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# SOME RESULTS ON THE EXISTENCE OF MULTIPLE POSITIVE SOLUTIONS FOR ITERATIVE SYSTEMS OF THE NONLINEAR SECOND-ORDER IMPULSIVE BOUNDARY VALUE PROBLEM

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Abstract. In this study, some criteria for the existence of multiple solutions of the iterative system of the second order nonlinear impulsive boundary value problem are presented with the help of the fixed point theorem on the cone. Then, the applicability of the results is emphasised with an example.

## 1. Introduction

Impulsive differential equations, unlike classical differential equations, are used to model systems that exhibit abrupt changes at certain points in time. These equations have important applications in fields as diverse as engineering, biology, economics and physics. In the references [**1**, **2**, **10**, **16**, **17**], a few specific examples of the application areas of impulsive differential equations can be as follows:

- It is used for modelling sudden changes in control systems. For example, these equations are used when a robot arm must suddenly stop or change direction.
- It is used for modelling sudden market movements or economic shocks. For example, it can be used to model the effects of sudden interest rate changes or financial crises.

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- It is used to model sudden increases or decreases of populations, especially situations such as migration, epidemics or sudden environmental changes.
- It is used for modelling drug administration. For example, it can be used to model the effects of drug doses given to a patient at certain intervals.
- In electrical circuits, it is used for modelling sudden voltage or current changes. In particular, impulsive equations are used in switching circuits.

Many authors have worked on the existence of solutions of impulsive boundary value problems. In fact, in most studies, especially the existence of positive solutions has been investigated. The existence of positive solutions is a critical issue in many mathematical and applied fields. For impulsive BVPs, positive solutions indicate that the system is stable and sustainable under certain conditions. Positive solutions refer to the transition of a system from a given initial state to a positive state or a sustainable positive state. in this context, [**4**, **5**, **7**, **13**] studies can be cited.

Moreover, some authors have considered the system form of impulsive differential equations, especially discrete systems. In this paper, the boundary value problem is considered as an iterative system, unlike the discrete system. It is clear that the iterative system is much more complicated than the discrete system. In iterative systems, starting from a starting point, a solution or sequence is generated through repeated steps. At each step, the next step is calculated using the results obtained from the previous step. [**3**, **6**, **8**, **9**, **11**, **12**, **14**] studies can also be given as examples of iterative systems and systems.

In this work, we consider the following iterative system of nonlinear second-order impulsive boundary value problem (IBVP), inspired by the aforementioned result:

$$
\begin{cases}
\kappa_i''(t) + \eta_i q_i(t) p_i(\kappa_{i+1}(t)) = 0, \ 1 \leqslant i \leqslant n, \ t \in J = [0,1] \\
\kappa_{n+1}(t) = \kappa_1(t), \\
\triangle \kappa_i|_{t=t_j} = \eta_i \bar{I}_{ij}(\kappa_{i+1}(t_j)), \ t \neq t_j, j = 1,2,...,k, \\
\triangle \kappa_i'|_{t=t_j} = -\eta_i \tilde{I}_{ij}(\kappa_{i+1}(t_j)), \\
\alpha_1 \kappa_i(0) - \alpha_2 \kappa_i'(0) = 0, \\
\alpha_3 \kappa_i(1) + \alpha_4 \kappa_i'(1) = 0
\end{cases} \tag{1.1}
$$

where $t \neq t_j$, $j = 1,2,...,k$ such that $0 < t_1 < t_2 < ... < t_k < 1$. Furthermore, for $i = 1,2,...,n$, the functions $\triangle \kappa_i$ and $\triangle \kappa_i'$ at the point $t = t_j$ stand for the jump of $\kappa_i(t)$ and $\kappa_i'(t)$ at the point $t = t_j$, i.e.,

$$
\triangle \kappa_i|_{t=t_j} = \kappa_i(t_j^+) - \kappa_i(t_j^-), \quad \triangle \kappa_i'|_{t=t_j} = \kappa_i'(t_j^+) - \kappa_i'(t_j^-),
$$

where the values $\kappa_i(t_j^+)$, $\kappa_i'(t_j^+)$ state the right-hand limit of $\kappa_i(t)$ and $\kappa_i'(t)$ at the point $t = t_j$, $j = 1,2,...,k$ and similarly $\kappa_i(t_j^-)$, $\kappa_i'(t_j^-)$ state left-hand limit of $\kappa_i(t)$ and $\kappa_i'(t)$ at the point $t = t_j$, $j = 1,2,...,k$.

Throughout this article we assume that the following conditions are met.

($C1$) Let $\eta_i$ be a positive number for all $i = 1,2,\ldots,n$. Given that $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0,\infty)$, we have $\alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 > 0$,

($C2$) For $1 \leqslant i \leqslant n$, the function $p_i : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous,

($C3$) For $1 \leqslant i \leqslant n$, the function $q_i \in C([0,1], \mathbb{R}^+)$ does not vanish identically on any closed subinterval of $[0,1]$,

($C4$) Let $\bar{I}_{ij} \in C(\mathbb{R}, \mathbb{R}^+)$ and $\tilde{I}_{ij} \in C(\mathbb{R}, \mathbb{R}^+)$ be bounded functions. For any nonnegative number $\lambda$ and for $t < t_j$ where $j = 1, 2, \ldots, k$ and $1 \leqslant i \leqslant n$, the inequality $[\alpha_4 + \alpha_3(1 - t_j)]\tilde{I}_{ij}(\lambda) > \alpha_3 \bar{I}_{ij}(\lambda)$ holds.

The primary tool used in this study is the fixed point theorem with reference to [**15**], aiming to identify certain conditions for the existence of multiple positive solutions in the iterative system of nonlinear second-order IBVP (1.1). To the best of the authors' knowledge, the existence of multiple positive solutions for this iterative system has not been previously investigated. By establishing some criteria for the existence of these solutions, we contribute to the existing literature.

The structure of this paper is as follows: In Section 2, we introduce several definitions and fundamental lemmas essential for understanding our main results. Section 3 provides criteria for the existence of multiple positive solutions for the iterative system of IBVP (1.1). Finally, in Section 4, we illustrate the application of our main results with an example.

## 2. Preliminaries

In this section, we start by introducing fundamental definitions in Banach spaces, followed by several supplementary lemmas that will be utilized later.

Define $J' = J \backslash \{t_1, t_2, ..., t_k\}$. The space $C(J)$ denotes the Banach space of all continuous mappings $\kappa : J \to \mathbb{R}$ equipped with the norm $\|\kappa\| = \sup_{t \in J} |\kappa(t)|$. The space $PC(J)$ consists of functions $\kappa : J \to \mathbb{R}$ such that $\kappa \in C(J')$, $\kappa(t_j^+)$ and $\kappa(t_j^-)$ exist, and $\kappa(t_j^-) = \kappa(t_j)$ for $j = 1, 2, ..., k$. $PC(J)$ is also a Banach space with the norm $\|\kappa\|_{PC} = \sup_{t \in J} |\kappa(t)|$. Let $\mathbb{B} = PC(J) \cap C^2(J')$. A function $(\kappa_1, ..., \kappa_n) \in \mathbb{B}^n$ is considered a solution of the second-order IBVP (1.1)'s iterative system if it satisfies the conditions of the second-order IBVP (1.1)'s iterative system.

Initially, we will address the case where $i = 1$ in the second-order IBVP (1.1). Consequently, we will present the solution $\kappa_1$ for the second-order IBVP (2.1). Subsequently, having determined $\kappa_1$, we proceed to find $\kappa_n$. Continuing in this manner, we can sequentially determine $\kappa_{n-1}$, $\kappa_{n-2}$, and so forth, eventually reaching $\kappa_2$. Thus, we obtain the solution $(\kappa_1, ..., \kappa_n)$ for the second-order IBVP (1.1)'s iterative system.

Assume that $g \in C[0,1]$, then we deal with the following second-order IBVP:

$$\begin{cases} \kappa_1''(t) + g(t) = 0, \ t \in J = [0,1], t \neq t_j, j = 1, 2, ..., k, \\ \triangle \kappa_1|_{t=t_j} = \eta_1 \bar{I}_{1j}(\kappa_2(t_j)), \\ \triangle \kappa_1'|_{t=t_j} = -\eta_1 \tilde{I}_{1j}(\kappa_2(t_j)), \\ \alpha_1 \kappa_1(0) - \alpha_2 \kappa_1'(0) = 0, \\ \alpha_3 \kappa_1(1) + \alpha_4 \kappa_1'(1) = 0. \end{cases} \tag{2.1}$$

The solutions of the corresponding homogeneous equation are defined by $\zeta$ and $\bar{\zeta}$. Under the initial conditions

$$\begin{cases} \zeta(0)=\alpha_2, & \zeta'(0)=\alpha_1, \\ \bar{\zeta}(1)=\alpha_4, & \bar{\zeta}'(1)=-\alpha_3, \end{cases} \tag{2.2}$$

we have

$$\kappa_1''(t)=0, \quad t\in[0,1]. \tag{2.3}$$

Using the initial conditions (2.2), we can deduce from equation (2.3) for $\zeta$ and $\bar{\zeta}$ the following equations:

$$\zeta(t)=\alpha_2+\alpha_1 t, \quad \bar{\zeta}(t)=\alpha_4+\alpha_3(1-t). \tag{2.4}$$

Set

$$\mu := \alpha_1\alpha_4+\alpha_1\alpha_3+\alpha_2\alpha_3. \tag{2.5}$$

LEMMA 2.1. *Let (C1)-(C4) hold. If $\kappa_1 \in \mathbb{B}$ is a solution of the equation*

$$\kappa_1(t)=\int_0^1 H(t,s)g(s)ds+\sum_{j=1}^{k} W_{1j}(t,t_j), \tag{2.6}$$

*where*

$$H(t,s)=\frac{1}{\mu}\begin{cases} (\alpha_2+\alpha_1 s)[\alpha_4+\alpha_3(1-t)], & s\leqslant t, \\ (\alpha_2+\alpha_1 t)[\alpha_4+\alpha_3(1-s)], & t\leqslant s, \end{cases} \tag{2.7}$$

(2.8)

$$W_{1j}(t,t_j)=$$

$$\frac{1}{\mu}\begin{cases} (\alpha_2+\alpha_1 t)[-\alpha_3\eta_1\bar{I}_{1j}(\kappa_2(t_j))+(\alpha_4+\alpha_3(1-t_j))\eta_1\tilde{I}_{1j}(\kappa_2(t_j))], & t<t_j, \\ (\alpha_4+\alpha_3(1-t))[\alpha_1\eta_1\bar{I}_{1j}(\kappa_2(t_j))+(\alpha_2+\alpha_1 t_j)\eta_1\tilde{I}_{1j}(\kappa_2(t_j))], & t_j\leqslant t, \end{cases}$$

*then $\kappa_1$ is a solution of the IBVP* (2.1).

PROOF. Let $\kappa_1$ satisfies the equation (2.6), then we get

$$\kappa_1(t)=\int_0^1 H(t,s)g(s)ds+\sum_{j=1}^{k} W_{1j}(t,t_j),$$

i.e.,

$$\begin{aligned} \kappa_1(t) \quad = \quad & \frac{1}{\mu}\int_0^t(\alpha_2+\alpha_1 s)[\alpha_4+\alpha_3(1-t)]g(s)ds+\frac{1}{\mu}\int_t^1(\alpha_2+\alpha_1 t)[\alpha_4+\alpha_3(1-s)]g(s)ds \\ & +\frac{1}{\mu}\sum_{0<t_j<t}(\alpha_4+\alpha_3(1-t))[\alpha_1\eta_1\bar{I}_{1j}(\kappa_2(t_j))+(\alpha_2+\alpha_1 t_j)\eta_1\tilde{I}_{1j}(\kappa_2(t_j))] \\ & +\frac{1}{\mu}\sum_{t<t_j<1}(\alpha_2+\alpha_1 t)[-\alpha_3\eta_1\bar{I}_{1j}(\kappa_2(t_j))+(\alpha_4+\alpha_3(1-t_j))\eta_1\tilde{I}_{1j}(\kappa_2(t_j))], \end{aligned}$$

$$\begin{aligned}\kappa_1'(t) &= \frac{1}{\mu}\int_0^t(-\alpha_3)(\alpha_2+\alpha_1 s)g(s)ds+\frac{1}{\mu}\int_t^1\alpha_1[\alpha_4+\alpha_3(1-s)]g(s)ds\\ &+\frac{1}{\mu}\sum_{0<t_j<t}(-\alpha_3)[\alpha_1\eta_1\bar{I}_{1j}(\kappa_2(t_j))+(\alpha_2+\alpha_1 t_j)\eta_1\tilde{I}_{1j}(\kappa_2(t_j))]\\ &+\frac{1}{\mu}\sum_{t<t_j<1}\alpha_1[-\alpha_3\eta_1\bar{I}_{1j}(\kappa_2(t_j))+(\alpha_4+\alpha_3(1-t_j))\eta_1\tilde{I}_{1j}(\kappa_2(t_j))].\end{aligned}$$

Thus

$$\begin{aligned}\kappa_1''(t) &= \frac{1}{\mu}(-\alpha_3 t-(\alpha_4+\alpha_3(1-t)))g(t)\\ &= -g(t),\end{aligned}$$

i.e.,

$$\kappa_1''(t)+g(t)=0.$$

Since

$$\begin{aligned}\kappa_1(0) &= \frac{1}{\mu}\int_0^1\alpha_2[\alpha_4+\alpha_3(1-s)]g(s)ds\\ &+\frac{1}{\mu}\sum_{j=1}^k\alpha_2[-\alpha_3\eta_1\bar{I}_{1j}(\kappa_2(t_j))+(\alpha_4+\alpha_3(1-t_j))\eta_1\tilde{I}_{1j}(\kappa_2(t_j))]\end{aligned}$$

and

$$\begin{aligned}\kappa_1'(0) &= \frac{1}{\mu}\int_0^1\alpha_1[\alpha_4+\alpha_3(1-s)]g(s)ds\\ &+\frac{1}{\mu}\sum_{j=1}^k\alpha_1[-\alpha_3\eta_1\bar{I}_{1j}(\kappa_2(t_j))+(\alpha_4+\alpha_3(1-t_j))\eta_1\tilde{I}_{1j}(\kappa_2(t_j))],\end{aligned}$$

we get

$$\alpha_1\kappa_1(0)-\alpha_2\kappa_1'(0)=0. \tag{2.9}$$

Since

$$\begin{aligned}\kappa_1(1) &= \frac{1}{\mu}\int_0^1(\alpha_2+\alpha_1 s)(\alpha_3+\alpha_4)g(s)ds\\ &+\frac{1}{\mu}\sum_{j=1}^k(\alpha_3+\alpha_4)[\alpha_1\eta_1\bar{I}_{1j}(\kappa_2(t_j))+(\alpha_2+\alpha_1 t_j)\eta_1\tilde{I}_{1j}(\kappa_2(t_j))]\end{aligned}$$

and

$$\begin{aligned}\kappa_1'(1) &= \frac{1}{\mu}\int_0^1 (-\alpha_3)(\alpha_2+\alpha_1 s)g(s)ds \\ &+\frac{1}{\mu}\sum_{j=1}^{k}(-\alpha_3)[\alpha_1\eta_1\bar{I}_{1j}(\kappa_2(t_j))+(\alpha_2+\alpha_1 t_j)\eta_1\tilde{I}_{1j}(\kappa_2(t_j))],\end{aligned}$$

we get

$$\alpha_3\kappa_1(1)+\alpha_4\kappa_1^{'}(1)=0. \tag{2.10}$$

The conditions of the IBVP (2.1) are satisfied as indicated by equations (2.9) and (2.10). □

Lemma 2.2. *Suppose that the conditions (C1)-(C4) are satisfied. For $\kappa_1(t)\in\mathbb{B}$ with $g(t)\geqslant 0$, the solution $\kappa_1(t)$ of the second-order IBVP (2.1) fulfills $\kappa_1(t)\geqslant 0$ for $t\in J$.*

Proof. First, note that the Green function $H(t,s)$ is non-negative for $t,s\in J\times J$. Furthermore, given that $\bar{I}_{1j}(\kappa_1(t_j))$ and $\tilde{I}_{1j}(\kappa_1(t_j))$ are non-negative, it follows that $W_{1j}(t,t_j)$ is also non-negative. Consequently, $\kappa_1(t)$ is positive for $t\in[0,1]$. □

Lemma 2.3. *Assuming that (C1)-(C4) hold, the solution $\kappa_1(t)\in\mathbb{B}$ of the second-order IBVP (2.1) satisfies $\kappa_1'(t)\geqslant 0$ for $t\in J$.*

Proof. Suppose the inequality $\kappa_1'(t)<0$ is satisfied. Given that $\kappa_1(t)$ is monotonically decreasing on $J$, it can be ascertained that $\kappa_1(1)$ is strictly less than $\kappa_1(0)$.
From the boundary conditions of the IBVP (2.1), we have

$$\kappa_1(1)<\frac{\alpha_2}{\alpha_1}\kappa_1'(0).$$

This implies that

$$\kappa_1(1)<\frac{\alpha_2}{\alpha_1}\kappa_1'(0)<0.$$

The last inequality contradicts the Lemma 2.2. That is, our assumption is wrong. Thus, $\kappa_1'(t)\geqslant 0$ for $t\in J$.

□

Lemma 2.4. *Assume that (C1)-(C4) hold, then for any $t,s\in J$, we conclude the following inequality*

$$H(s,s)\geqslant H(t,s)\geqslant 0. \tag{2.11}$$

Proof. From equation (2.7), it is easily obtained. □

LEMMA 2.5. *Assuming that (C1)-(C4) hold, then for any* $t, s \in J$, *and* $\tau \in \left(0, \frac{1}{2}\right)$, *we obtain the following inequality*

$$H(s,s) \leqslant \frac{1}{\gamma} H(t,s) \tag{2.12}$$

*where* $\gamma := \min\left\{\frac{\alpha_2 + \alpha_1 \tau}{\alpha_2 + \alpha_1}, \frac{\alpha_4 + \alpha_3 \tau}{\alpha_4 + \alpha_3}\right\}$.

PROOF. From the definition of $H(t,s)$, we can conclude that for any $t$ belonging to the interval $[\tau, 1-\tau]$, we obtain

$$\frac{H(t,s)}{H(s,s)} = \begin{cases} \dfrac{\alpha_2 + \alpha_1 t}{\alpha_2 + \alpha_1 s}, & t \leqslant s, \\ \\ \dfrac{\alpha_4 + \alpha_3(1-t)}{\alpha_4 + \alpha_3(1-s)}, & s \leqslant t \end{cases}$$

$$\geqslant \begin{cases} \dfrac{\alpha_2 + \alpha_1 \tau}{\alpha_2 + \alpha_1}, & t \leqslant s, \\ \\ \dfrac{\alpha_4 + \alpha_3 \tau}{\alpha_4 + \alpha_3}, & s \leqslant t \end{cases}$$

$$\geqslant \min\left\{\frac{\alpha_2 + \alpha_1 \tau}{\alpha_2 + \alpha_1}, \frac{\alpha_4 + \alpha_3 \tau}{\alpha_4 + \alpha_3}\right\}$$

$$:= \gamma.$$

□

Consider the set $\mathcal{K}$ defined as $\mathcal{K} = \{\kappa_1(t) \in \mathbb{B} : \kappa_1(t)$ is nonnegative, nondecreasing, and concave on $J\}$. It follows that $\mathcal{K}$ constitutes a cone within $PC(J)$.

LEMMA 2.6. *Assume that the conditions (C1)-(C4) are satisfied. Then, for* $\kappa_1(t) \in \mathcal{K}$ *and* $\tau \in \left(0, \frac{1}{2}\right)$, *we conclude the following inequality*

$$\|\kappa_1\|_{PC} \leqslant \frac{1}{\tau} \min_{t \in [\tau, 1-\tau]} \kappa_1(t). \tag{2.13}$$

PROOF. Given that $\kappa_1(t) \in \mathcal{K}$, it can be deduced that $\kappa_1(t)$ is concave on $J$. Consequently, we can assert that the minimum value of $\kappa_1(t)$ for $t$ in the interval $[\tau, 1-\tau]$ is achieved at $\kappa_1(\tau)$, and the norm $\|\kappa_1\|_{PC}$ is equal to the supremum of $|\kappa_1(t)|$ on $J$, which is attained at $\kappa_1(1)$. Since the graph of $\kappa_1$ exhibits a concave downward shape on $J$, we obtain

$$\frac{\kappa_1(\tau) - \kappa_1(0)}{\tau - 0} \geqslant \frac{\kappa_1(1) - \kappa_1(0)}{1 - 0},$$

In other words, we have $\tau\kappa_1(1) + (1-\tau)\kappa_1(0) \leqslant \kappa_1(\tau)$. Consequently, we can deduce that $\tau\kappa_1(1) \leqslant \kappa_1(\tau)$, that is, $\kappa_1(1) \leqslant \frac{1}{\tau}\kappa_1(\tau)$. This concludes the proof. □

We note that

$$\begin{aligned}\kappa_1(t) =&\eta_1 \int_0^1 H(t,s_1)q_1(s_1)p_1\Bigg(\eta_2 \int_0^1 H(s_1,s_2)q_2(s_2)p_2\Bigg(\eta_3 \int_0^1 H(s_2,s_3)q_3(s_3)p_3 ...\\ &p_{n-1}\Bigg(\eta_n \int_0^1 H(s_{n-1},s_n)q_n(s_n)p_n(\kappa_1(s_n))ds_n + \sum_{j=1}^{k} W_{nj}(s_{n-1},t_j)\Bigg)ds_{n-1}\\ &+\sum_{j=1}^{k} W_{n-1,j}(s_{n-2},t_j)\Bigg)ds_{n-2} + ... + \sum_{j=1}^{k} W_{3j}(s_2,t_j)\Bigg)ds_2\\ &+\sum_{j=1}^{k} W_{2j}(s_1,t_j)\Bigg)ds_1 + \sum_{j=1}^{k} W_{1j}(t,t_j),\end{aligned}$$

$$\begin{aligned}\kappa_i(t) =&\eta_i \int_0^1 H(t,s)q_i(s)p_i(\kappa_{i+1}(s))ds + \sum_{j=1}^{k} W_{ij}(t,t_j),\ t \in J,\\ \kappa_{n+1}(t) =& \kappa_1(t)\end{aligned}$$

and

$$W_{ij}(t,t_j) = \frac{1}{\mu}\begin{cases}(\alpha_2+\alpha_1 t)[-\alpha_3\eta_i \bar{I}_{ij}(\kappa_{i+1}(t_j)) + (\alpha_4+\alpha_3(1-t_j))\eta_i \tilde{I}_{ij}(\kappa_{i+1}(t_j))], & t < t_j,\\ (\alpha_4+\alpha_3(1-t))[\alpha_1\eta_i \bar{I}_{ij}(\kappa_{i+1}(t_j)) + (\alpha_2+\alpha_1 t_j)\eta_i \tilde{I}_{ij}(\kappa_{i+1}(t_j))], & t_j \leqslant t,\end{cases}$$

if and only if the $n$-tuple $(\kappa_1(t), \kappa_2(t), ..., \kappa_n(t))$ is a solution of the iterative system of the IBVP (1.1).

## 3. Main results

The fixed point theorem presented below is fundamental and plays a crucial role in proving our main result.

DEFINITION 3.1. Let $\mathbb{B}$ be a Banach space. For a nonnegative continuous function $\gamma$ defined on a cone $\mathcal{K} \subset \mathbb{B}$, we define the set $\mathcal{K}(\gamma, c) = \{x \in \mathcal{K} : \gamma(x) < c\}$ for each $c > 0$.

To determine some conditions on the existence of at least three positive solutions for the iterative system of the IBVP (1.1), we will apply the following fixed point theorem [**15**].

LEMMA 3.1. *[**15**] Let $\mathcal{K}$ be a cone in a real Banach space $\mathbb{B}$. Suppose $\psi$, $\theta$, and $\varphi$ are three increasing, nonnegative, and continuous functionals on $\mathcal{K}$, satisfying the conditions for some $v > 0$ and $\tilde{M} > 0$ such that:*

$$\psi(x) \leqslant \theta(x) \leqslant \varphi(x), \quad \|x\| \leqslant \tilde{M}\psi(x)$$

*for all $x$ in $\overline{\mathcal{K}(\psi, \ell_3)}$.*

*Assume there exists a completely continuous operator $T : \overline{\mathcal{K}(\psi, \ell_3)} \to \mathcal{K}$ and $0 < \ell_1 < \ell_2 < \ell_3$ such that:*

(i) *$\psi(Tx) < \ell_3$ for all $x \in \partial P(\psi, \ell_3)$;*
(ii) *$\theta(Tx) > \ell_2$ for all $x \in \partial P(\theta, \ell_2)$;*
(iii) *$P(\varphi, \ell_1) \neq \emptyset$, and $\varphi(Tx) < \ell_1$ for all $x \in \partial P(\varphi, \ell_1)$.*

*Then, the operator $T$ has at least three fixed points, $x_1$, $x_2$, and $x_3$ in $\overline{\mathcal{K}(\psi, \ell_3)}$ such that:*

$$0 \leqslant \varphi(x_1) < \ell_1 < \varphi(x_2), \quad \theta(x_2) < \ell_2 < \theta(x_3), \quad \psi(x_3) < \ell_3.$$

Now, we define the following operator $\mathbb{B} \to \mathbb{B}$, for $\kappa_1 \in \mathbb{B}$, by

$$\begin{aligned} T\kappa_1(t) = \eta_1 \int_0^1 H(t, s_1) q_1(s_1) p_1 \Bigg( \eta_2 \int_0^1 H(s_1, s_2) q_2(s_2) p_2 \Bigg( \eta_3 \int_0^1 H(s_2, s_3) q_3(s_3) p_3 ... \\ p_{n-1} \Bigg( \eta_n \int_0^1 H(s_{n-1}, s_n) q_n(s_n) p_n(\kappa_1(s_n)) ds_n + \sum_{j=1}^{k} W_{nj}(s_{n-1}, t_j) \Bigg) ds_{n-1} \\ + \sum_{j=1}^{k} W_{n-1,j}(s_{n-2}, t_j) \Bigg) ds_{n-2} + ... + \sum_{j=1}^{k} W_{3j}(s_2, t_j) \Bigg) ds_2 \\ + \sum_{j=1}^{k} W_{2j}(s_1, t_j) \Bigg) ds_1 + \sum_{j=1}^{k} W_{1j}(t, t_j). \end{aligned} \tag{3.1}$$

Given (C1)-(C4), along with Lemmas 2.2, 2.3, and the definition of $T$, it is evident that for $\kappa_1(t) \in \mathcal{K}$, $T\kappa_1(t) \geqslant 0$, $(T\kappa_1)'(t) \geqslant 0$ *and* $T\kappa_1(t)$ is concave on $J$. Consequently, $T(\mathcal{K}) \subset \mathcal{K}$. Additionally, the *Arzela-Ascoli Theorem* demonstrates that the operator $T$ is completely continuous. We now focus on investigating the relevant fixed points of $T$ that are located within the cone $\mathcal{K}$.

We investigate the presence of at least three positive solutions for the impulsive boundary value problem (1.1) using the fixed point theorem discussed in [**15**].

Let $\tau \in \left(0, \frac{1}{2}\right)$ and define the increasing, nonnegative continuous functionals $\psi, \theta$ and $\varphi$ on $\mathcal{K}$ by

$$\psi(\kappa) = \max_{t \in [0,\tau]} \kappa(t) = \kappa(\tau),$$

$$\theta(\kappa) = \min_{t \in [\tau, 1-\tau]} \kappa(t) = \kappa(\tau),$$

$$\varphi(\kappa) = \max_{t \in [0,1]} \kappa(t) = \kappa(1).$$

For each $\kappa \in \mathcal{K}$, it is clear that

$$\psi(\kappa) = \theta(\kappa) \leqslant \varphi(\kappa).$$

Additionally, with the help of the Lemma 2.6, for each $\kappa \in \mathcal{K}$,

$$\|\kappa\|_{PC} \leqslant \frac{1}{\tau} \kappa(\tau) = \frac{1}{\tau} \psi(\kappa).$$

The notations that follow are provided for the convenience.
Suppose

$$\Omega_i = \eta_i \gamma \int_{\tau}^{1-\tau} H(s,s) q_i(s) ds,$$

$$\Lambda_i = \eta_i \left[ \int_0^1 H(s,s) q_i(s) ds + \frac{k}{\mu}(2\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4) \right].$$

THEOREM 3.1. *Assume that conditions (C1)-(C4) are satisfied. Let there exist positive numbers $\ell_1 < \ell_2 < \ell_3$ such that*

$$\ell_1 < \frac{\ell_1}{\tau} < \ell_2 < \frac{\Lambda_i}{\Omega_i} \ell_2 < \ell_3$$

*for $i = 1,2,...,n$ and $\tau \in \left(0, \frac{1}{2}\right)$. And for $i = 1,2,...,n$ and $j = 1,2,...,k$, assume that the functions $p_i$, $\bar{I}_{ij}$ and $\tilde{I}_{ij}$ satisfies the following conditions:*

(a) $p_i(t, \kappa_{i+1}(t)) < \frac{\ell_3}{\Lambda_i}$, $\bar{I}_{ij}(\kappa_{i+1}(t_j)) < \frac{\ell_3}{\Lambda_i}$, $\tilde{I}_{ij}(\kappa_{i+1}(t_j)) < \frac{\ell_3}{\Lambda_i}$ *for all* $(t, \kappa_{i+1}) \in [0,1] \times \left[0, \frac{\ell_3}{\tau}\right]$,

(b) $p_i(t, \kappa_{i+1}(t)) > \frac{\ell_2}{\Omega_i}$, *for all* $(t, \kappa_{i+1}) \in [\tau, 1-\tau] \times \left[\ell_2, \frac{\ell_3}{\tau}\right]$,

(c) $p_i(t, \kappa_{i+1}(t)) < \frac{\ell_1}{\Lambda_i}$, $\bar{I}_{ij}(\kappa_{i+1}(t_j)) < \frac{\ell_1}{\Lambda_i}$, $\tilde{I}_{ij}(\kappa_{i+1}(t_j)) < \frac{\ell_1}{\Lambda_i}$ *for all* $(t, \kappa_{i+1}) \in [0,1] \times \left[0, \frac{\ell_1}{\tau}\right]$.

*Then, the operator $T$ has at least three fixed points, $\kappa_i$, $\bar{\kappa}_i$ and $\tilde{\kappa}_i \in \overline{\mathcal{K}(\psi, \ell_3)}$ such that*

$$0 \leqslant \psi(\kappa_i) < \ell_1 < \psi(\bar{\kappa}_i),\ \theta(\bar{\kappa}_i) < \ell_2 < \theta(\tilde{\kappa}_i),\ \varphi(\tilde{\kappa}_i) < \ell_3,\ for\ i = 1,2,...,n.$$

PROOF. We identify the completely continuous operator $T$ as in (3.1). Thus, it is straightforward to verify that $T : \overline{\mathcal{K}(\psi, \ell_3)} \to \mathcal{K}$. Next, we demonstrate that all the criteria of Lemma 3.1 are met. To verify condition $(i)$ of Lemma 3.1, we select $\kappa_1 \in \mathcal{K}(\psi, \ell_3)$. Then $\psi(\kappa_1) = \ell_3$, i.e., $\psi(\kappa_1) = \max_{t \in [0,\tau]} \kappa_1(t) = \kappa_1(\tau) = \ell_3$, this means $0 \leqslant \kappa_1(t) \leqslant \ell_3$ for all $t \in [0, \tau]$.

If we recall that $\|\kappa_1\| \leqslant \frac{1}{\tau}\psi(\kappa_1) = \frac{1}{\tau}\ell_3 = \frac{\ell_3}{\tau}$. So, we get $0 \leqslant \kappa_1 \leqslant \frac{\ell_3}{\tau}$ for all $t \in [0,1]$.

Now, we can demonstrate that $\kappa_2(t) \in \left[0, \frac{\ell_3}{\tau}\right]$ for $t \in [0,1]$. For all $(s_{n-1}, \kappa_n) \in [0,1] \times \left[0, \frac{\ell_3}{\tau}\right]$, by using the assumption (a) in Theorem 3.1 and the Lemma 2.5, we obtain

$$\begin{aligned}
&\eta_n \int_0^1 H(s_{n-1}, s_n) q_n(s_n) p_n(\kappa_1(s_n)) ds_n + \sum_{j=1}^{k} W_{nj}(s_{n-1}, t_j) \\
&\geqslant \eta_n \int_0^1 H(s_{n-1}, s_n) q_n(s_n) p_n(\kappa_1(s_n)) ds_n \\
&\geqslant \eta_n \gamma \int_\tau^{1-\tau} H(s_n, s_n) q_n(s_n) p_n(\kappa_1(s_n)) ds_n \\
&\geqslant 0,
\end{aligned}$$

and by using the assumtion (a) in Theorem 3.1, we get

$$\begin{aligned}
&\eta_n \int_0^1 H(s_{n-1}, s_n) q_n(s_n) p_n(\kappa_1(s_n)) ds_n + \sum_{j=1}^{k} W_{nj}(s_{n-1}, t_j) \\
&\leqslant \eta_n \int_0^1 H(s_n, s_n) q_n(s_n) p_n(\kappa_1(s_n)) ds_n + \eta_n \frac{k}{\mu}(2\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4) \\
&\quad \cdot \max\{\bar{I}_{nj}(\kappa_1(t_j)), \tilde{I}_{nj}(\kappa_1(t_j))\} \\
&\leqslant \frac{\ell_3}{\Lambda_n} \eta_n \left[ \int_0^1 H(s_n, s_n) q_n(s_n) ds_n + \frac{k}{\mu}(2\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4) \right] \\
&= \ell_3 \\
&\leqslant \frac{\ell_3}{\tau}.
\end{aligned}$$

Then, we have $\kappa_n(t) \in \left[0, \frac{\ell_3}{\tau}\right]$ for all $t \in [0,1]$. By continuing with this bootstrapping argument, we can establish that $\kappa_2(t) \in \left[0, \frac{\ell_3}{\tau}\right]$ for all $t \in [0,1]$.

For all $(t, \kappa_1) \in [0,1] \times \left[0, \frac{\ell_3}{\tau}\right]$, by using the assumption (a) in Theorem 3.1, we obtain

$$\begin{aligned}
\psi(T\kappa_1) &= \max_{t \in [0,\tau]} T\kappa_1(t) \\
&= (T\kappa_1)(\tau) \\
&= \eta_1 \int_0^1 H(\tau, s) q_1(s) p_1(\kappa_2(s)) ds + \sum_{j=1}^{k} W_{1j}(\tau, t_j) \\
&\leqslant \frac{\ell_3}{\Lambda_1} \eta_1 \left[ \int_0^1 H(s,s) q_1(s) ds + \frac{k}{\mu}(2\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4) \right] \\
&= \ell_3.
\end{aligned}$$

Secondly, we show that (ii) of th Lemma 3.1 is satisfied. For this, we take $\kappa_1 \in \partial\mathcal{K}(\theta, \ell_2)$. Then, $\theta(\kappa_1) = \min_{t \in [\tau, 1-\tau]} \kappa_1(t) = \kappa_1(\tau) = \ell_2$, this means $\kappa_1(t) \geqslant \ell_2$

for all $t \in [\tau, 1-\tau]$. Noticing that, $\|\kappa_1\| \leqslant \frac{1}{\tau}\psi(\kappa_1) = \frac{1}{\tau}\theta(\kappa_1) = \frac{\ell_2}{\tau} \leqslant \frac{\ell_3}{\tau}$ for all $t \in [0,1]$. That is, we get $\ell_2 \leqslant \kappa_1(t) \leqslant \frac{\ell_3}{\tau}$ for $t \in [\tau, 1-\tau]$.

Now, we can show that $\kappa_2(t) \in \left[\ell_2, \frac{\ell_3}{\tau}\right]$ for $t \in [\tau, 1-\tau]$. For all $(s_{n-1}, \kappa_n) \in [\tau, 1-\tau] \times \left[\ell_2, \frac{\ell_3}{\tau}\right]$, by using the assumption (b) in Theorem 3.1, we obtain

$$\begin{aligned}
&\eta_n \int_0^1 H(s_{n-1}, s_n) q_n(s_n) p_n(\kappa_1(s_n)) ds_n + \sum_{j=1}^{k} W_{nj}(s_{n-1}, t_j)\\
&\geqslant \eta_n \int_0^1 H(s_{n-1}, s_n) q_n(s_n) p_n(\kappa_1(s_n)) ds_n\\
&\geqslant \eta_n \gamma \int_\tau^{1-\tau} H(s_n, s_n) q_n(s_n) p_n(\kappa_1(s_n)) ds_n\\
&\geqslant \frac{\ell_2}{\Omega_n}\left[\eta_n \delta \int_\tau^{1-\tau} H(s_n, s_n) q_n(s_n) ds_n\right]\\
&= \ell_2,
\end{aligned}$$

and by using the assumtion (a) in Theorem 3.1, we get

$$\begin{aligned}
&\eta_n \int_0^1 H(s_{n-1}, s_n) q_n(s_n) p_n(\kappa_1(s_n)) ds_n + \sum_{j=1}^{k} W_{nj}(s_{n-1}, t_j)\\
&\leqslant \eta_n \int_0^1 H(s_n, s_n) q_n(s_n) p_n(\kappa_1(s_n)) ds_n + \eta_n \frac{k}{\mu}(2\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4)\\
&\quad \cdot \max\{\bar{I}_{nj}(\kappa_1(t_j)), \tilde{I}_{nj}(\kappa_1(t_j))\}\\
&\leqslant \frac{\ell_3}{\Lambda_n}\eta_n\left[\int_0^1 H(s_n, s_n) q_n(s_n) ds_n + \frac{k}{\mu}(2\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4)\right]\\
&= \ell_3\\
&\leqslant \frac{\ell_3}{\tau}.
\end{aligned}$$

For all $(t, \kappa_1) \in [\tau, 1-\tau] \times \left[\ell_2, \frac{\ell_3}{\tau}\right]$, by using the assumption (b) in Theorem 3.1, we obtain

$$\begin{aligned}
\theta(T\kappa_1) &= \min_{t \in [\tau, 1-\tau]} T\kappa_1(t)\\
&= (T\kappa_1)(\tau)\\
&= \eta_1 \int_0^1 H(\tau, s) q_1(s) p_1(\kappa_2(s)) ds + \sum_{j=1}^{k} W_{1j}(\tau, t_j)
\end{aligned}$$

$$
\begin{aligned}
&\geqslant \eta_1 \gamma \int_\tau^{1-\tau} H(s,s) q_1(s) p_1(\kappa_2(s)) ds \\
&\geqslant \frac{\ell_2}{\Omega_1} \eta_1 \left[ \gamma \int_\tau^{1-\tau} H(s,s) q_1(s) ds \right] \\
&= \ell_2.
\end{aligned}
$$

Lastly, we show that the condition (iii) of the Lemma 3.1 is satisfied. We note that $\kappa_1(t) = \dfrac{\ell_1}{3}$ for $t \in J$ is a member of the set $\partial\mathcal{K}(\varphi, \ell_1)$. And so $\mathcal{K}(\varphi, \ell_1) \neq \emptyset$. Now, let $\kappa_1 \in \partial\mathcal{K}(\varphi, \ell_1)$. Then, $\varphi(\kappa_1) = \max\limits_{t\in[0,1]} \kappa_1(t) = \kappa_1(1) = \ell_1$. This implies $0 \leqslant \kappa_1(t) \leqslant \ell_1,\ t \in [0,1]$. Noticing that $\|\kappa_1\| \leqslant \dfrac{1}{\tau}\psi(\kappa_1) \leqslant \dfrac{1}{\tau}\varphi(\kappa_1) = \dfrac{\ell_1}{\tau}$. We get, $0 \leqslant \kappa_1(t) \leqslant \dfrac{\ell_1}{\tau}$ for $t \in [0,1]$. Now, we can demonstrate that $\kappa_2(t) \in \left[0, \dfrac{\ell_1}{\tau}\right]$ for $t \in [0,1]$. For all $(s_{n-1}, \kappa_n) \in [0,1] \times \left[0, \dfrac{\ell_1}{\tau}\right]$, by using the assumtion (b) in Theorem 3.1, we conclude that

$$
\begin{aligned}
&\eta_n \int_0^1 H(s_{n-1}, s_n) q_n(s_n) p_n(\kappa_1(s_n)) ds_n + \sum_{j=1}^k W_{nj}(s_{n-1}, t_j) \\
&\geqslant \eta_n \int_0^1 H(s_{n-1}, s_n) q_n(s_n) p_n(\kappa_1(s_n)) ds_n \\
&\geqslant \eta_n \gamma \int_\tau^{1-\tau} H(s_n, s_n) q_n(s_n) p_n(\kappa_1(s_n)) ds_n \\
&\geqslant 0,
\end{aligned}
$$

and by using the assumtion (c) in Theorem 3.1, we get

$$
\begin{aligned}
&\eta_n \int_0^1 H(s_{n-1}, s_n) q_n(s_n) p_n(\kappa_1(s_n)) ds_n + \sum_{j=1}^k W_{nj}(s_{n-1}, t_j) \\
&\leqslant \eta_n \int_0^1 H(s_n, s_n) q_n(s_n) p_n(\kappa_1(s_n)) ds_n + \eta_n \frac{k}{\mu}(2\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4) \\
&\quad \cdot \max\{\bar{I}_{nj}(\kappa_1(t_j)), \tilde{I}_{nj}(\kappa_1(t_j))\} \\
&\leqslant \frac{\ell_1}{\Lambda_n} \eta_n \left[ \int_0^1 H(s_n, s_n) q_n(s_n) ds_n + \frac{k}{\mu}(2\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4) \right] \\
&= \ell_1 \\
&\leqslant \frac{\ell_1}{\tau}.
\end{aligned}
$$

Then, we have $\kappa_n(t) \in \left[0, \frac{\ell_1}{\tau}\right]$ for all $t \in [0,1]$. By continuing with this bootstrapping argument, we can establish that $\kappa_2(t) \in \left[0, \frac{\ell_1}{\tau}\right]$ for all $t \in [0,1]$.

For all $(t, \kappa_1) \in [0,1] \times \left[0, \frac{\ell_1}{\tau}\right]$, by using the assumption (c) in Theorem 3.1, we obtain

$$\begin{aligned}\varphi(T\kappa_1) &= \max_{t\in[0,1]} T\kappa_1(t)\\ &= (T\kappa_1)(1)\\ &= \eta_1 \int_0^1 H(1,s)q_1(s)p_1(\kappa_2(s))ds + \sum_{j=1}^{k} W_{1j}(1,t_j)\\ &\leqslant \frac{\ell_1}{\Lambda_1}\eta_1\left[\int_0^1 H(s,s)q_1(s)ds + \frac{k}{\mu}(2\alpha_1+\alpha_2)(\alpha_3+\alpha_4)\right]\\ &= \ell_1.\end{aligned}$$

As a consequence, all conditions of Lemma 3.1 are satisfied. Hence, the operator $T$ has at least three fixed points $\kappa_1(t)$, $\bar{\kappa}_1$ and $\tilde{\kappa}_1 \in \overline{\mathcal{K}(\psi,\ell_3)}$. As a result, by setting $\kappa_{n+1}(t) = \kappa_1(t)$, $\bar{\kappa}_{n+1}(t) = \bar{\kappa}_1(t)$ and $\tilde{\kappa}_{n+1}(t) = \tilde{\kappa}_1(t)$, we obtain the existence of at least three positive solutions $(\kappa_1, \kappa_2, ..., \kappa_n)$, $(\bar{\kappa}_1, \bar{\kappa}_2, ..., \bar{\kappa}_n)$ and $(\tilde{\kappa}_1, \tilde{\kappa}_2, ..., \tilde{\kappa}_n)$ for the IBVP (1.1) given iteratively by

$$\kappa_r(t) = \eta_r \int_0^1 H(t,s)q_r(s)p_r(\kappa_{r+1}(s))ds + \sum_{j=1}^{k} W_{rj}(t,t_j), \quad r = n, n-1, ..., 1,$$

$$\bar{\kappa}_r(t) = \eta_r \int_0^1 H(t,s)q_r(s)p_r(\bar{\kappa}_{r+1}(s))ds + \sum_{j=1}^{k} W_{rj}(t,t_j), \quad r = n, n-1, ..., 1$$

and

$$\tilde{\kappa}_r(t) = \eta_r \int_0^1 H(t,s)q_r(s)p_r(\tilde{\kappa}_{r+1}(s))ds + \sum_{j=1}^{k} W_{rj}(t,t_j), \quad r = n, n-1, ..., 1$$

such that

$$0 \leqslant \varphi(\kappa_i) < \ell_1 < \varphi(\bar{\kappa}_i), \quad \theta(\bar{\kappa}_i) < \ell_2 < \theta(\tilde{\kappa}_i), \quad \psi(\tilde{\kappa}_i) < \ell_3$$

for $i = 1, 2, ..., n$. The proof of Theorem 3.1 is completed. □

## 4. An example

EXAMPLE 4.1. In the iterative system of the IBVP (1.1), suppose that $n = 3$, $k = 1$, $q_i(t) = 1 = \eta_i$ for $i = 1, 2, 3$, $\alpha_1 = 3$, $\alpha_2 = 2$, $\alpha_3 = 4$, $\alpha_4 = 1$ and

$t_1 = \dfrac{1}{3}$, i.e.,

$$(4.1)\qquad \begin{cases} \kappa_i''(t) + p_i(\kappa_{i+1}(t)) = 0,\ t \in J = [0,1], t \neq \frac{1}{3},\ i = 1,2,3, \\ \kappa_4 = \kappa_1(t), \\ \triangle \kappa_i|_{t=\frac{1}{3}} = \bar{I}_{i1}\left(\kappa_{i+1}\left(\dfrac{1}{3}\right)\right), \\ \triangle \kappa_i'|_{t=\frac{1}{3}} = -\tilde{I}_{i1}\left(\kappa_{i+1}\left(\dfrac{1}{3}\right)\right), \\ 3\kappa_i(0) - 2\kappa_i'(0) = 0, \\ 4\kappa_i(1) + \kappa_i'(1) = 0. \end{cases}$$

where

$$p_1(\kappa_2) = \begin{cases} 0,01, & \kappa_2 \in [0,25], \\ 128\kappa_2 - 3199,99, & \kappa_2 \in [25,30], \\ 640,01, & \kappa_2 \in [30, 15 \times 10^5], \end{cases}$$

$$p_2(\kappa_3) = \begin{cases} 0,003, & \kappa_3 \in [0,25], \\ 130\kappa_3 - 3249,997, & \kappa_3 \in [25,30], \\ 650,003, & \kappa_3 \in [30, 15 \times 10^5], \end{cases}$$

$$p_3(\kappa_1) = \begin{cases} 0,01, & \kappa_1 \in [0,25], \\ 127\kappa_1 - 3174,99, & \kappa_1 \in [25,30], \\ 635,01, & \kappa_1 \in [30, 15 \times 10^5]. \end{cases}$$

For $\kappa_i \geqslant 0$, $(i = 1,2,3)$

$$\bar{I}_{11}(\kappa_2) = \frac{\kappa_2}{10000}, \quad \bar{I}_{21}(\kappa_3) = \frac{\kappa_3}{6000}, \quad \bar{I}_{31}(\kappa_1) = \frac{\kappa_1}{8000},$$

$$\tilde{I}_{11}(\kappa_2) = \frac{\kappa_2}{5000}, \quad \tilde{I}_{21}(\kappa_3) = \frac{\kappa_3}{3000}, \quad \tilde{I}_{31}(\kappa_1) = \frac{\kappa_1}{4000}.$$

By simple calculation, we get $\mu = 23, \theta(t) = 3 + 6t, \phi(t) = 9 - 6t$ and

$$H(t,s) = \frac{1}{23}\begin{cases} (2+3s)(5-4t), & s \leqslant t, \\ (2+3t)(5-4s), & t \leqslant s. \end{cases}$$

Choosing the number $\tau = \dfrac{2}{5}$. With the help of the calculations, we have $\gamma = \min\left\{\dfrac{2+3\frac{2}{5}}{2+3}, \dfrac{1+4\frac{2}{5}}{1+4}\right\} = \min\left\{\dfrac{16}{25}, \dfrac{13}{25}\right\} = \dfrac{13}{25}$.

Also, we obtain $\Omega_i = \dfrac{6799}{143750} \approx 0,047297$ and $\Lambda_i = \dfrac{42339}{46} \approx 920,413043$ for $i = 1,2,3$. Taking $\ell_1 = 10,\ \ell_2 = 30$ and $\ell_3 = 6 \times 10^5$, we get

$$\ell_1 = 10 < \frac{\ell_1}{\tau} = 25 < \ell_2 = 30 < \frac{\Lambda_i}{\Omega_i}\ell_2 = 583803,68436 < \ell_3 = 6 \times 10^5.$$

It is clear that (C1)-(C4) has been satisfied. Next, we show that the all conditions of the Theorem 3.1 are also satisfied.

Firstly, we show that the condition $(a)$ of the Theorem 3.1 is satisfied for $i = 1,2,3$. For $(t,\kappa_{i+1}) \in [0,1] \times \left[0, 15 \times 10^5\right]$, we get $p_i(\kappa_{i+1}(t)) \leqslant 650,003 < \frac{\ell_3}{\Lambda_i} \approx 651,88124$, $\bar{I}_{ij}(\kappa_{i+1}(t_1)), \tilde{I}_{ij}(\kappa_{i+1}(t_1)) < 500 < \frac{\ell_3}{\Lambda_i} \approx 651,88124$.

Secondly, we show that the condition $(b)$ of the Theorem 3.1 is satisfied for $i = 1,2,3$. For $(t,\kappa_{i+1}) \in [\frac{2}{5}, \frac{3}{5}] \times \left[30, 15 \times 10^5\right]$, we get $p_i(\kappa_{i+1}(t)) \geqslant 635,01 > \frac{\ell_2}{\Omega_i} \approx 634,284453$.

Lastly, we show that the condition $(c)$ of the Theorem 3.1 is satisfied for $i = 1,2,3$. For $(t,\kappa_{i+1}) \in [0,1] \times \left[0, 25\right]$, we get $p_i(\kappa_{i+1}(t)) \leqslant 0,01 < \frac{\ell_1}{\Lambda_i} \approx 0,010864$, $\bar{I}_{ij}(\kappa_{i+1}(t_1)), \tilde{I}_{ij}(\kappa_{i+1}(t_1)) < 0,008333 < \frac{\ell_1}{\Lambda_i} \approx 0,010864$.

Then, all of Theorem 3.1's criteria are satisfied. Hence, $T$ has at least three fixed points $\kappa_1(t)$, $\bar{\kappa}_1(t)$ and $\tilde{\kappa}_1(t) \in \overline{\mathcal{K}(\psi, 6 \times 10^5)}$. As a result, by setting $\kappa_4(t) = \kappa_1(t)$, $\bar{\kappa}_4(t) = \bar{\kappa}_1(t)$ and $\tilde{\kappa}_4(t) = \tilde{\kappa}_1(t)$, we obtain the existence of at least three positive solutions $(\kappa_1, \kappa_2, \kappa_3)$, $(\bar{\kappa}_1, \bar{\kappa}_2, \bar{\kappa}_3)$ and $(\tilde{\kappa}_1, \tilde{\kappa}_2, \tilde{\kappa}_3)$ for the IBVP (4.1) given iteratively by

$$\kappa_r(t) = \eta_r \int_0^1 H(t,s) q_r(s) p_r(\kappa_{r+1}(s)) ds + \sum_{j=1}^{k} W_{rj}(t,t_j), \quad r = 3,2,1,$$

$$\bar{\kappa}_r(t) = \eta_r \int_0^1 H(t,s) q_r(s) p_r(\bar{\kappa}_{r+1}(s)) ds + \sum_{j=1}^{k} W_{rj}(t,t_j), \quad r = 3,2,1$$

and

$$\tilde{\kappa}_r(t) = \eta_r \int_0^1 H(t,s) q_r(s) p_r(\tilde{\kappa}_{r+1}(s)) ds + \sum_{j=1}^{k} W_{rj}(t,t_j), \quad r = 3,2,1$$

such that

$$0 \leqslant \varphi(\kappa_i) < 10 < \varphi(\bar{\kappa}_i), \quad \theta(\bar{\kappa}_i) < 30 < \theta(\tilde{\kappa}_i), \quad \psi(\tilde{\kappa}_i) < 6 \times 10^5$$

for $i = 1,2,3$.

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