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FIXED k-CASSINI OVAL RESULTS ON METRIC SPACES

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Abstract. In this paper, we generalize the well-known Cassini curves to metric spaces by introducing the concept of k-Cassini ovals. We provide various examples of k-Cassini ovals in different metric spaces, accompanied by illustrative shapes. Furthermore, we present the existence and uniqueness theorems for the transformation $T : X \to X$, establishing conditions under which T preserves the k-Cassini oval within the metric space (X, d). Finally, we demonstrate an application of this framework to the $LReLU$ (Leaky Rectified Linear Unit) function.

1. Introduction and background

Fixed Point Theory is a fundamental area of mathematics, particularly in the fields of analysis, topology, and geometry. The central idea is simple: a fixed point of a function T is a fixed point x in the domain such that

$$Tx = x.$$

This concept has profound applications in various branches of mathematics, as well as in applied fields like economics, computer science, and engineering. Several important theorems in fixed point theory guarantee the existence of fixed points under certain conditions. Here are some of the most well-known theorems:

- Banach Fixed Point Theorem [**2**],
- Brouwer Fixed Point Theorem [**5**],
- Schauder Fixed Point Theorem [**19**],
- Kakutani Fixed Point Theorem [**12**].

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In addition to theoretical studies, the application areas of fixed point theory are also important. Some examples of application areas include:

- Mathematical Economics,
- Numerical Methods,
- Differential Equations,
- Dynamical Systems,
- Computer Science,
- Control Theory,
- Topology and Geometry,
- Optimization.

Fixed point theory is a vast and powerful area of mathematics with a wide range of applications in both theoretical and applied contexts. From proving the existence of solutions in differential equations to applications in economics, computer science, and control theory, fixed point results provide the foundation for solving many complex problems. Key theorems such as Banach's Contraction Theorem, Brouwer's Fixed Point Theorem, and Kakutani's Theorem have had a profound impact on fields ranging from game theory to numerical analysis.

When the number of fixed points of a given function $T : X \to X$ is more than one, investigating a geometric interpretation of the set of fixed points has recently brought a geometric perspective to fixed point theory. This approach was initially explored under the fixed circle problem [**17**] and later expanded under the fixed figure problem ([**20**] and the references therein).

The study of the invariance of different geometric shapes under the fixed figure problem has become an important aspect of the geometry of fixed point theory. Each geometric shape studied sheds light on a new area of research. For example, concepts such as fixed ellipses, fixed hyperbolas, fixed Cassini curves, and fixed k-ellipses have been explored, and with the help of these concepts, new results regarding fixed figures from different perspectives have been contributed to the literature ([**1**], [**7**], [**9**], [**20**] and the references therein).

A Cassini oval (or Cassini curve) is a type of plane curve defined by a specific equation in terms of distances to two fixed points (foci). The family of k-Cassini ovals is a generalized version of this concept, which depends on a parameter k that alters the shape of the curve. A k-Cassini oval can be understood as a generalization of the standard Cassini oval, where a parameter k is introduced to scale or transform the equation of the oval. Specifically, the k-Cassini oval may be defined through a variation of the original equation, which results in a family of curves that continuously change their shape depending on the value of k (see, [**3**], [**4**] and [**6**] for more details).

Considering all the works mentioned above, in this paper, we aim to derive new fixed figure theorems using the concept of the k-Cassini oval. To achieve this, we first provide examples of k-Cassini ovals on different metric space structures, ensuring that the concept is thoroughly understood. Then, using the concept of the k-Cassini oval, new fixed figure theorems are stated and proven. These proven theorems are the existence and uniqueness theorems for the fixed k-Cassini oval.

Furthermore, to demonstrate the consistency of the obtained theoretical results, providing examples and a theorem excluding the identity function are presented. Finally, to show the applicability of the theoretical results, an application to activation functions is provided.

2. Main results

In this section, we give some fixed k-Cassini oval theorems on metric spaces with necessary examples.

DEFINITION 2.1. Let (X, d) be a metric space. Then the k-Cassini oval (or k-Cassini curve) is defined by

$$C[x_1, \ldots, x_k; r] = \left\{ x \in X : \prod_{i=1}^{k} d(x, x_i) = r \right\}.$$

REMARK 2.1. If we take $k = 1$ then we get a circle and if we take $k = 2$ then we get a Cassini oval on metric spaces.

EXAMPLE 2.1. Let $\left(X = \mathbb{R}^2, d\right)$ be a metric space with the metric $d : X \times X \to \mathbb{R}$ defined as

$$d(a, b) = |u_1 - v_1| + |u_2 - v_2|,$$

such that $a = (u_1, u_2), b = (v_1, v_2) \in X$. Let us define two 3-Cassini ovals with multi-focal points

$$\{x_1 = (-1/2, 0), x_2 = (0, 0), x_3 = (0, 1)\} \text{ for } C_1$$

and

$$\{x_1 = (-1, 0), x_2 = (0, 0), x_3 = (0, 1)\} \text{ for } C_2$$

as follows (see Figure 1) :

$$C_1[x_1, x_2, x_3; r] = \{p(x, y) \in X : (|x + 1/2| + |y|)(|x| + |y|)(|x| + |y - 1|) = r\},$$
$$C_2[x_1, x_2, x_3; r] = \{p(x, y) \in X : (|x + 1| + |y|)(|x| + |y|)(|x| + |y - 1|) = r\}.$$

EXAMPLE 2.2. Let $\left(X = \mathbb{R}^2, d\right)$ be a metric space with the metric $d : X \times X \to \mathbb{R}$ defined as

$$d(a, b) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2},$$

such that $a = (u_1, u_2), b = (v_1, v_2) \in X$. Let us define a 3-Cassini oval for $x_1 = (3, 0)$, $x_2 = (0, 0)$, $x_3 = (0, 4)$ as follows (see Figure 2) :

$$C[x_1, x_2, x_3; r] = \left\{ p(x, y) \in X : \sqrt{(x - 3)^2 + y^2}.\sqrt{x^2 + y^2}.\sqrt{x^2 + (y - 4)^2} = r \right\}.$$

EXAMPLE 2.3. Let $\left(X = \mathbb{R}^2, d\right)$ be a metric space with the metric $d : X \times X \to \mathbb{R}$ defined as

$$d(a, b) = \max\{|u_1 - v_1|, |u_2 - v_2|\},$$

such that $a = (u_1, u_2), b = (v_1, v_2) \in X$. Let us define a 3-Cassini oval for $x_1 = (1, 0)$, $x_2 = (0, 0)$, $x_3 = (0, 1)$ as follows (see Figure 3) :

$$C[x_1, x_2, x_3; r] = \{p(x, y) \in X : \max\{|x - 1|, |y|\} . \max\{|x|, |y|\} . \max\{|x|, |y - 1|\} = r\}.$$

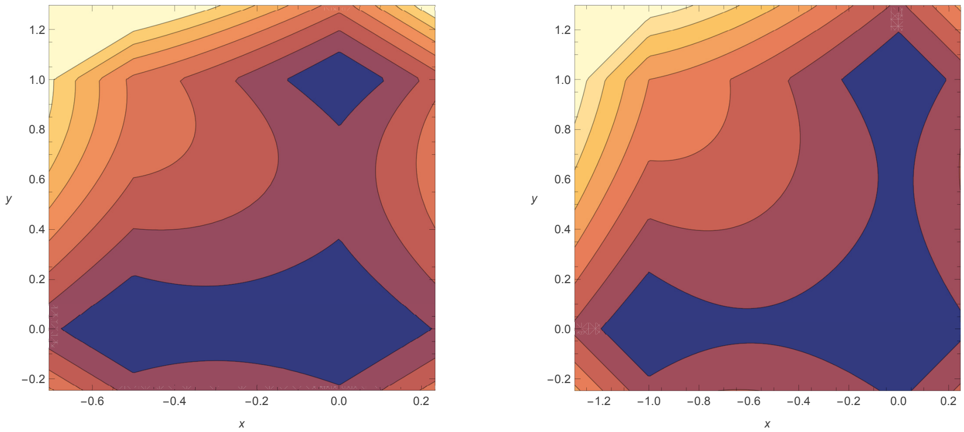

FIGURE 1. The 3-Cassini ovals C_1 (left) and C_2 (right)

EXAMPLE 2.4. Let $\left(X = \mathbb{R}^3, d\right)$ be a metric space with the metric $d : X \times X \to \mathbb{R}$ defined as

$$d(a,b) = |u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|,$$

such that $a = (u_1, u_2, u_3), b = (v_1, v_2, v_3) \in X$. Let us define a 3-Cassini oval for $x_1 = (1,0,0), x_2 = (0,1,0), x_3 = (0,0,1)$ as follows (see Figure 4) :

$$\begin{aligned} C[x_1, x_2, x_3; r] &= \{p(x,y,z) \in X : (|x-1| + |y| + |z|) . (|x| + |y-1| + |z|) \\ &\qquad . (|x| + |y| + |z-1|) = r\}. \end{aligned}$$

EXAMPLE 2.5. Let $\left(X = \mathbb{R}^3, d\right)$ be a metric space with the metric $d : X \times X \to \mathbb{R}$ defined as

$$d(a,b) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2},$$

such that $a = (u_1, u_2, u_3), b = (v_1, v_2, v_3) \in X$. Let us define a 3-Cassini oval for $x_1 = (5,0,0), x_2 = (0,2,0), x_3 = (0,0,1)$ as follows (see Figure 5) :

$$C[x_1, x_2, x_3; r] = \left\{ p(x,y,z) \in X : \sqrt{(x-5)^2 + y^2 + z^2} \right.$$

$$\left. .\sqrt{x^2 + (y-2)^2 + z^2} . \sqrt{x^2 + y^2 + (z-1)^2} = r \right\}.$$

EXAMPLE 2.6. Let $\left(X = \mathbb{R}^3, d\right)$ be a metric space with the metric $d : X \times X \to \mathbb{R}$ defined as

$$d(a,b) = \sqrt[4]{(u_1 - v_1)^4 + (u_2 - v_2)^4 + (u_3 - v_3)^4},$$

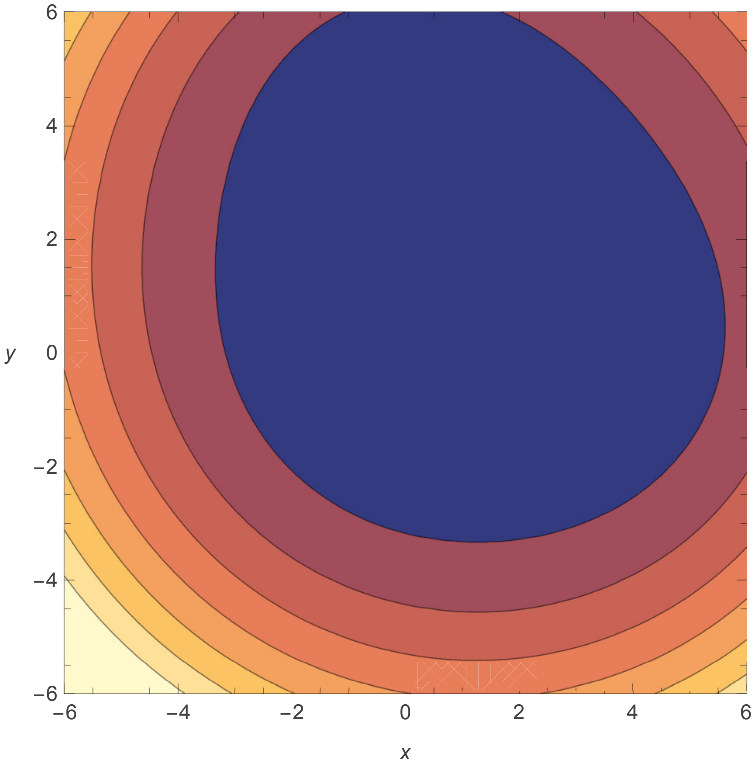

FIGURE 2. The 3-Cassini oval for $x_1 = (3, 0)$, $x_2 = (0, 0)$, $x_3 = (0, 4)$.

such that $a = (u_1, u_2, u_3)$, $b = (v_1, v_2, v_3) \in X$. Let us define a 3-Cassini oval for $x_1 = (-1, 0, 0)$, $x_2 = (1, 0, 0)$, $x_3 = (0, 1, 0)$ as follows (see Figure 6) :

$$C[x_1, x_2, x_3; r] = \Big\{ p(x, y, z) \in X : \sqrt[4]{(x+1)^4 + y^4 + z^4} . \sqrt[4]{(x-1)^4 + y^4 + z^4} . \sqrt[4]{x^4 + (y-1)^4 + z^4} = r \Big\}.$$

EXAMPLE 2.7. Let $(X = \mathbb{R}^2, d)$ be a metric space with the metric $d : X \times X \to \mathbb{R}$ defined as

$$d(a, b) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2},$$

such that $a = (u_1, u_2)$, $b = (v_1, v_2) \in X$. Let us define a 4-Cassini oval for $x_1 = (1, 0)$, $x_2 = (0, 0)$, $x_3 = (0, 2)$, $x_4 = (-3, 1)$ as follows (see Figure 7) :

$$C[x_1, x_2, x_3, x_4; r] = \Big\{ p(x, y) \in X : \sqrt{(x-1)^2 + y^2} . \sqrt{x^2 + y^2} . \sqrt{x^2 + (y-2)^2} . \sqrt{(x+3)^2 + (y-1)^2} = r \Big\}.$$

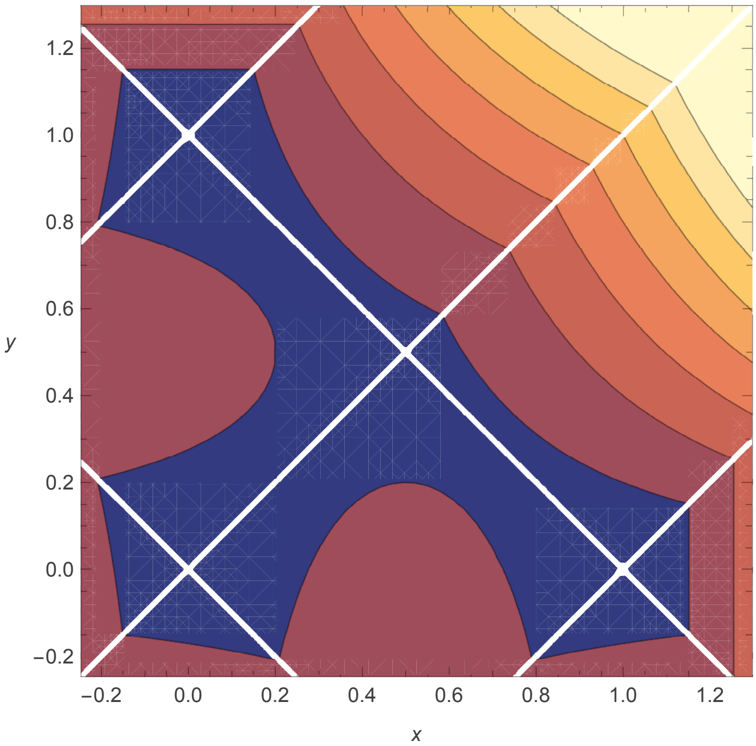

FIGURE 3. The 3-Cassini oval for $x_1 = (1, 0)$, $x_2 = (0, 0)$, $x_3 = (0, 1)$.

EXAMPLE 2.8. Let $(X = \mathbb{R}^2, d_1)$ and $(X = \mathbb{R}^2, d_2)$ be two metric spaces with the metrics $d_{1,2} : X \times X \to \mathbb{R}$ defined as

$$d_1(a, b) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2},$$

$$d_2(a, b) = |u_1 - v_1| + |u_2 - v_2|,$$

such that $a = (u_1, u_2), b = (v_1, v_2) \in X$. Now, let us define two 4-Cassini ovals for $x_1 = (1, 1)$, $x_2 = (-1, 1)$, $x_3 = (-1, -1)$, $x_4 = (1, -1)$ as follows (see Figure 8) :

$$C_1[x_1, x_2, x_3, x_4; r] = \left\{p(x, y) \in X : \sqrt{(x-1)^2 + (y-1)^2}.\sqrt{(x+1)^2 + (y-1)^2}\right.$$
$$\left..\sqrt{(x+1)^2 + (y+1)^2}.\sqrt{(x-1)^2 + (y+1)^2} = r\right\},$$

$$C_2[x_1, x_2, x_3, x_4; r] = \{p(x, y) \in X : (|x-1| + |y-1|).(|x+1| + |y-1|)$$
$$.(|x+1| + |y+1|).(|x-1| + |y+1|) = r\}.$$

Notice that C_1 also contains a sinusoidal spiral when $r = a^4 = 4$; since $x_1, ..., x_4$ are chosen as corners of a regular n-gon (a square in this case). Here $a = \sqrt{2}$ denotes the radius of the square with respect to the d_1 metric.

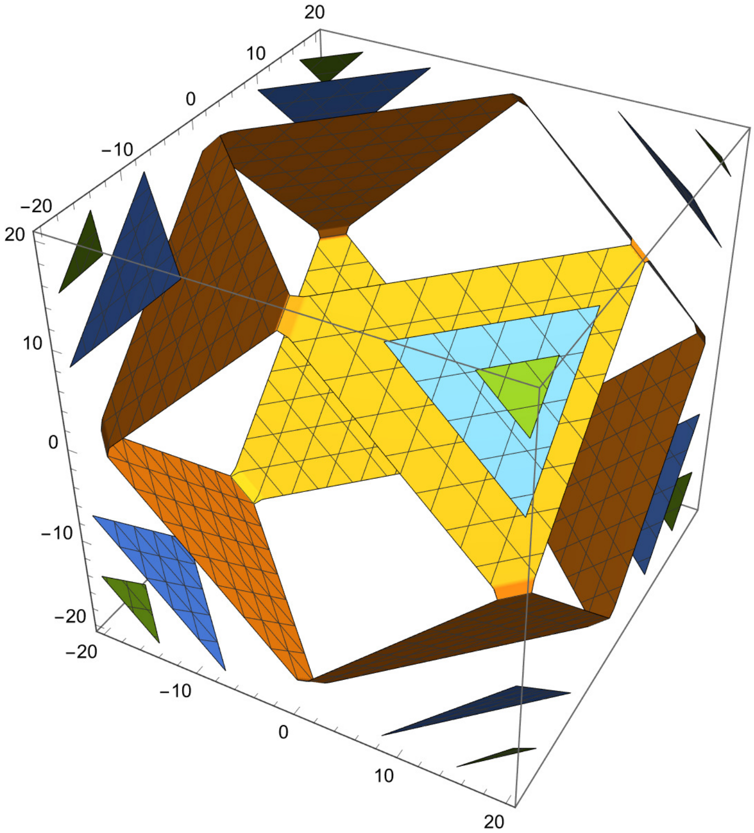

FIGURE 4. The 3-Cassini oval for $x_1 = (1, 0, 0)$, $x_2 = (0, 1, 0)$, $x_3 = (0, 0, 1)$.

PROPOSITION 2.1. *Let (X, d) be a metric space, and $C[x_1, \ldots, x_k; r]$ and $C'[x'_1, \ldots, x'_k; r']$ two k-Cassini ovals. Then there exists at least one self-mapping $T : X \to X$ such that T fixes the k-Cassini ovals $C = C[x_1, \ldots, x_k; r]$ and $C' = C'[x'_1, \ldots, x'_k; r']$.*

PROOF. Let us define the self mapping $T : X \to X$ as

$$Tx = \begin{cases} x, & x \in C \cup C' \\ \alpha, & \text{otherwise} \end{cases}$$

for all $x \in X$, where α is a constant such that $\prod_{i=1}^{k} d(\alpha, x_i) \neq r$ and $\prod_{i=1}^{k} d(\alpha, x'_i) \neq r'$. It is clear that for all $x \in C \cup C'$, x is a fixed point of T. As a result, T fixes both C and C' as a whole. □

Proposition 2.1 can be generalized as follows:

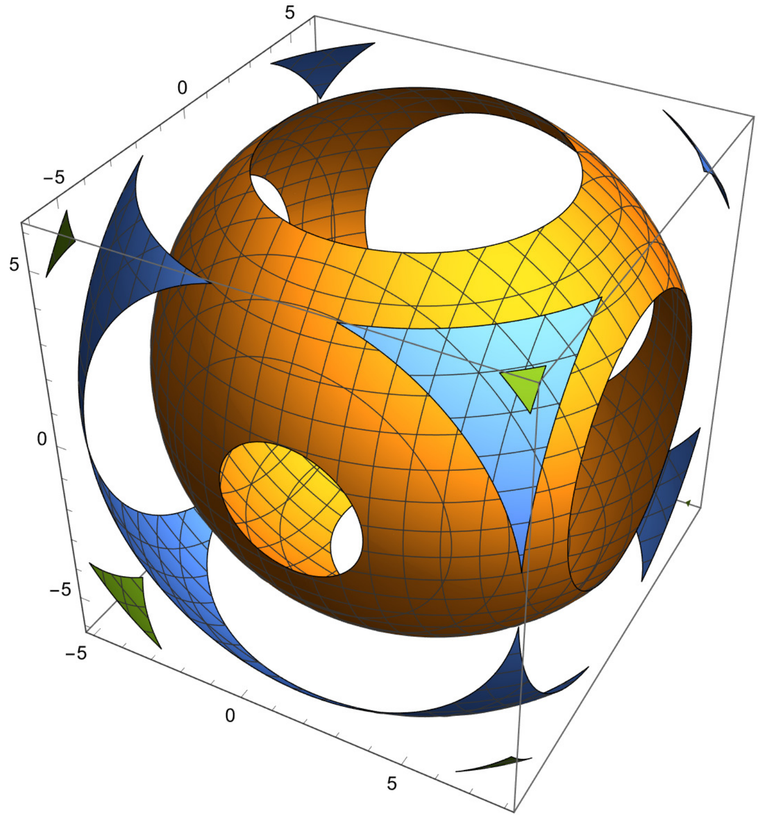

FIGURE 5. The 3-Cassini oval for $x_1 = (5,0,0)$, $x_2 = (0,2,0)$, $x_3 = (0,0,1)$.

PROPOSITION 2.2. *Let (X, d) be a metric space, and $C[x_1, \ldots, x_k; r]$, ..., $C^n[x_1^n, \ldots, x_k^n; r^n]$ any k-Cassini ovals. Then there exists at least one self-mapping $T : X \to X$ such that T fixes the k-Cassini ovals $C[x_1, \ldots, x_k; r]$, ..., $C^n[x_1^n, \ldots, x_k^n; r^n]$.*

PROOF. By the similar arguments used in the proof of Proposition 2.1, it can be easily seen. □

From Propositions 2.1 and 2.2, it is important to search the existence and uniqueness conditions of the fixed k-Cassini ovals. We can say that the fixed k-Cassini oval does not have to be unique as seen from the above propositions. For this purpose, we give the following theorems:

THEOREM 2.1. *Let (X, d) be a metric space and $C[x_1, \ldots, x_k; r]$ any k-Cassini oval on X. Let us define the mapping $\mu : X \to [0, \infty)$ as*

$$\mu(x) = \prod_{i=1}^{k} d(x, x_i),$$

for all $x \in X$. If there exists a self-mapping $T : X \to X$ such that

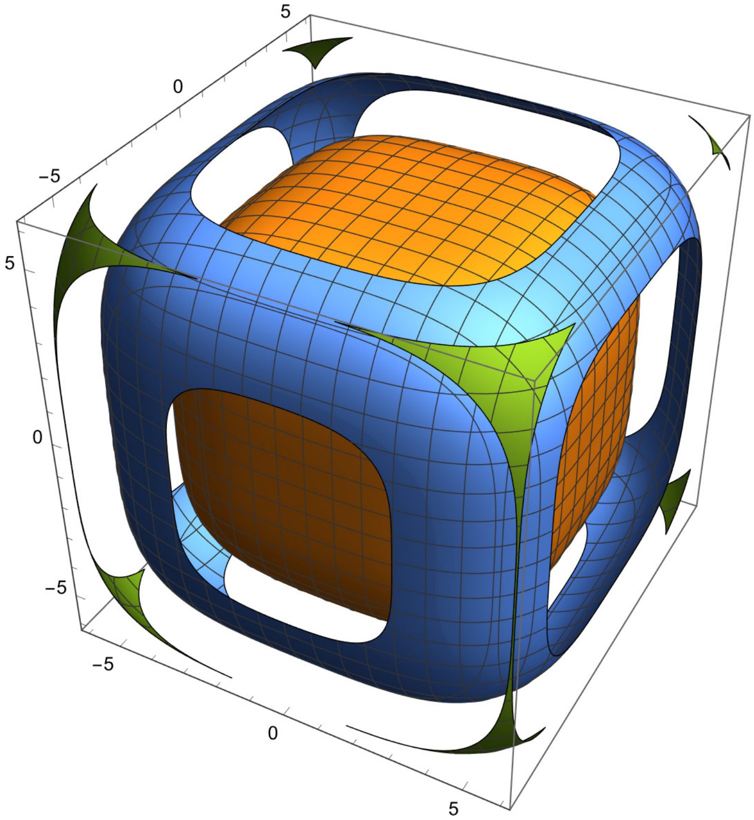

FIGURE 6. The 3-Cassini oval for $x_1 = (-1, 0, 0)$, $x_2 = (1, 0, 0)$, $x_3 = (0, 1, 0)$.

(C_k1) *$d(x, Tx) \leqslant \mu(x) - \mu(Tx)$ for all $x \in C[x_1, \ldots, x_k; r]$,*

(C_k2) *$\prod_{i=1}^{k} d(Tx, x_i) \geqslant r$ for all $x \in C[x_1, \ldots, x_k; r]$,*

(C_k3) *$d(Tx, Ty) \leqslant hd(x, y)$ for all $x \in C[x_1, \ldots, x_k; r]$, $y \in X - C[x_1, \ldots, x_k; r]$ and some $h \in (0, 1)$;*

then $C[x_1, \ldots, x_k; r]$ is the unique fixed k-Cassini oval of T.

PROOF. Now, we prove the existence of a fixed k-Cassini oval of T. Let $x \in C[x_1, \ldots, x_k; r]$. Using the hypothesis, we obtain

$$\begin{aligned} d(x, Tx) &\leqslant \mu(x) - \mu(Tx) \\ &= \prod_{i=1}^{k} d(x, x_i) - \prod_{i=1}^{k} d(Tx, x_i) \\ &= r - \prod_{i=1}^{k} d(Tx, x_i) \leqslant r - r = 0 \end{aligned}$$

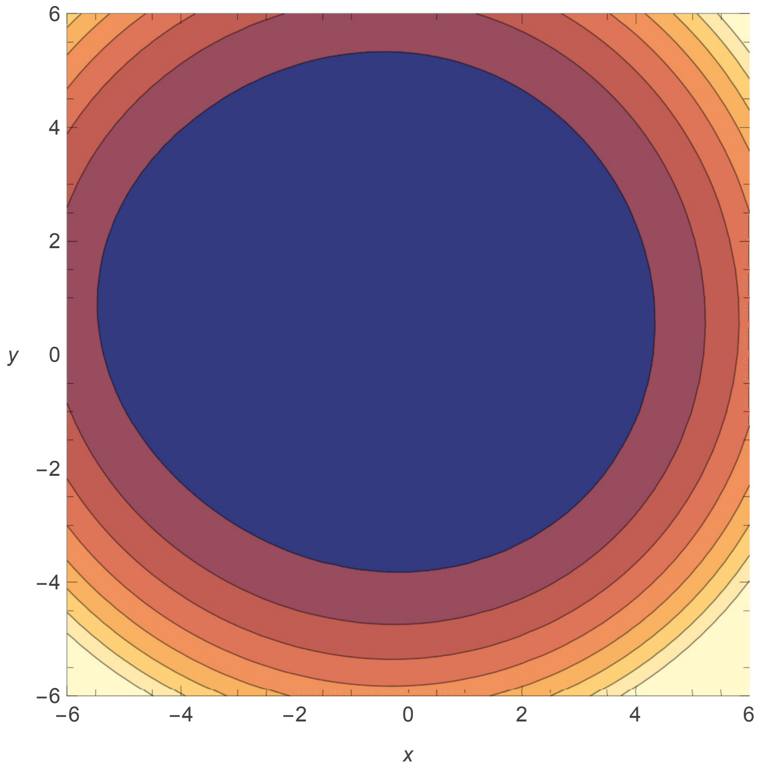

FIGURE 7. The 4-Cassini oval for $x_1 = (1, 0)$, $x_2 = (0, 0)$, $x_3 = (0, 2)$, $x_4 = (-3, 1)$.

and so x is a fixed point of T. Hence, $C[x_1, \ldots, x_k; r]$ is a fixed k-Cassini oval of T.

Finally, we show the uniqueness of a fixed k-Cassini oval. On the contrary, we assume that $C[x_1', \ldots, x_k'; r']$ is another fixed k-Cassini oval of T. Let $x \in C[x_1, \ldots, x_k; r]$ and $y \in C[x_1', \ldots, x_k'; r']$ such that $x \neq y$. Using the condition (C_k3), we get

$$d(Tx, Ty) = d(x, y) \leqslant hd(x, y),$$

a contradiction with $h \in (0, 1)$. It should be $x = y$. Consequently, $C[x_1, \ldots, x_k; r]$ is a unique fixed k-Cassini oval of T. □

REMARK 2.2. (i) Theorem 2.1 generalizes the given fixed-circle theorem in [**17**] and the given fixed Cassini curve result in [**7**].

(ii) The condition (C_k1) guarantees that Tx is not in the exterior of the k-Cassini oval $C = C[x_1, \ldots, x_k; r]$ for each $x \in C$ and the condition (C_k2) guarantees that Tx is not in the interior of the k-Cassini oval C for each $x \in C$. So we have $T(C) \subseteq C$.

(iii) The condition (C_k3) can be considered as Banach type contractive condition.

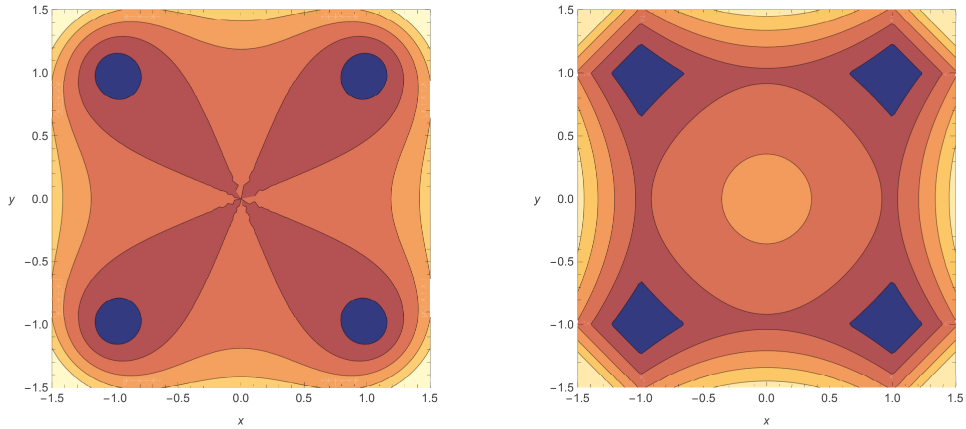

FIGURE 8. The 4-Cassini ovals C_1 (left) and C_2 (right)

EXAMPLE 2.9. Let (X, d) be a metric space, $C_1 = C[x_1, \ldots, x_k; r]$, $C_2 = C[x_1', \ldots, x_k'; r']$ any two k-Cassini ovals and α a constant such that

$$\prod_{i=1}^{k} d(\alpha, x_i) \neq r \text{ and } \prod_{i=1}^{k} d(\alpha, x_i') \neq r'.$$

Let us define the self-mapping $T : X \to X$ as

$$Tx = \begin{cases} x, & x \in C_1 \cup C_2 \\ \alpha, & \text{otherwise} \end{cases}$$

for all $x \in X$. Then T satisfies the conditions (C_k1) and (C_k2), but T does not satisfy the condition (C_k3). T fixes two k-Cassini ovals C_1 and C_2.

In the following example, we see that T has a unique fixed k-Cassini oval.

EXAMPLE 2.10. Let (X, d) be a metric space, $C = C[x_1, \ldots, x_k; r]$ any k-Cassini oval and α a constant such that

$$3d(x, \alpha) < d(x, y),$$

for all $x \in C$ and $y \in X - C$. Let us define the self mapping $T : X \to X$ as

$$Tx = \begin{cases} x, & x \in C \\ \alpha, & \text{otherwise} \end{cases}$$

for all $x \in X$. Then T satisfies the conditions (C_k1), (C_k2) and (C_k3). Hence C is a unique fixed k-Cassini oval of T.

THEOREM 2.2. *Let (X, d) be a metric space, $C = C[x_1, \ldots, x_k; r]$ any k-Cassini oval on X and the mapping $\mu : X \to [0, \infty)$ defined as in Theorem 2.1. If there exists a self-mapping $T : X \to X$ such that*

$(C_k'1)$ $d(x, Tx) \leqslant \mu(x) + \mu(Tx) - 2r$ *for all* $x \in C[x_1, \ldots, x_k; r]$,

($C_k'2$) $\prod_{i=1}^{k} d(Tx, x_i) \leqslant r$ *for all* $x \in C[x_1, \ldots, x_k; r]$,

($C_k'3$) $d(Tx, Ty) \leqslant h \max\{d(x, y), d(Tx, x), d(Ty, y)\}$ *for all* $x \in C[x_1, \ldots, x_k; r]$, $y \in X - C[x_1, \ldots, x_k; r]$ *and some* $h \in (0, 1)$,

then $C[x_1, \ldots, x_k; r]$ *is a unique fixed* k*-Cassini oval of* T.

Proof. Now, we prove the existence of a fixed k-Cassini oval of T. Let $x \in C[x_1, \ldots, x_k; r]$. Using the hypothesis, we get

$$\begin{aligned} d(x, Tx) &\leqslant \mu(x) + \mu(Tx) - 2r \\ &= \prod_{i=1}^{k} d(x, x_i) + \prod_{i=1}^{k} d(Tx, x_i) - 2r \\ &= r + \prod_{i=1}^{k} d(Tx, x_i) - 2r \\ &= \prod_{i=1}^{k} d(Tx, x_i) - r \leqslant r - r = 0 \end{aligned}$$

and so x is a fixed point of T. Hence $C[x_1, \ldots, x_k; r]$ is a fixed k-Cassini oval of T.

Finally, we show the uniqueness of $C[x_1, \ldots, x_k; r]$. Let us assume $C'[x_1', \ldots, x_k'; r']$ is another fixed k-Cassini oval of T. Let $x \in C[x_1, \ldots, x_k; r]$ and $y \in C'[x_1', \ldots, x_k'; r']$ such that $x \neq y$. Using the condition ($C_k'3$), we obtain

$$\begin{aligned} d(Tx, Ty) &= d(x, y) \leqslant h \max\{d(x, y), d(Tx, x), d(Ty, y)\} \\ &= hd(x, y), \end{aligned}$$

a contradiction with $h \in (0, 1)$. Thus we have $x = y$. Consequently, $C[x_1, \ldots, x_k; r]$ is unique. □

Remark 2.3. i) Theorem 2.2 generalizes the given fixed-circle theorem in [**17**] and the given fixed-Cassini curve result in [**7**].

ii) The condition ($C_k'1$) guarantees that Tx is not in the interior of the k-Cassini oval $C = C[x_1, \ldots, x_k; r]$ for each $x \in C$ and the condition ($C_k'2$) guarantees that Tx is not in the exterior of the k-Cassini oval C for each $x \in C$. So we have $T(C) \subseteq C$.

Example 2.11. Let (X, d) be a metric space, $C = C[x_1, \ldots, x_k; r]$ any k-Cassini oval and α a constant such that

$$d(y, \alpha) < d(x, y)$$

and

$$2d(x, \alpha) < d(x, y),$$

for all $x \in C$ and $y \in X - C$. Let us define the self mapping $T : X \to X$ as

$$Tx = \begin{cases} x, & x \in C \\ \alpha, & \text{otherwise} \end{cases}$$

for all $x \in X$. Then T satisfies the conditions ($C_k'1$), ($C_k'2$) and ($C_k'3$). So C is a unique fixed k-Cassini oval of T.

THEOREM 2.3. *Let (X,d) be a metric space, $C=C[x_1,\ldots,x_k;r]$ any k-Cassini oval on X and the mapping $\mu : X \to [0,\infty)$ defined as in Theorem 2.1. If there exists a self-mapping $T : X \to X$ satisfying the conditions (C_k1), (C_k3) and*

$(C_k''2)$ *$\beta d(x,Tx)+\prod_{i=1}^{k} d(Tx,x_i) \geqslant r$ for each $x \in C[x_1,\ldots,x_k;r]$ and some $\beta \in [0,1)$,*

then $C[x_1,\ldots,x_k;r]$ is a unique fixed k-Cassini oval of T.

PROOF. We prove the existence of a fixed k-Cassini oval of T. Let $x \in C[x_1,\ldots,x_k;r]$. Using the hypothesis, we find

$$\begin{aligned} d(x,Tx) &\leqslant \mu(x)-\mu(Tx) \\ &= \prod_{i=1}^{k} d(x,x_i) - \prod_{i=1}^{k} d(Tx,x_i) \\ &= r - \prod_{i=1}^{k} d(Tx,x_i) \\ &\leqslant r - r + \beta d(x,Tx) = \beta d(x,Tx) \end{aligned}$$

and since $\beta \in [0,1)$, then it should be $x = Tx$. Hence $C[x_1,\ldots,x_k;r]$ is a fixed k-Cassini oval of T.

Using similar arguments given in the proof of Theorem 2.1, the uniqueness of a fixed k-Cassini oval can be easily seen. □

REMARK 2.4. *i*) Theorem 2.3 generalizes the given fixed-circle theorem in [**17**] and the given fixed-Cassini curve result in [**7**].

ii) The condition $(C_k''2)$ implies that Tx can lie on or the exterior or interior of the k-Cassini oval.

iii) If we consider Example 2.10, then the conditions of Theorem 2.3 are satisfied by T defined in Example 2.10.

THEOREM 2.4. *Let (X,d) be a metric space and $C[x_1,\ldots,x_k;r]$ any k-Cassini oval on X. Let us define the mapping $\phi : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ as*

$$\phi(x) = \begin{cases} x-r, & x>0 \\ 0, & x=0 \end{cases}$$

for all $x \in \mathbb{R}^+ \cup \{0\}$. If there exists a self-mapping $T : X \to X$ satisfying

$(C_k'''1)$ *$\prod_{i=1}^{k} d(Tx,x_i) = r$ for each $x \in C[x_1,\ldots,x_k;r]$,*

$(C_k'''2)$ *$d(Tx,Ty) > r$ for each $x,y \in C[x_1,\ldots,x_k;r]$ with $x \neq y$,*

$(C_k'''3)$ *$d(Tx,Ty) \leqslant d(x,y) - \phi(d(x,Tx))$ for each $x,y \in C[x_1,\ldots,x_k;r]$,*

$(C_k'''4)$ *$d(Tx,Ty) < \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$ for each $x \in C[x_1,\ldots,x_k;r]$ and $y \in X - C[x_1,\ldots,x_k;r]$,*

then $C[x_1,\ldots,x_k;r]$ is a unique fixed k-Cassini oval of T.

PROOF. To prove the existence of a fixed k-Cassini oval of T, we suppose that $x \in C[x_1,\ldots,x_k;r]$. Let x be any point in X such that $x \neq Tx$.Using the condition $(C_k'''2)$, we have

$$d\left(Tx, T^2(x)\right) > r, \tag{2.1}$$

and using ($C_k'''3$), we get

$$\begin{aligned} d\left(Tx,T^{2}\left(x\right)\right) &\leqslant d\left(x,Tx\right)-\phi\left(d\left(x,Tx\right)\right) \\ &= d\left(x,Tx\right)-d\left(x,Tx\right)+r=r, \end{aligned}$$

a contradiction with (2.1). Hence it should be $x = Tx$ and so $C[x_1, \dots, x_k; r]$ is a fixed k-Cassini oval of T.

Finally, we show the uniqueness of the fixed k-Cassini oval $C[x_1, \dots, x_k; r]$ of T. To do this, let $C'[x_1', \dots, x_k'; r']$ be another fixed k-Cassini oval of T, $x \in C[x_1, \dots, x_k; r]$ and $y \in C'[x_1', \dots, x_k'; r']$ such that $x \neq y$. By ($C_k'''4$), we get

$$\begin{aligned} d\left(Tx,Ty\right) &= d\left(x,y\right)<\max\left\{d\left(x,y\right),d\left(x,Tx\right),d\left(y,Ty\right),d\left(x,Ty\right),d\left(y,Tx\right)\right\} \\ &= \max\left\{d\left(x,y\right),0,0,d\left(x,y\right),d\left(y,x\right)\right\} \\ &= d\left(x,y\right), \end{aligned}$$

a contradiction. It should be $x = y$. Consequently, $C[x_1, \dots, x_k; r]$ is a unique fixed k-Cassini oval of T. □

Remark 2.5. (i) Theorem 2.4 is a generalization of a fixed-circle theorem in [**17**].

(ii) The condition ($C_k'''4$) can be considered as Rhoades type contractive condition [**18**].

Example 2.12. Let $X = \{-1, 0, 1, 5, 120, 150\}$ be the usual metric space. Let us take a 3-Cassini oval $C[-1, 0, 1; 120]$ such as

$$C\left[-1,0,1;120\right]=\left\{x\in X:\ \left|x+1\right|.\left|x\right|.\left|x-1\right|=120\right\}=\left\{5\right\}.$$

Let us define the self-mapping $T : X \to X$ as

$$Tx=5,$$

for all $x \in X$. Then T satisfies the conditions of Theorem 2.4 and so $C[-1, 0, 1; 120]$ is a unique fixed 3-Cassini oval of T.

Now we investigate a contraction that excludes the identity map $I_X : X \to X$ defined by $I_X(x) = x$ for all $x \in X$ in Theorems 2.1,2.2,2.3,2.4. Initially, we provide a condition that is satisfied exclusively by the identity map I_X as follows:

Theorem 2.5. *Let (X, d) be a metric space, $C[x_1, \dots, x_k; r]$ any k-Cassini oval on X and the mapping $\mu : X \to [0, \infty)$ defined as in Theorem 2.1. T satisfies the condition*

(I_{ex})

$$d\left(x,Tx\right)\leqslant\frac{k}{k+1}\frac{\left|\mu\left(x\right)-\mu\left(Tx\right)\right|}{\max\left\{D_{1},D_{2}\right\}}$$

for all $x \in X$ if and only if $T = I_X$. Here we denote

$$D_{1}=\sum_{m=0}^{k-2}\left\{\left[d\left(x,Tx\right)\right]^{k-m-1}.\sum_{1\leqslant i_{1}<i_{2}<\ldots<i_{m}\leqslant k}\left[d\left(Tx,x_{i_{1}}\right).d\left(Tx,x_{i_{2}}\right).\cdots.d\left(Tx,x_{i_{m}}\right)\right]\right\}$$

$$+\sum_{\substack{m=k-1\\1\leqslant i_1<i_2<\ldots<i_m\leqslant k}}\left[d\left(Tx,x_{i_1}\right).d\left(Tx,x_{i_2}\right).....d\left(Tx,x_{i_m}\right)\right]$$

and

$$D_2=\sum_{m=0}^{k-2}\left\{\left[d\left(x,Tx\right)\right]^{k-m-1}.\sum_{1\leqslant i_1<i_2<\ldots<i_m\leqslant k}\left[d\left(x,x_{i_1}\right).d\left(x,x_{i_2}\right).....d\left(x,x_{i_m}\right)\right]\right\}$$
$$+\sum_{\substack{m=k-1\\1\leqslant i_1<i_2<\ldots<i_m\leqslant k}}\left[d\left(x,x_{i_1}\right).d\left(x,x_{i_2}\right).....d\left(x,x_{i_m}\right)\right].$$

Proof. Let T satisfy the condition (I_{ex}) and $x\neq Tx$ for some $x\in X$. Firstly, notice that $D_1>0$ and $D_2>0$, considering the fact that $x_1,x_2,...,x_k$ are distinct points and for $m=k-1$, the last parts of D_1 and D_2 have at least one positive term. In this case, we arrive at $\max\{D_1,D_2\}>0$, which does not lead to a division by zero error. One can directly distribute the following terms in left hand side as

$$\prod_{i=1}^{k}(a+b_i)=\sum_{m=0}^{k-1}a^{k-m}\sum_{1\leqslant i_1<i_2<\ldots<i_m\leqslant k}b_{i_1}.b_{i_2}.....b_{i_m}+\prod_{i=1}^{k}b_i\ ; \tag{2.2}$$

for $a\geqslant 0$, $b_i\geqslant 0$, $i\in\{1,2,...,k\}$. Here, let at most one of the b_i be 0; otherwise, all $b_i>0$. Our selection of numbers will become clear as we explore two options:

(i) If $\mu(x)\geqslant\mu(Tx)$, using (I_{ex}) and writing $a=d(x,Tx)$, $b_i=d(Tx,x_i)$ in equation (2.2), we have

$$\begin{aligned}(k+1)\,d(x,Tx) &\leqslant k\frac{\mu(x)-\mu(Tx)}{\max\{D_1,D_2\}}\leqslant k\frac{\prod_{i=1}^{k}d(x,x_i)-\prod_{i=1}^{k}d(Tx,x_i)}{D_1}\\ &\leqslant k\frac{\prod_{i=1}^{k}\left[d(x,Tx)+d(Tx,x_i)\right]-\prod_{i=1}^{k}d(Tx,x_i)}{D_1}\\ &= k\frac{d(x,Tx).D_1}{D_1}=k.d(x,Tx);\end{aligned}$$

a contradiction with $x\neq Tx$. Notice that when we factor out $d(x,Tx)$, we carefully separate the last term from the sum in order to avoid the indeterminate form $\left[0^0\right]$.

(ii) If $\mu(x)\leqslant\mu(Tx)$, using (I_{ex}) and writing $a=d(Tx,x)$, $b_i=d(x,x_i)$ in equation (2.2), we obtain

$$\begin{aligned}(k+1)\,d(x,Tx) &\leqslant k\frac{\mu(Tx)-\mu(x)}{\max\{D_1,D_2\}}\leqslant k\frac{\prod_{i=1}^{k}d(Tx,x_i)-\prod_{i=1}^{k}d(x,x_i)}{D_2}\\ &\leqslant k\frac{\prod_{i=1}^{k}\left[d(Tx,x)+d(x,x_i)\right]-\prod_{i=1}^{k}d(x,x_i)}{D_2}\\ &= k\frac{d(x,Tx).D_2}{D_2}=k.d(x,Tx);\end{aligned}$$

Again, this contradicts $x\neq Tx$.Hence, the assumption $x\neq Tx$ for some $x\in X$ is wrong. So $x=Tx$ for all $x\in X$, that is, $T=I_X$. The converse statement is trivial. □

REMARK 2.6. If a self-mapping $T : X \to X$ satisfies the conditions of Theorems 2.1 to 2.4, but does not satisfy the condition (I_{ex}), then we exclude the identity map. As conclusion, we can state the following final theorem:

THEOREM 2.6. *Let (X, d) be a metric space, $C[x_1, \ldots, x_k; r]$ any k-Cassini oval on X and the mapping $\mu : X \to [0, \infty)$ defined as in Theorem 2.1. If T satisfies the conditions of the Theorems 2.1 to 2.4, and in addition*

$$d(x, Tx) > \frac{k}{k+1} \frac{|\mu(x) - \mu(Tx)|}{\max\{D_1, D_2\}}$$

for some $x \in X$, then T is not the identity map and $C[x_1, \ldots, x_k; r]$ is the unique fixed k-Cassini oval of T.

For $k = 2$, the k-Cassini oval becomes the well-known Cassini oval and all of our results are valid. So we can give the following corollary for $k = 2$:

COROLLARY 2.1. *Let (X, d) be a metric space, $C[x_1, x_2; r]$ any Cassini oval on X and the mapping $\mu : X \to [0, \infty)$ defined as*

$$\mu(x) = d(x, x_1)\, d(x, x_2).$$

If T satisfies the conditions of the Theorems 2.1 to 2.4 (for $k = 2$), and in addition

$$d(x, Tx) > \frac{2}{3} \frac{|\mu(x) - \mu(Tx)|}{\max\{D_1, D_2\}}$$

for some $x \in X$, then T is not the identity map and $C[x_1, x_2; r]$ is the unique fixed Cassini oval of T. Here we denote

$$D_1 = d(x, Tx) + [d(Tx, x_1) + d(Tx, x_2)]$$

and

$$D_2 = d(x, Tx) + [d(x, x_1) + d(x, x_2)].$$

If we consider 3-Cassini ovals, we can see that

$$\begin{aligned} D_1 &= d(x, Tx)^2 + d(x, Tx) . [d(Tx, x_1) + d(Tx, x_2) + d(Tx, x_3)] \\ &\quad + [d(Tx, x_1) . d(Tx, x_2) + d(Tx, x_1) . d(Tx, x_3) + d(Tx, x_2) . d(Tx, x_3)] \end{aligned}$$

and

$$\begin{aligned} D_2 &= d(x, Tx)^2 + d(x, Tx) . [d(x, x_1) + d(x, x_2) + d(x, x_3)] \\ &\quad + [d(x, x_1) . d(x, x_2) + d(x, x_1) . d(x, x_3) + d(x, x_2) . d(x, x_3)]. \end{aligned}$$

The rest is generated from the general formulas in Theorem 2.5. To clarify, for $m = 0$, since the multiplicative identity is 1, we interpret

$$\sum_{1 \leqslant i_1 < i_2 < \ldots < i_m \leqslant k} b_{i_1} . b_{i_2} \ldots . b_{i_m}$$

as equal to 1, in our theorems. It is also noteworthy to mention that for $k = 1$, if we iterate backwards and consider $D_1 = D_2 = 1$; our results contain the fixed-circle identity exclusion theorem too.

3. An application to the Leaky Rectified Linear Unit (Leaky ReLU)

In this section, we give an application to the leakly rectified linear unit activation function to show the importance of the obtained results. The leakly rectified lienar unit activation function is defined as follows [**13**]:

$$LReLU(x) = \begin{cases} 0,01x, & x \leqslant 0 \\ x, & x > 0 \end{cases} \tag{3.1}$$

Let us take $X = \mathbb{R}$ with the usual metric. Then we get

$$\begin{aligned} C[1,3,5;15] &= \{x \in X : |x-1|.|x-3|.|x-5| = 15\} \\ &= \{0,6\}. \end{aligned}$$

The activation function $LReLU(x)$ satisfies the conditions (C_k1) and (C_k2) of Theorem 2.1 for $k=3$, $x_1 = 1$, $x_1 = 3$, $x_1 = 5$, $r = 15$, $\mu(x) = |x-1|.|x-3|.|x-5|$. Hence $C[1,3,5;15]$ is a fixed 3-Cassini oval of $T = LReLU$. $LReLU(x)$ does not satisfy (C_k3), so $C[1,3,5;15]$ is not unique.

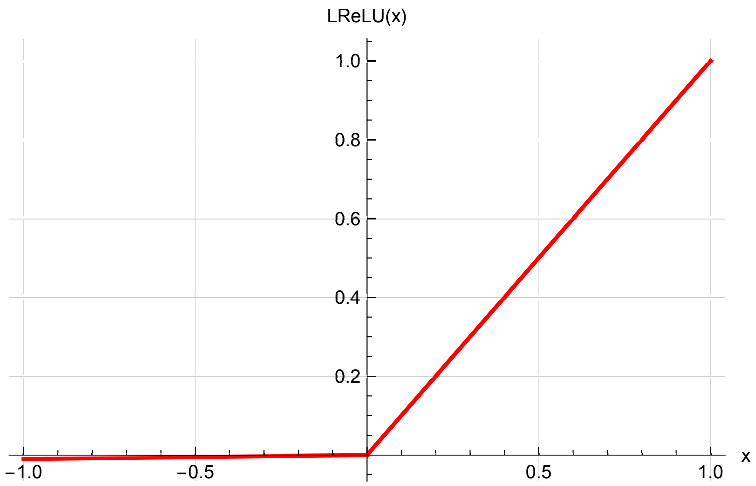

FIGURE 9. The $LReLU(x)$ Function

It is obvious that $LReLU(x) \neq I_X$. Let us see that for $x = -1 \in X = \mathbb{R}$, the condition (I_{ex}) is invalid. Since $T(-1) = -0,01$, we have

$$d(-1, T(-1)) = |-1 - (-0.01)| = 0,99$$

$$\mu(x) = \mu(-1) = |-1-1|.|-1-3|.|-1-5| = 48,$$

$$\mu(Tx) = \mu(-0,01) = |-0,01-1|.|-0,01-3|.|-0,01-5| = 15,185501$$

$$\begin{aligned} D_1 &= d(x,Tx)^2 + d(x,Tx).[d(Tx,x_1)+d(Tx,x_2)+d(Tx,x_3)] \\ &\quad + [d(Tx,x_1).d(Tx,x_2)+d(Tx,x_1).d(Tx,x_3)+d(Tx,x_2).d(Tx,x_3)] \\ &= (0,99)^2 + (0,99).[(1,01)+(3,01)+(5,01)] \\ &\quad + [(1,01).(3,01)+(1,01).(5,01)+(3,01).(5,01)] \\ &= 33,1001 \end{aligned}$$

$$\begin{aligned} D_2 &= d(x,Tx)^2 + d(x,Tx).[d(x,x_1)+d(x,x_2)+d(x,x_3)] \\ &\quad + [d(x,x_1).d(x,x_2)+d(x,x_1).d(x,x_3)+d(x,x_2).d(x,x_3)] \\ &= (0,99)^2 + (0,99).[2+4+6] \\ &\quad + [2.4+2.6+4.6] \\ &= 56,8601 \end{aligned}$$

$$\frac{k}{k+1}\frac{|\mu(x)-\mu(Tx)|}{\max\{D_1,D_2\}} = \frac{3}{4}\frac{|48-15,185501|}{56,8601} \approx 0,432832$$

As a result, for $x=-1$, we find

$$d(x,Tx) = 0,99 > 0,432832 \approx \frac{k}{k+1}\frac{|\mu(x)-\mu(Tx)|}{\max\{D_1,D_2\}}.$$

So, the fact that $LReLU(x) \neq I_X$ is also validated by satisfying the opposite of (I_{ex}).

References

1. H. Aytimur, Ş. Güvenç, and N. Taş, New fixed figure results with the notion of k-ellipse. *Mathematica Moravica,* **27** (1) (2023), 37–52.
2. S. Banach, Sur les operations dans les ensembles abstraitset leur application aux equations integrals. *Fundamenta Mathematicae*, **3** (1992), 133–181.
3. W. Blaschke, *Geometrie der gewebe*, Springer-Verlag, 1923.
4. H. Brezis, *Functional analysis, sobolev spaces and partial differential equations*, Springer, 2010.
5. L. E. J. Brouwer, Über die Topologie der Produkträume [On the topology of product spaces]. *Mathematische Annalen*, **71** (1911), 97–115.
6. G. Cassini, *Nova methodus pro solvenda propositio in curvis*, Acta Eruditorum, 1690.
7. G. Z. Erçınar, *Some geometric properties of fixed points*, Ph.D. Thesis, Eskişehir Osmangazi University, 2020.
8. S. K. Ghosh, Ö. Ege, J. Ahmad, A. Aloqaily, and N. Mlaiki, On elliptic valued b-metric spaces and some new fixed point results with an application. *AIMS Mathematics*, **9** (7) (2024), 17184–17204.
9. M. Joshi, A. Tomar, and S. K. Padaliya, Fixed point to fixed ellipse in metric spaces and discontinuous activation function. *Applied Mathematics E-Notes*, **21** (2021), 225–237.
10. M. Joshi, A. Tomar, H. A. Nabwey, and R. George, On unique and nonunique fixed points and fixed circles in Mbv-metric space and application to cantilever beam problem. *J. Funct. Spaces,* (2021), 6681044, 15 pages.
11. M. Joshi and A. Tomar, On unique and nonunique fixed points in metric spaces and application to chemical sciences. *J. Funct. Spaces,* (2021), 5525472, 11 pages.
12. S. Kakutani, A generalization of Brouwer's fixed-point theorem. *Duke Mathematical Journal,* **8 (3)** (1941), 457–459.
13. A. L. Maas, A. Y. Hannun, and A. Y. Ng, Rectifier nonlinearities improve neural network acoustic models. *Proc. ICML.* **30** (1) (2013).

14. G. Mani, S. Haque, A. J. Gnanaprakasam, Ö. Ege, and N. Mlaiki, The study of bicomplex-valued controlled metric spaces with applications to fractional differential equations. *Mathematics*, **11** (12) (2023), 2742, 1–19.
15. G. Nallaselli, A. J. Gnanaprakasam, G. Mani, Ö. Ege, D. Santina, and N. Mlaiki, A study on fixed point techniques under the $\alpha - F$-convex contraction with an application. *Axioms*, **12** (2) (2023), 139, 1–18.
16. G. Nallaselli, A. J. Gnanaprakasam, G. Mani, and Ö. Ege, Novel results of an orthogonal $(\alpha - F)$-convex contraction mappings. *Rocky Mountain Journal of Mathematics*, **54** (5) (2024), 1411–1425.
17. N. Y. Özgür, and N. Taş, Some fixed-circle theorems on metric spaces. *Bull. Malays. Math. Sci. Soc.* **42** (4) (2019), 1433–1449.
18. B. E. Rhoades, A comparison of various definitions of contractive mappings. *Transactions of the American Mathematical Society*, **266** (1977), 257–290.
19. J. Schauder, Zur theorie der topologischen abbildungen [On the theory of topological mappings]. *Mathematische Annalen*, **102** (1930), 169–188.
20. N. Taş and N. Özgür, New fixed-figure results on metric spaces. in: Debnath, P., Srivastava, H.M., Kumam, P., Hazarika, B. (eds) *Fixed Point Theory and Fractional Calculus.* Forum for Interdisciplinary Mathematics. Springer, Singapore, 2022.

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