially ordered set.

Definition 4.1. A bipolar-valued fuzzy SBG-ideal $f = (L; f^+, f^-)$ of the SBG-algebra $\mathcal{L} = (L; |)$ is called normal if there exist elements $a, x \in L$ such that $f^-(a) = -1$ and $f^+(x) = 1$.

Theorem 4.1. *A bipolar fuzzy SBG-ideal $f = (L; f^+, f^-)$ of the SBG-algebra $\mathcal{L} = (L; |)$ is normal if and only if $f^-(0) = -1$ and $f^+(0) = 1$.*

Proof. Assume that $f = (L, f^+, f^-)$ is a normal bipolar-valued fuzzy SBG-ideal of $\mathcal{L} = (L, |)$. By definition, there exist elements $a, x \in L$ such that $f^-(a) = -1$ and $f^+(x) = 1$. Since $f^-(0) \leqslant f^-(a) = -1$ and $f^+(0) \geqslant f^+(x) = 1$, it follows that $f^-(0) = -1$ and $f^+(0) = 1$. The converse is straightforward and evident. □

Given two bipolar-valued fuzzy sets $f = (L; f^+, f^-)$ and $g = (L; g^+, g^-)$ in L, the inclusion relation $\Subset$ between them is defined as follows:

$$f \Subset g \Leftrightarrow (\forall x \in L)(f^-(x) \geqslant g^-(x) \text{ and } f^+(x) \leqslant g^+(x)).$$

In this case, we say that $g = (L; g^+, g^-)$ is larger than $f = (L; f^+, f^-)$.

Theorem 4.2. *Let $f = (L; f^+, f^-)$ be a bipolar-valued fuzzy set in L. Define the bipolar-valued fuzzy set $f_* = (L; f_*^-, f_*^+)$ in L by*

$$f_*^-(a) = f^-(a) - 1 - f^-(0) \quad \text{and} \quad f_*^+(x) = f^+(x) + 1 - f^+(0)$$

for all $(a, x) \in L \times L$. Then, $f = (L; f^+, f^-)$ is a bipolar-valued fuzzy SBG-ideal of the algebra $\mathcal{L} = (L; |)$ if and only if $f_* = (L; f_*^-, f_*^+)$ is a bipolar-valued fuzzy SBG-ideal of $\mathcal{L} = (L; |)$. Furthermore, $f_* = (L; f_*^-, f_*^+)$ is normal and larger than $f = (L; f^+, f^-)$.

Proof. Assume that $f = (L, f^+, f^-)$ is a bipolar-valued fuzzy SBG-ideal of $\mathcal{L} = (L, |)$. Let $x, y \in L$ such that $x \leqslant y$. We can express $f_*^-(x)$ as $f^-(x) - 1 - f^-(0)$, which implies that

$$f_*^-(x) \geqslant f^-(0) - 1 - f^-(0) = f_*^-(0),$$

and similarly, $f_*^+(x) = f^+(x) + 1 - f^+(0) \leqslant f^+(0) + 1 - f^+(0) = f_*^+(0)$.

For any $x, y \in L$, we have:

$$f_*^-(x) = f^-(x) - 1 - f^-(0) \leqslant \max\{f^-(x^y|x^y), f^-(y)\} - 1 - f^-(0),$$

which simplifies to:

$$f_*^-(x) \leqslant \max\{f^-(x^y|x^y)-1-f^-(0), f^-(y)-1-f^-(0)\} = \max\{f_*^-(x^y|x^y), f_*^-(y)\}.$$

Similarly, we have:

$$f_*^+(x) = f^+(x) + 1 - f^+(0) \geqslant \min\{f^+(x^y|x^y), f^+(y)\} + 1 - f^+(0),$$

which simplifies to:

$$f_*^+(x) \geqslant \min\{f^+(x^y|x^y)+1-f^+(0), f^+(y)+1-f^+(0)\} = \min\{f_*^+(x^y|x^y), f_*^+(y)\}.$$

Thus, $f_* = (L; f_*^-, f_*^+)$ is a bipolar-valued fuzzy SBG-ideal of $\mathcal{L} = (L; |)$.

Conversely, suppose that $f_* = (L; f_*^-, f_*^+)$ is a bipolar-valued fuzzy SBG-ideal of $\mathcal{L} = (L; |)$. From the definitions, we have $f_*^-(0) = f^-(0) - 1 - f^-(0) \leqslant f_*^-(a) = f^-(a) - 1 - f^-(0)$ and $f_*^+(0) = f^+(0) + 1 - f^+(0) \geqslant f_*^+(x) = f^+(x) + 1 - f^+(0)$ for all $(a, x) \in L \times L$. Therefore, it follows that $f^-(0) \leqslant f^-(x)$ and $f^+(0) \geqslant f^+(x)$ for all $x \in L$.

Additionally, for all $(a, x), (b, y) \in L$, we have:

$$f_*^-(a) \leqslant \max\{f_*^-(a^b|a^b), f_*^-(b)\},$$

which simplifies to:

$$\begin{aligned} f_*^-(a) = f^-(a) - 1 - f^-(0) &\leqslant \max\{f^-(a^b|a^b) - 1 - f^-(0), f^-(b) - 1 - f^-(0)\} \\ &= \max\{f^-(a^b|a^b), f^-(b)\} - 1 - f^-(0). \end{aligned}$$

Similarly, we have:

$$f_*^+(x) \geqslant \min\{f_*^+(x^y|x^y), f_*^+(y)\},$$

which simplifies to:

$$\begin{aligned} f_*^+(x) = f^+(x) + 1 - f^+(0) &\geqslant \min\{f^+(x^y|x^y) + 1 - f^+(0), f^+(y) + 1 - f^+(0)\} \\ &= \min\{f^+(x^y|x^y), f^+(y)\} + 1 - f^+(0). \end{aligned}$$

Thus, $f = (L; f^+, f^-)$ is a bipolar-valued fuzzy SBG-ideal of $\mathcal{L} = (L; |)$. Moreover, since $f_*^-(0) = f^-(0) - 1 - f^-(0) = -1$ and $f_*^+(0) = f^+(0) + 1 - f^+(0) = 1$, it follows that $f_* = (L; f_*^-, f_*^+)$ is normal. Additionally, we have $f_*^-(x) = f^-(x) - 1 - f^-(0) \leqslant f^-(x)$ and $f_*^+(x) = f^+(x) + 1 - f^+(0) \geqslant f^+(x)$ for all $x \in L$, showing that $f_* = (L; f_*^-, f_*^+)$ is larger than $f = (L; f^+, f^-)$. □

THEOREM 4.3. *Let $f = (L; f^+, f^-)$ be a bipolar-valued fuzzy SBG-ideal of the SBG-algebra $L = (L; |)$. The set f is a normal bipolar-valued fuzzy SBG-ideal if and only if $f_* = f$. That is, $f^-(x) = f_*^-(x)$ and $f^+(x) = f_*^+(x)$ for all $x \in L$.*

PROOF. Let $f = (L; f^+, f^-)$ be a bipolar-valued fuzzy SBG-ideal of $\mathcal{L} = (L; |)$. We clearly attain that $f_* = (L, f_*^-, f_*^+)$ is a normal bipolar-valued fuzzy SBG-ideal of $\mathcal{L} = (L; |)$. Therefore, it is evident that if $f_* = f$, then $f = (L; f^+, f^-)$ must be normal.

Conversely, if $f = (L; f^+, f^-)$ is normal, then for all $x \in L$ we have $f_*^-(x) = f^-(x) - 1 - f^-(0) = f^-(x)$ and $f_*^+(x) = f^+(x) + 1 - f^+(0) = f^+(x)$. Thus, $f_* = f$. □

Proposition 4.1. *Let $f = (L, f^+, f^-)$ and $g = (L; g^+, g^-)$ be bipolar-valued fuzzy SBG-ideals of $\mathcal{L} = (L; |)$ such that $f \Subset g$. If $f^-(0) = g^-(0)$ and $f^+(0) = g^+(0)$, then $L_{f_F} \Subset L_{g_F}$.*

Proof. The result follows directly. $\square$

Theorem 4.4. *Let $f = (L; f^+, f^-)$ be a bipolar-valued fuzzy SBG-ideal of $L = (L; |)$. Then f is normal if and only if there exists a bipolar-valued fuzzy SBG-ideal $g = (L; g^+, g^-)$ of $\mathcal{L} = (L; |)$ such that $g_* \Subset f$.*

Proof. The necessity is straightforward because if $f = (L; f^+, f^-)$ is normal, then $f_* = f$.

Conversely, assume that there exists a bipolar-valued fuzzy SBG-ideal $g = (L; g^-, g^+)$ of $\mathcal{L} = (L, |)$ such that $g_* \Subset f$. Then, $-1 = g_*^-(0) \geqslant f^-(0)$ and $1 = g_*^+(0) \leqslant f^+(0)$. Thus, $f^-(0) = -1$ and $f^+(0) = 1$, which implies that $f = (L; f^+, f^-)$ is normal. $\square$

Theorem 4.5. *Given a bipolar-valued fuzzy set $f = (L; f^+, f^-)$ in L, consider an increasing mapping $\ell = (\ell^-, \ell^+) : [-1, f^-(0)] \times [0, f^+(0)] \to [-1, 0] \times [0, 1]$. If $f = (L; f^+, f^-)$ is a bipolar-valued fuzzy SBG-ideal of $\mathcal{L} = (L; |)$, then the bipolar-valued fuzzy set $f_\ell = (L; f_\ell^-, f_\ell^+)$ in L defined by $f_\ell^-(a) = \ell^-(f^-(a))$ and $f_\ell^+(x) = \ell^+(f^+(x))$ for all $(a, x) \in L \times L$ is a bipolar-valued fuzzy SBG-ideal of $\mathcal{L} = (L; |)$. Moreover, if $f_\ell^-(0) = -1$ and $f_\ell^+(0) = 1$, then $f_\ell = (L; f_\ell^-, f_\ell^+)$ is normal, and for all $(s, t) \in [-1, f^-(0)] \times [0, f^+(0)]$, if $\ell^-(s) \leqslant s$ and $\ell^+(t) \geqslant t$, then $f \Subset f_\ell$.*

Proof. Assume that $f = (L; f^+, f^-)$ is a bipolar-valued fuzzy SBG-ideal of $L = (L; |)$. Let $x, y \in L$ be such that $x \leqslant y$. Then $f_\ell^-(x) = \ell^-(f^-(x)) \geqslant \ell^-(f^-(0)) = f_\ell^-(0)$ and $f_\ell^+(x) = \ell^+(f^+(x)) \leqslant \ell^+(f^+(0)) = f_\ell^+(0)$. For every $x, y \in L$, we have

$$\begin{aligned} f_\ell^-(x) &= \ell^-(f^-(x)) \\ &\leqslant \ell^-(\max\{f^-(x^y|x^y), f^-(y)\}) \\ &= \max\{\ell^-(f^-(x^y|x^y)), \ell^-(f^-(y))\} \\ &= \max\{f_\ell^-(x^y|x^y), f_\ell^-(y)\}, \end{aligned}$$

and

$$\begin{aligned} f_\ell^+(x) &= \ell^+(f^+(x)) \\ &\geqslant \ell^+(\min\{f^+(x^y|x^y), f^+(y)\}) \\ &= \min\{\ell^+(f^+(x^y|x^y)), \ell^+(f^+(y))\} \\ &= \min\{f_\ell^+(x^y|x^y), f_\ell^+(y)\}. \end{aligned}$$

Therefore, it is obtain that $f_\ell = (L; f_\ell^-, f_\ell^+)$ is a bipolar-valued fuzzy SBG-ideal of $\mathcal{L} = (L; |)$. If $f_\ell^-(0) = -1$ and $f_\ell^+(0) = 1$, then $f_\ell = (L; f_\ell^-, f_\ell^+)$ is normal by Theorem 4.3 2.1. Let $(s, t) \in [-1, f^-(0)] \times [0, f^+(0)]$ be such that $\ell^-(s) \leqslant s$ and $\ell^+(t) \geqslant t$. Then $f_\ell^-(x) = \ell^-(f^-(x)) \leqslant f^-(x)$ and $f_\ell^+(x) = \ell^+(f^+(x)) \geqslant f^+(x)$ for all $x \in L$. Hence $f \Subset f_\ell$. $\square$

Theorem 4.6. *Let $f = (L; f^+, f^-)$ be a normal bipolar-valued fuzzy SBG-ideal of $\mathcal{L} = (L; |)$ such that $f^-(a) \neq f^-(0)$ and $f^+(x) \neq f^+(0)$ for some $(a, x) \in L \times L$.*

If $f = (L; f^+, f^-)$ is a maximal element of $(\mathcal{N}_F(L), \Subset)$, then it is described as follows:

$$f^- : L \to [-1, 0], \quad a \mapsto \begin{cases} -1 & \text{if } a = 1, \\ 0 & \text{otherwise}, \end{cases}$$

$$f^+ : L \to [0, 1], \quad x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{otherwise}, \end{cases}$$

where $\mathcal{N}_F(L)$ is the set of all normal bipolar-valued fuzzy SBG-ideals of $\mathcal{L} = (L; |)$.

Proof. It is evident that $(\mathcal{N}_F(L), \Subset)$ is a partially ordered set (poset). Suppose that $f = (L; f^+, f^-)$ is a maximal element in $(\mathcal{N}_F(L), \Subset)$. Since $f = (L; f^+, f^-)$ is normal, it follows that $f^-(0) = -1$ and $f^+(0) = 1$. Let $(a, x) \in L \times L$ such that $f^-(a) \neq f^-(0)$ and $f^+(x) \neq f^+(0)$. If $f^-(a) \neq 0$ and $f^+(x) \neq 0$, then there exist $(c, z) \in L \times L$ such that $-1 < f^-(c) < 0$ and $0 < f^+(z) < 1$. Define a bipolar-valued fuzzy set $g = (L; g^-, g^+)$ on L as follows:

$$g^- : L \to [-1, 0], \quad a \mapsto \frac{1}{2}(f^-(a) + f^-(c)),$$

$$g^+ : L \to [0, 1], \quad x \mapsto \frac{1}{2}(f^+(x) + f^+(z)).$$

For all $x, y \in L$, we have:

$$g^-(x) = \frac{1}{2}(f^-(x) + f^-(c)) \geqslant \frac{1}{2}(f^-(y) + f^-(c)) = g^-(y),$$

and

$$g^+(x) = \frac{1}{2}(f^+(x) + f^+(z)) \leqslant \frac{1}{2}(f^+(y) + f^+(z)) = g^+(y).$$

Thus, for every $x, y \in L$, we also have:

$$\begin{aligned} g^-(x) = \frac{1}{2}(f^-(x) + f^-(c)) &\leqslant \frac{1}{2}\max\{f^-(x^y|x^y), f^-(y)\} + f^-(c) \\ &= \max\{g^-(x^y|x^y), g^-(y)\}, \end{aligned}$$

and

$$\begin{aligned} g^+(x) = \frac{1}{2}(f^+(x) + f^+(z)) &\geqslant \frac{1}{2}\min\{f^+(x^y|x^y), f^+(y)\} + f^+(z) \\ &= \min\{g^+(x^y|x^y), g^+(y)\}. \end{aligned}$$

Thus, $g = (L; g^-, g^+)$ is a bipolar-valued fuzzy SBG-ideal of $\mathcal{L} = (L; |)$, and $g_* = (L; g_*^-, g_*^+)$ is a normal bipolar-valued fuzzy SBG-ideal of $\mathcal{L} = (L; |)$. Moreover:

$$\begin{aligned} g_*^-(x) &= g^-(x) - 1 - g^-(0) = \frac{1}{2}(f^-(x) + f^-(c)) - 1 - \frac{1}{2}(f^-(0) + f^-(c)) \\ &= \frac{1}{2}(f^-(x) - 1) \leqslant f^-(x), \end{aligned}$$

and

$$\begin{aligned} g_*^+(x) &= g^+(x) + 1 - g^+(0) = \frac{1}{2}(f^+(x) + f^+(z)) + 1 - \frac{1}{2}(f^+(0) + f^+(z)) \\ &= \frac{1}{2}(f^+(x) + 1) \geqslant f^+(x), \end{aligned}$$

for all $x \in L$. Thus, $f \Subset g_*$, which contradicts the assumption that $f = (L; f^+, f^-)$ is a maximal element in $(\mathcal{N}_F(L), \Subset)$. Therefore, we must have $(f^-(a), f^+(x)) = (0, 0)$ for all $(a, x) \in L \times L$ such that $f^-(a) \neq -1$ and $f^+(x) \neq 1$. Consequently, it is verified that $f = (L; f^-, f^+)$ is a bipolar fuzzy SBG-ideal of $\mathcal{L}$ by Definition 2.8. □

5. Conclusion

In this manuscript, we introduced and examined the concept of bipolar fuzzy implicative SBG-ideals within the framework of SBG-algebras. We began by defining bipolar fuzzy implicative SBG-ideals and establishing conditions involving both the negative and positive components of bipolar fuzzy sets. Our work demonstrated that every bipolar fuzzy implicative SBG-ideal is inherently a bipolar fuzzy SBG-ideal. Through the exploration of key propositions and theorems, we proved that these implicative ideals are preserved under various operations, such as homomorphisms and (α, β)-translations. Additionally, we analyzed the relationship between these ideals and cuts, showing that nonempty cuts form implicative SBG-ideals. We also discussed the connection between bipolar fuzzy sub-implicative SBG-ideals and implicative ones, particularly in medial SBG-algebras, where bipolar fuzzy implicative SBG-ideals serve as sub-implicative ideals. Lastly, we demonstrated that these implicative ideals are preserved under surjective homomorphisms, further reinforcing their structural robustness.

Furthermore, we extended the study to include the concept of normal bipolar-valued fuzzy SBG-ideals in SBG-algebras. We characterized the conditions under which a bipolar-valued fuzzy SBG-ideal can be considered normal and explored the equivalence between the normality of these ideals and specific conditions on their membership functions. Additionally, we examined the relationship between normal bipolar-valued fuzzy SBG-ideals and their larger counterparts, employing a novel construction based on inclusion relations. We concluded by proving maximality conditions for normal bipolar-valued fuzzy SBG-ideals, providing insights into their behavior within a partially ordered set.

Future work will aim to extend these findings by investigating more complex algebraic structures and exploring the computational applications of these ideals in areas such as artificial intelligence and decision theory. Additionally, further research will delve into the connections between normal bipolar fuzzy implicative SBG-ideals and other algebraic systems, with a particular focus on their practical relevance in technological applications.

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